## A BINOMIAL IDENTITY PROOF WITH PICTURES

MIKE ZABROCKI

I set myself a high goal in class and stated that I could prove the following binomial identity in just four sentences.

$$
\binom{2(2 n+1)}{2 n+1}=\sum_{k=0}^{n}\left[\binom{2 n+2-k}{2 n+1-2 k}\binom{2 n-1+k}{2 k}+\binom{2 n+2-k}{2 n-2 k}\binom{2 n+k}{2 k+1}\right]
$$

My sentences are going to be long (but precise), refer to the diagrams below, and use the identity that we proved in class that the number of paths in an $r \times m$ rectangle is $\binom{m+r}{r}$.


Proof. Every path from the SW corner to the NE corner of an $(2 n+1) \times(2 n+1)$ rectangle must either pass through a horizontal segment in row $2 k$ and connecting column $k+1$ to $k+2$ for $0 \leq k \leq n$ or it passes through a vertex in row $2 k+1$ and column $k+2$ for $0 \leq k \leq n$ because if it passes through the vertex at row $2 k$ and column $k+2$ then it passes through a point that is further E than all edges in rows $2 i$ and and columns $i+1 \rightarrow i+2$ and all vertices in rows $2 i+1$ and columns $i+2$ for $0 \leq i<k$ and it is passes through a point which is further N than all vertices in rows $2 \ell+1$ and column $\ell+2$ and horizontal segments in rows $2 \ell$ and column $\ell+2$ for $k<\ell \leq 2 n+1$ and so cannot pass through any of the other highlighted vertices or horizontal segments.


The number of paths that pass through the segment in row $2 k$ and whose left edge is at $k+1$ and right edge at $k+2$ consists of a path in a rectangle of height $2 n+1-2 k$ and width $k+1$, followed by the horizontal segment, followed by a path in a rectangle to the NE corner of height $2 k$ and width $2 n+1-(k+2)$, so there are

$$
\binom{(2 n+1-2 k)+(k+1)}{2 n+1-2 k}\binom{(2 n+1-(k+2))+2 k}{2 k}=\binom{2 n+2-k}{2 n+1-2 k}\binom{2 n-1+k}{2 k}
$$

such paths.


The number of paths that pass through the vertex in row $2 k+1$ and column $k+2$ consists of a path in a rectangle of height $2 n+1-(2 k+1)$ and width $k+2$ followed by a path from the vertex to the NE corner in a rectangle of height $2 k+1$ and width $2 n+1-(k+2)$, hence there are

$$
\binom{(2 n-2 k)+(k+2)}{2 n-2 k}\binom{(2 n-1-k)+(2 k+1)}{2 k+1}=\binom{2 n-k+2}{2 n-2 k}\binom{2 n+k}{2 k+1}
$$

such paths.
Since there are $\binom{2(2 n+1)}{2 n+1}$ total paths in a $(2 n+1) \times(2 n+1)$ square we have that the total number of paths is equal to the number of paths which pass through the horizontal segments plus the number of paths which pass through the highlighted vertices for each $0 \leq k \leq n$, so that

$$
\binom{2(2 n+1)}{2 n+1}=\sum_{k=0}^{n}\binom{2 n+2-k}{2 n+1-2 k}\binom{2 n-1+k}{2 k}+\sum_{k=0}^{n}\binom{2 n+2-k}{2 n-2 k}\binom{2 n+k}{2 k+1}
$$

