A BINOMIAL IDENTITY PROOF WITH PICTURES

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I set myself a high goal in class and stated that I could prove the following binomial identity in just four sentences.

$$\binom{2(2n+1)}{2n+1} = \sum_{k=0}^{n} \left[\binom{2n+2-k}{2n+1-2k} \binom{2n-1+k}{2k} + \binom{2n+2-k}{2n-2k} \binom{2n+k}{2k+1} \right]$$

My sentences are going to be long (but precise), refer to the diagrams below, and use the identity that we proved in class that the number of paths in an $r \times m$ rectangle is $\binom{m+r}{r}$.



Proof. Every path from the SW corner to the NE corner of an $(2n+1) \times (2n+1)$ rectangle must either pass through a horizontal segment in row 2k and connecting column k + 1 to k + 2 for $0 \le k \le n$ or it passes through a vertex in row 2k + 1 and column k + 2 for $0 \le k \le n$ because if it passes through the vertex at row 2k and column k + 2 then it passes through a point that is further E than all edges in rows 2i and and columns $i + 1 \rightarrow i + 2$ and all vertices in rows 2i + 1 and columns i + 2 for $0 \le i < k$ and it is passes through a point which is further N than all vertices in rows $2\ell + 1$ and column $\ell + 2$ and horizontal segments in rows 2ℓ and column $\ell + 2$ for $k < \ell \le 2n + 1$ and so cannot pass through any of the other highlighted vertices or horizontal segments.



The number of paths that pass through the segment in row 2k and whose left edge is at k + 1 and right edge at k + 2 consists of a path in a rectangle of height 2n + 1 - 2k and width k + 1, followed by the horizontal segment, followed by a path in a rectangle to the NE corner of height 2k and width 2n + 1 - (k + 2), so there are

$$\binom{(2n+1-2k)+(k+1)}{2n+1-2k}\binom{(2n+1-(k+2))+2k}{2k} = \binom{2n+2-k}{2n+1-2k}\binom{2n-1+k}{2k}$$

such paths.



The number of paths that pass through the vertex in row 2k+1 and column k+2 consists of a path in a rectangle of height 2n+1-(2k+1) and width k+2 followed by a path from the vertex to the NE corner in a rectangle of height 2k+1 and width 2n+1-(k+2), hence there are

$$\binom{(2n-2k)+(k+2)}{2n-2k}\binom{(2n-1-k)+(2k+1)}{2k+1} = \binom{2n-k+2}{2n-2k}\binom{2n+k}{2k+1}$$

such paths.

Since there are $\binom{2(2n+1)}{2n+1}$ total paths in a $(2n+1) \times (2n+1)$ square we have that the total number of paths is equal to the number of paths which pass through the horizontal segments plus the number of paths which pass through the highlighted vertices for each $0 \le k \le n$, so that

$$\binom{2(2n+1)}{2n+1} = \sum_{k=0}^{n} \binom{2n+2-k}{2n+1-2k} \binom{2n-1+k}{2k} + \sum_{k=0}^{n} \binom{2n+2-k}{2n-2k} \binom{2n+k}{2k+1}.$$