HOMEWORK ASSIGNMENT NO. 6

DATE GIVEN: MARCH 19, 2012 DUE: DO IT BUT DON'T HAND IT IN

(1) Below are listed relations on sets. Tell whether they are transitive, reflexive and/or symmetric. Explain with a short sentence for each why or why not. If they do not satisfy a property then you should provide a counter-example.
   (a) $R_1 = \{(x, y) : x, y \in \mathbb{R}, x = \pm y \text{ or } \pm 2y\}$ as a relation on $\mathbb{R}$
   (b) $R_2 = \{(x, y) : x, y \in \mathbb{R}, x = \pm\frac{1}{2}y \text{ or } \pm 2y\}$ as a relation on $\mathbb{R}$
   (c) $R_3 = \{(x, y) : x, y \in \mathbb{R}, |x - y| \leq 2\}$ as a relation on $\mathbb{R}$
   (d) $R_4 = \{(x, y) : x, y \in \mathbb{Z}, \exists r \in \mathbb{Z}, x - y = r^2\}$ as a relation on integers.
   (e) $R_5 = \{(x, y) : x, y \in \mathbb{Z}, \exists r \in \mathbb{Z}, x + y = r^2\}$ as a relation on integers.

(2) A relation is called *anti-symmetric* if $(x, y) \in R$ implies that $(y, x) \notin R$. Which of the examples of relations above are anti-symmetric? Explain your answer.

(3) The following questions are be about words in the letters a and b. A word w will be represented as a sequence of letters $w_1w_2 \cdots w_n$ where n is the length of the word and w_i is either a or b. If u and v are words of length n and r (respectively) then $uv = u_1u_2 \cdots u_nv_1v_2 \cdots v_r$ is the concatenation of the words.

   A word in the letters a and b are called balanced if it has the same number of a's as b's. A word is called Catalan if $w = w_1w_2 \ldots w_n$ is a balanced word, and $w_1w_2 \ldots w_k$ has at least as many a's as b's for each k between 1 and n (e.g. $aabb$ and $abab$ are Catalan, but $abbb$ and $abba$ are not).

   The following statements are either true or false for all words w and u in a's and b's. For each statement, if it is true, provide a proof; if it is false, provide a counter-example.
   (a) If w is balanced and $w = uv$, then u is balanced.
   (b) If w is Catalan and $w = uv$ where u is balanced, then u is Catalan.
   (c) If w is a Catalan word, then $aw_1w_2 \ldots w_nb$ is also a Catalan word.
   (d) If w is Catalan word and u is balanced, then wu is Catalan.
   (e) If $u = awb = u_1u_2...u_n$ where w is Catalan, then $u_1u_2...u_k$ is not Catalan for any $k < n$.