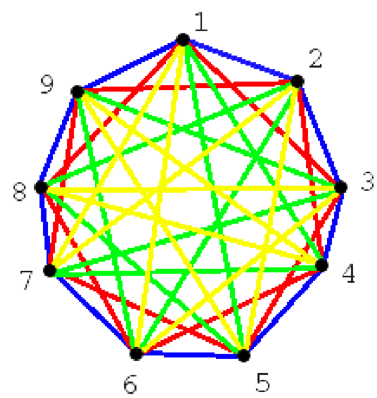
HOMEWORK #2 - MATH 3260

ASSIGNED: JANUARAY 30, 2003 DUE: FEBRUARY 15, 2002 AT 2:30PM

(1) (a) Give by listing the sequence of vertices 4 Hamiltonian cycles in K_9 no two of which have an edge in common.

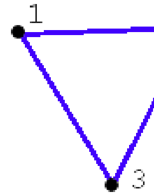
Solution: Here is one set of 4 Hamiltonian cycles. There are many different cycles that are also disjoint.

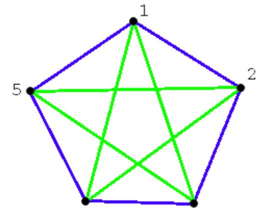
$$1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 5 \leftrightarrow 6 \leftrightarrow 7 \leftrightarrow 8 \leftrightarrow 9 \leftrightarrow 1$$
$$1 \leftrightarrow 3 \leftrightarrow 5 \leftrightarrow 7 \leftrightarrow 9 \leftrightarrow 2 \leftrightarrow 4 \leftrightarrow 6 \leftrightarrow 8 \leftrightarrow 1$$
$$1 \leftrightarrow 4 \leftrightarrow 7 \leftrightarrow 3 \leftrightarrow 9 \leftrightarrow 6 \leftrightarrow 2 \leftrightarrow 8 \leftrightarrow 5 \leftrightarrow 1$$
$$1 \leftrightarrow 6 \leftrightarrow 3 \leftrightarrow 8 \leftrightarrow 4 \leftrightarrow 9 \leftrightarrow 5 \leftrightarrow 2 \leftrightarrow 7 \leftrightarrow 1$$

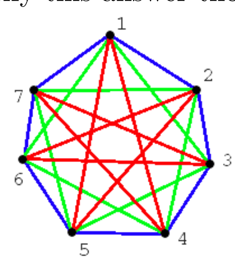


(b) What is the maximum number of edge disjoint Hamiltonian cycles in K_{2k+1}?

Solution: We can find 1 disjoint Hamiltonian cycle of K_3, 2 disjoint cycles of K_5, 3 disjoint cycles of K_7, and 4 disjoint cycles of K_9. This indicates that one can find k disjoint Hamiltonian cycles in K_{2k+1}. We need to justify this answer though.





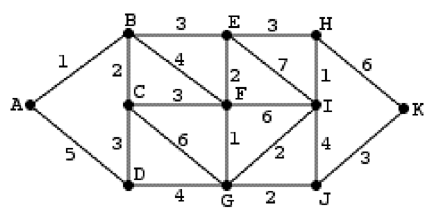


Certainly there can be no more than k disjoint Hamiltonian cycles because each cycle has $2k + 1$ edges and if there are k of them then we have used up $k(2k + 1)$ edges and there are only $k(2k+1)$ edges in K_{2k+1}. There could be less cycles, so we need to describe k of them.

I will describe here how to produce a Hamiltonian cycles for each integer between 1 and k which is relatively prime to $2k+1$. If $2k+1$ is prime then this produces k different cycles. If $2k+1$ is not prime then I don't know how to produce k different cycles (and for all I know they may not exist).

Let d equal a number 1 through k which is relatively prime to $2k+1$. The subgraph that has an edge from i to j if either $i-j-d$ is divisible by $2k+1$ or $j-i-d$ is divisible by $2k+1$ is a cycle because the first vertex is 1, the second $1+d$, the third $1+2d$, the fourth will be $1+3d$ $(mod\ 2k+1)$, etc. This produces $2k+1$ different vertices because if d is relatively prime to $2k+1$ then the numbers 0, d $(mod\ 2k+1)$, $2d$ $(mod\ 2k+1)$, etc. are all distinct (I have to appeal to some algebra to say why this is true) and so this produces a Hamiltonian cycle. I claim also that each of these Hamiltonian cycles are distinct. For each pair $i \leftrightarrow j$, $-2k \leq i-j \leq 2k$ and without loss of generality, assume $i > j$ and $1 \leq i-j \leq 2k$. If $1 \leq i-j \leq k$ then there is exactly one integer between 1 and k such that $i-j-d$ is divisible by $2k+1$ (namely $d = i-j$) and if $k+1 \leq i-j \leq 2k$ then $d = 2k+1-i+j$ is between 1 and k and $j-i-d$ is divisibly by $2k+1$. Therefore exactly one cycle corresponding to a particular d uses the edge $i \leftrightarrow j$.

(2) Find the shortest path from A to each of the other vertices in the weighted graph of the figure below. Draw a spanning tree rooted at A with smallest weight. Are there others?

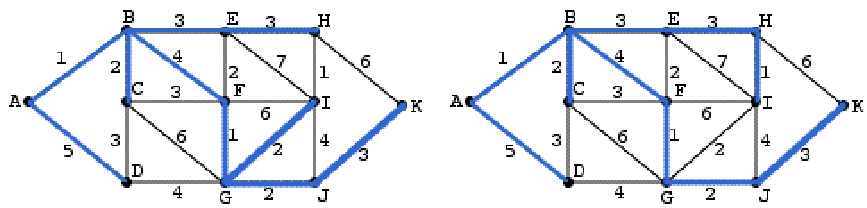


Solution: We will do a table as in the algorithm we did in class.

distance	*vertex*	A	B	C	D	E	F	G	H	I	J	K
0	A	0	1		5							
1	B	0	1	3	5	4	5					
3	C	0	1	3	5	4	5	9				
4	E	0	1	3	5	4	5	9	7	11		
5	F	0	1	3	5	4	5	6	7	11		
5	D	0	1	3	5	4	5	6	7	11		
6	G	0	1	3	5	4	5	6	7	8	8	
7	H	0	1	3	5	4	5	6	7	8	8	13
8	I	0	1	3	5	4	5	6	7	8	8	13
8	J	0	1	3	5	4	5	6	7	8	8	11
11	K	0	1	3	5	4	5	6	7	8	8	11

There are two possible shortest trees and this comes from the step in the algorithm when at a distance of 7 from A, the shortest path from A to I is already 8 (by passing through

G) and the edges from H to I will also create a path from A to I of length 8. Below are the only two minimal length spanning trees that are rooted at A.

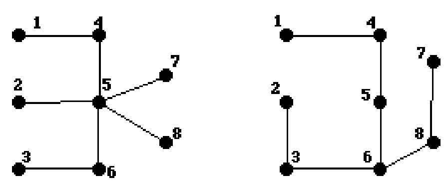


(3) Let T_1 and T_2 be spanning trees of a connected graph G.

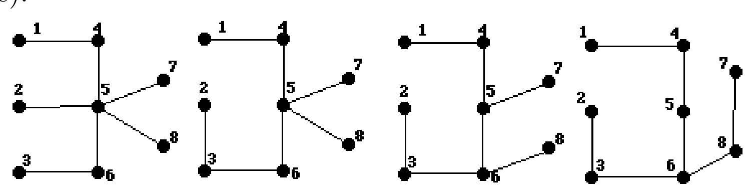
(a) If e is any edge of T_1, show that there exists an edge f of T_2 such that the graph with edge set equal to $E(T_1) - \{e\} \cup \{f\}$ (obtained from T_1 by replacing e by f) is also a spanning tree.

Solution: Let e be an edge of T_1. If e is also an edge of T_2 then $f = e$ has the property that the tree with edge set equal to $E(T_1) - \{e\} \cup \{f\}$ is the same as T_1 and so is a spanning tree. If this edge is not in T_2 then if it is added to the edge set of T_2 then it creates exactly one cycle (by theorem 9.1.vi). Take this cycle and add all edges into T_1 and find the smallest cycle which contains e (there is at least one since the whole cycle contains e). Take f to be an edge of this cycle which is not in T_1 (since T_1 has no cycles there must be at least one). Now the graph consisting of edge set $E(T_1) - \{e\} \cup \{f\}$ has $n - 1$ edges and no cycles and by theorem 9.1.ii is a tree.

(b) Transform the graph on the left to the one on the right by a sequence of trees each of which differs from the next by a single edge. Explain why this can be done for any two trees with the same vertex set.



Solution: Here is a sequence of trees which transforms the first tree into the second (it is not unique).



The reason why this always works on any two trees with the same vertex set is that we can apply the first part of this problem with any edge e which is not in the second tree. There is an edge f in the second tree which is not in the first and obtain a tree with edge set $E(T_1) - \{e\} \cup \{f\}$ that will have one more edge in common with the second tree and one edge which is not in common with the first. If we continue to apply this result, each successive tree has one more edge in common with the second graph and one less edge in common with the first graph until eventually the trees differ by exactly one edge.

(4) How many spanning trees does W_n have for $n \geq 4$?

Answer: Let a_n be the number of spanning trees of the wheel graph for $n \geq 4$, then

$$a_n = \left(\frac{3+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{3-\sqrt{5}}{2}\right)^{n-1} - 2$$

This is a pretty amazing formula, huh? Compute the first few and you will find $a_4 = 16$, $a_5 = 45$, $a_6 = 121$, $a_7 = 320$, etc. You will also find that $a_n = 3a_{n-1} - a_{n-2} + 2$.

Solution: Using the Matrix-Tree Theorem (Theorem 10.3 in your book) we see that a_n is equal to

$$a_n = det \begin{vmatrix} 3 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 3 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & -1 & 3 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix}$$

where the determinant is an $(n-1) \times (n-1)$ matrix. This is tough to compute, but it isn't too bad if you use the right notation. Let B_n be the $n \times n$ matrix with $b_{ii} = 3$ and $b_{ij} = -1$ if $|i-j| = 1$ and $b_{ij} = 0$ if $|i-j| > 1$. That is,

$$B_n = \begin{bmatrix} 3 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & -1 & 3 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 3 \end{bmatrix}$$

Expand the determinant that a_n is equal to about the first row and we have

$$a_n = 3 \cdot det\,|B_{n-2}| + \begin{vmatrix} -1 & -1 & 0 & \cdots & 0 \\ 0 & & & & \\ \vdots & & B_{n-3} & & \\ 0 & & & & \\ -1 & & & & \end{vmatrix} + (-1)^n \begin{vmatrix} -1 & & & & \\ 0 & & & & \\ \vdots & & B_{n-3} & & \\ 0 & & & & \\ -1 & 0 & \cdots & 0 & -1 \end{vmatrix}.$$

Expanding these last two determinants even further we can show that the relation is:

$$a_n = 3 \cdot det\,[B_{n-2}] - 2 \cdot det\,[B_{n-3}] - 2$$

Notice that the $det\,[B_n]$ also satisfies a recurrence

$$det\,[B_n] = 3 \cdot det\,[B_{n-1}] - det\,[B_{n-2}]$$

which we have found by expanding $det\,[B_n]$ about the first row. Now combine these two recurrences and we find that

$$\begin{aligned} a_n &= 3\cdot(3\cdot det\,[B_{n-3}] - det\,[B_{n-4}]) - 2\cdot(3\cdot det\,[B_{n-4}] - det\,[B_{n-5}]) - 2 \\ &= 3\cdot(3\cdot det\,[B_{n-3}] - 2\cdot det\,[B_{n-4}] - 2) - (3\cdot det\,[B_{n-4}] - 2\cdot det\,[B_{n-5}] - 2) + 2 \\ &= 3\cdot a_{n-1} - a_{n-2} + 2 \end{aligned}$$

Therefore we have shown that these numbers a_n satisfy a recurrence $a_n = 3\cdot a_{n-1} - a_{n-2} + 2$. But now look at the answer that was given above, because it too satisfies the same recurrence. If we assume that $a_k = \left(\frac{3+\sqrt{5}}{2}\right)^{k-1} + \left(\frac{3-\sqrt{5}}{2}\right)^{k-1} - 2$ for $k < n$, then since $\frac{7+3\sqrt{5}}{2} = \left(\frac{3+\sqrt{5}}{2}\right)^2$ and $\frac{7-3\sqrt{5}}{2} = \left(\frac{3-\sqrt{5}}{2}\right)^2$ we have then

$$\begin{aligned} a_n &= 3a_{n-1} - a_{n-2} + 2 \\ &= 3\left(\frac{3+\sqrt{5}}{2}\right)^{n-2} + 3\left(\frac{3-\sqrt{5}}{2}\right)^{n-2} - 6 - \left(\frac{3+\sqrt{5}}{2}\right)^{n-3} - \left(\frac{3-\sqrt{5}}{2}\right)^{n-3} + 2 + 2 \\ &= \left(\frac{3+\sqrt{5}}{2}\right)^{n-3}\left(\frac{9+3\sqrt{5}}{2} - 1\right) + \left(\frac{3-\sqrt{5}}{2}\right)^{n-3}\left(\frac{9-3\sqrt{5}}{2} - 1\right) - 2 \\ &= \left(\frac{3+\sqrt{5}}{2}\right)^{n-3}\left(\frac{7+3\sqrt{5}}{2}\right) + \left(\frac{3-\sqrt{5}}{2}\right)^{n-3}\left(\frac{7-3\sqrt{5}}{2}\right) - 2 \\ &= \left(\frac{3+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{3-\sqrt{5}}{2}\right)^{n-1} - 2 \end{aligned}$$

To justify that this works we still need to verify a couple of base cases (to make sure that an argument by induction works) for values of a_n and show that they satisfy the formula listed above, but this is not too difficult especially when we know that $a_4 = 16$ from Theorem 10.1 and with a little counting argument we can show that $a_5 = 45$.

(5) Prove that any tree which is semi-Hamiltonian is isomorphic to P_n.

A tree on n vertices has $n-1$ edges. Any graph with n vertices which is semi-Hamiltonian has a subgraph isomorphic to P_n that passes through each vertex exactly once and this path has $n-1$ edges. A tree, T, which is semi-Hamiltonian has the path as a subgraph and this path has the same number of edges and vertices as T and so is equal to the whole graph.