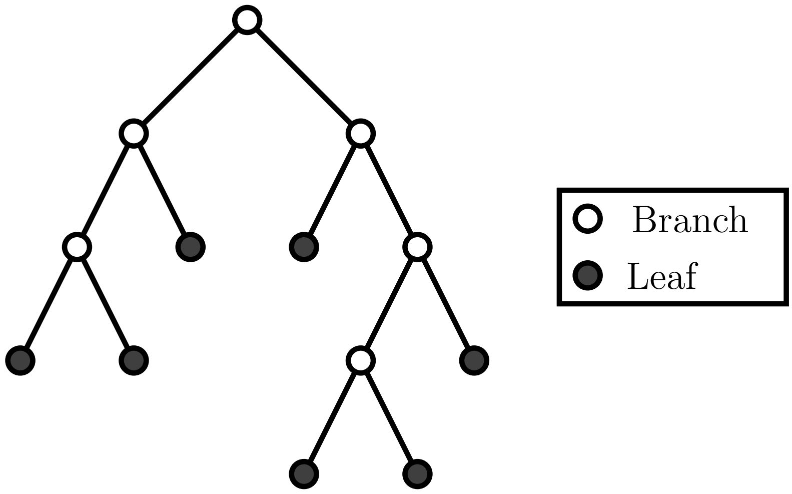
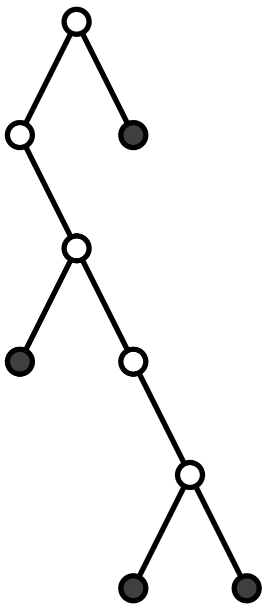
Binary Trees

Complete

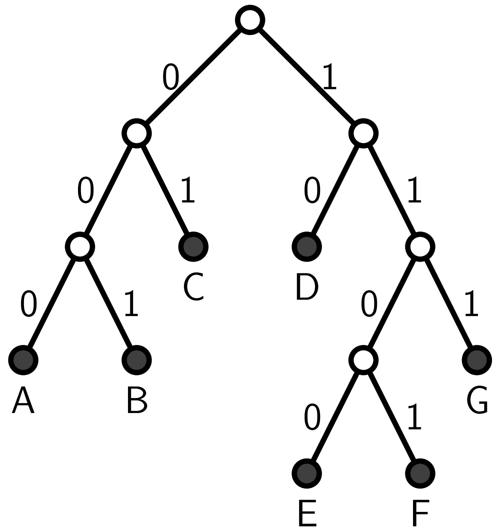


Incomplete



A Comma-Free Binary Code

Definition: A binary code is *comma-free* if no prefix of the code of a letter is the code of another letter.



A=000	B=001	C=01
D=10	E=1100	F=1101
G=111		

$$\text{File length} = 3N_A + 3N_B + 2N_C + 2N_D + 4N_E + 4N_F + 3N_G$$

Morse Code

A	•—	J	•———	S	•••
B	—•••	K	—•—	T	—
C	—•—•	L	•—••	U	••—
D	—••	M	——	V	•••—
E	•	N	—•	W	•——
F	••—•	O	———	X	—••—
G	——•	P	•——•	Y	—•——
H	••••	Q	——•—	Z	——••
I	••	R	•—•		

• $\leftrightarrow 0$ — $\leftrightarrow 10$ $Comma \leftrightarrow 11$

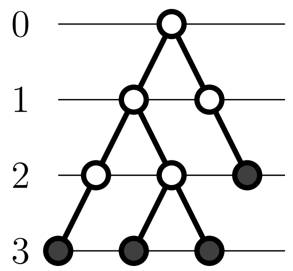
Expected code length

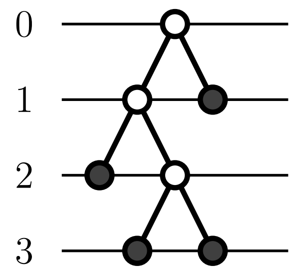
$$5N_A + 7N_B + 8N_C + \cdots + 9N_Y + 8N_Z$$

Using single letter english frequencies, the average number of bits per letter is

$$\frac{5 \cdot 73 + 7 \cdot 9 + 8 \cdot 30 + \cdots + 9 \cdot 19 + 8 \cdot 1}{1000} = 5.738$$

Leaf Heights





$$\frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^2} = 5/8 \qquad \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^1} = 1$$

Theorem 1 *The sequence of integers* $h_1, h_2, \ldots, h_n$ *are leaf heights of a binary tree if and only if*

$$\sum_{i=1}^{n} \frac{1}{2^{h_i}} \leq 1$$

with equality only if the tree is complete.

Expected Code Length

Theorem 2 *The best possible expected code length (bits per letter) is*

$$\boxed{H = \sum_{i=1}^{n} p_i \log_2 1/p_i}$$

Proof.

$$\text{Letter frequenices } N_1, N_2, \ldots, N_k \ (N = \textstyle\sum_{i=1}^{k} N_i)$$

$$\text{Code lengths } h_1, h_2, \ldots, h_k \text{ (from a binary tree)}$$

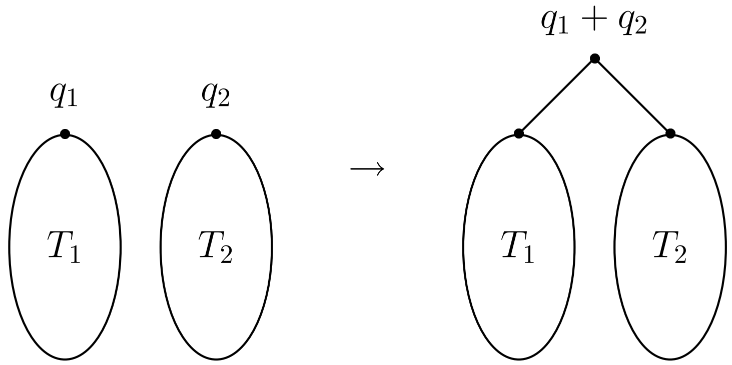
$$p_i = N_i/N \text{ and } q_i = 1/2^{h_i}$$

$$\begin{aligned}
\text{File length} &= \sum_{i=1}^{k} N_i h_i \\
&= \sum_{i=1}^{k} N_i \log_2 2^{h_i} \\
&= N \sum_{i=1}^{k} p_i \log_2 1/q_i \\
&\geq N \sum_{i=1}^{k} p_i \log_2 1/p_i = NH
\end{aligned}$$

Huffman Heights

Theorem 3 *A letter that occurs with probability p will be represented by a leaf with height $h \leq \lceil \log_2 1/p \rceil$ in the Huffman tree.*

Proof (by induction) Assume that the height of a particular leaf in T_1 is $h \leq \lceil \log_2 q_1/p \rceil$.



The height of that same leaf in the resulting tree is exactly $h+1$, which is bounded above by:

$$h + 1 \leq \left\lceil \log_2 \frac{q_1}{p} + 1 \right\rceil = \left\lceil \log_2 \frac{2q_1}{p} \right\rceil \leq \left\lceil \log_2 \frac{q_1 + q_2}{p} \right\rceil$$

Note that $q_2 \geq q_1$, as assumed in the construction of the Huffman code.

Expected Code Length

Theorem 4 *The Huffman Code yields expected code length within 1 of the entropy, H.*

Proof. Assume that $h_i = \lceil \log_2 1/p_i \rceil$.

$$\sum_{i=1}^{k} \frac{1}{2^{h_i}} = \sum_{i=1}^{k} \frac{1}{2^{\lceil \log_2 1/p_i \rceil}} \leq \sum_{i=1}^{k} \frac{1}{2^{\log_2 1/p_i}} = \sum_{i=1}^{k} \frac{1}{1/p_i} = 1$$

Therefore the sequence $h_1, h_2, \ldots h_k$ corresponds to a binary tree

$$\text{Expected code length (ECL)} = \sum_{i=1}^{k} p_i h_i = \sum_{i=1}^{k} p_i \lceil \log_2 1/p_i \rceil$$

$$H = \sum_{i=1}^{k} p_i \log_2 1/p_i \leq ECL \leq \sum_{i=1}^{k} p_i (\log_2 1/p_i + 1) = H + 1$$