HALF A PROOF OF PROBLEM # 15 ON PAGE 7

FROM CLASS JANUARY 26, 2004

To solve this problem I needed two Lemmas.

This is the problem # 10 on page 6. You should be able to solve it without referring to problem # 15.

Lemma 1.

$$F_{n+1}^2 = F_{n+2}F_n + (-1)^{n+1}$$

The following is either another exercise or the definition of L_n. I don't have the text at the moment to look it up.

Lemma 2.

$$L_n = F_{n-1} + F_{n+1}$$

Theorem 1. *For all $n \geq 1$,*

$$F_{2n} = L_n F_n$$

$$F_{2n+1} = F_{n+1}L_n + (-1)^n$$

Proof. We will prove these two statements by induction simultaneously. They are true for $n = 1$ since $F_2 = 1 = F_1 L_1$ and $F_3 = 2 = F_2 L_1 + 1$. Now assume that for a fixed n we have $F_{2n} = L_n F_n$ and $F_{2n+1} = F_{n+1}L_n + (-1)^n$.

Now for all k we have $F_k = F_{k-1} + F_{k-2}$ by definition. Therefore,

$$\begin{aligned}
F_{2n+2} &= F_{2n+1} + F_{2n} && \text{definition of } F_{2n+2} \\
&= F_{n+1}L_n + (-1)^n + L_n F_n && \text{inductive assumtion} \\
&= (F_{n+1} + F_{n-1})(F_{n+1} + F_n) + (-1)^n && \text{Lemma 2} \\
&= F_{n+1}^2 + F_{n+1}F_n + F_{n-1}F_{n+1} + F_n F_{n-1} + (-1)^n \\
&= F_{n+2}F_n + (-1)^{n+1} + F_{n+1}F_n + F_{n-1}F_{n+1} + F_n F_{n-1} + (-1)^n && \text{Lemma 1} \\
&= F_{n+2}F_n + (F_n + F_{n-1})F_n + F_{n-1}F_{n+1} + F_n F_{n-1} && \text{definition of } F_{n+1} \\
&= F_{n+2}F_n + F_n F_n + F_{n+1}F_{n-1} + F_n F_{n-1} + F_n F_{n-1} \\
&= F_{n+2}F_n + F_n F_n + F_{n+2}F_{n-1} + F_n F_{n-1} && \text{definition of } F_{n+2} \\
&= (F_{n+2} + F_n)(F_n + F_{n-1}) \\
&= L_{n+1}F_{n+1} && \text{Lemma 2}
\end{aligned}$$

Similarly we have

$$\begin{aligned}
F_{2n+3} &= F_{2n+2} + F_{2n+1} && \text{definition of } F_{2n+2} \\
&= L_{n+1}F_{n+1} + F_{n+1}L_n + (-1)^n && \text{inductive assumption and previous result} \\
&= \cdots && \text{fill in the missing proof here} \\
&= F_{n+2}L_{n+1} + (-1)^{n+1}
\end{aligned}$$

□