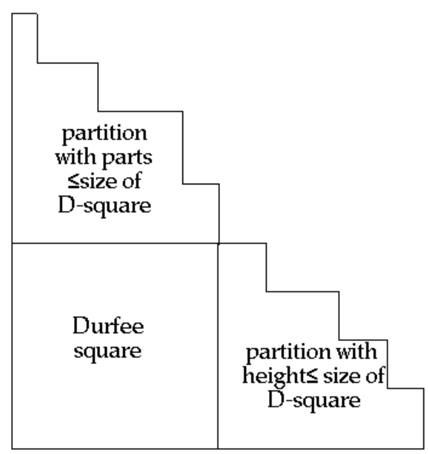
PROOF OF A GENERATING FUNCTION EXPRESSION

MARCH 23, 2004

I wanted to show you all that the three equations that I asked you to prove are not unrealistic. You have all of the information you need in order to prove them. I will do this by showing you how to prove the first one.

Look at the following diagram for a partition divided into pieces. The total length of the partition is the length of the Durfee square plus the length of the partition whose parts are less than or equal to the size of the square.



Note that

$$z^k q^{k^2} \prod_{i=1}^{k} \frac{1}{(1-q^i)(1-zq^i)}$$

is the generating function for a Durfee square of size k (with $q^{\text{size of Durfee square}} z^{\text{length of square}}$, a partition of height less than or equal to k (weighted by $q^{\text{size of the partition}}$), and a partition with parts smaller than or equal to k (weighted by $q^{\text{size of the partition}} z^{\text{length of the partition}}$).

Now every partition weighted by $z^{\text{length of the partition}} q^{\text{size of the partition}}$ (with generating function $\prod_{i\geq 1} \frac{1}{1-zq^i}$) is either empty or it has a Durfee square of size 1, or it has a Durfee square of size 2, or it has a Durfee square of size 3, etc. Therefore

$$\prod_{i\geq 1} \frac{1}{1-zq^i} = 1 + \sum_{k\geq 1} \frac{z^k q^{k^2}}{\prod_{i=1}^{k}(1-q^i)(1-zq^i)}$$

The other two identities are extremely similar. The formula,

$$\prod_{i\geq 1} \frac{1}{1-zq^i} = 1 + \sum_{k\geq 1} \frac{z^k q^k}{\prod_{i=1}^{k}(1-q^i)}$$

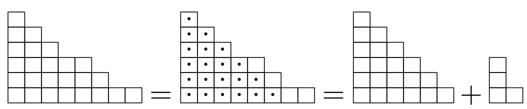
says that every partition is either empty or it has height 1, or it has height 2, or it has height 3, etc.

The formula,

$$\prod_{i\geq 1}(1+zq^i) = 1 + \sum_{k\geq 1} \frac{z^k q^{k(k+1)/2}}{\prod_{i=1}^{k}(1-q^i)}$$

says that every partition with distinct parts consists of a staircase of length k and a partition of height less than or equal to k.

For example the partition $8+6+5+3+2+1$ that has distinct parts can be drawn and we notice that it contains a staircase partition $6+5+4+3+2+1$ and what remains is a regular partition $2+1+1$.



The formula is equivalent to the remark that EVERY partition with distinct parts contains some staircase partition and a partition of height less than or equal to the staircase. The weight of $q^{k(k+1)/2}$ comes from $k+(k-1)+(k-2)+\cdots+2+1 = k(k+1)/2$.