WORKSHEET VI: COMBINATORIAL PROBLEMS AND GENERATING FUNCTIONS

MARCH 2, 2006

On the Forum, call these "GF COUNTING PROBLEM #xxx"

Translate the following combinatorial problems depending on the unknown n into generating functions expressions in the variable q. Use a computer or other means to find the specified coefficient.

(1) The number of ways are there of distributing n identical jelly beans among four children:
   (a) without restriction
   (b) With one child getting at least 10 jelly beans and another child getting at most 10 jelly beans.
   Coefficient of q^{40}.

(2) The number of integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = n$ with
   (a) $x_i \geq 0$
   (b) $x_i > 0$
   (c) $x_i \geq i$ (for each $i = 1, 2, 3, 4, 5$)
   Coefficient of q^{28}.

(3) The number of integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 \leq n$ with $x_i \geq 0$. (hint: build on problem (2) (a)) Coefficient of q^{28}.

(4) The number of integer solutions to $x_1+x_2+x_3+x_4+x_5 = m$ with $m \leq n$ and $m \equiv n \ (mod\ 2)$ and with $x_i \geq 0$. (hint: build on problem (2) (a)) Coefficient of q^{28}.

(5) The number of ways to distribute identical balls into n distinct boxes.
   Coefficient of q^k.

(6) The number of ways to distribute n identical balls into 6 boxes with the first two boxes collectively having *at most* four balls.
   Coefficient of q^8.

(7) How many ways are there of making change for n cents in
   (a) 1952 pennies, 1959 pennies and 1964 nickles?
   (b) 1952 pennies, 1959 pennies, 1964 nickles, and 1971 quarters?
   Coefficient of q^{35}.

(8) The number of selections of n marbles from a group of 5 reds, 4 blues.
   Coefficient of q^7.

(9) The number of selections of n marbles from a group of 24 reds, 19 blues.
   Coefficient of q^{30}.

(10) The number of selections of n marbles from a group of 5 reds, 4 blues, and 2 pinks.
   Coefficient of q^5.

(11) The number of selections of n marbles from a group of 20 reds, 35 blues, and 33 pinks.
   Coefficient of q^{50}.

(12) Selections of n apples from 4 types with at least 2 apples of each type.
   Coefficient of q^{12}.

(13) Selections of n jelly beans from 4 different types with an even number of each type and not more than 8 of any one type.
Coefficient of q^{20}.

(14) Distributions of n black chips into 5 distinct boxes.
Coefficient of q^{30}.

(15) Distributions of n red balls into 6 distinct boxes with at least 2 balls in each box.
Coefficient of q^{18}.

(16) Distributions of n markers into 4 distinct boxes with the same number of markers in the first and second boxes.
Coefficient of q^{20}.

(17) The number of election outcomes if there are 3 candidates and n voters. If in addition, one of the three candidates receives at least 15 votes, how does your answer change?
Coefficient q^{30}.

(18) The number of election outcomes in the race for class president are there if there are 5 candidates and n students in the class and
  (a) Every candidate receives at least two votes.
  (b) One candidate receives at most one vote and all the other receive at least two votes.
  (c) No candidate receives more than 20 votes.
  (d) Exactly three of the candidates have the same number of votes and they have at least 10 each.

Coefficient of q^{40}.

(19) The number of numbers between 0 and $9,999$ (inclusive). that thave a sum of digits
  (a) equal to n.
  (b) less than or equal to n.

Coefficient of q^7.

(20) The number of integer solutions are there to the equation $x_1+x_2+x_3+x_4 \leq n$ with $x_i \geq i$.
Coefficient of q^{55}.

(21) The number of non-negative integer solutions to the equation $2x_1 + 2x_2 + x_3 + x_4 = n$.
Coefficient of q^{12}.

(22) The number of non-negative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = n$ with
  (a) $x_i \leq 10$
  (b) $x_1 = 2x_2$

Coefficient of q^{20}

(23) The number of ways of distributing n oranges in 3 different boxes such that there are at most 8 oranges in each box.
Coefficient of q^{15}.

(24) Create a generating function in two variables x and q with $\sum_{n\geq 0, m\geq 0} a_{n,m} q^n x^m$ for the numbers $a_{n,m}$ which are the number of ways of distributing r identical objects in n distinct boxes so that exactly m boxes are empty.