GENERATING FUNCTIONS FROM SEQUENCES THAT SATISFY A RECURSION

JANUARY 19, 2010

The following problems describe a sequence $a_0, a_1, a_2, a_3, \ldots$ which are defined by a recursion. That recursion can be turned into a generating function and that is the main goal of this set of problems.

- The first step is to calculate the first few terms of the sequence to make sure you understand the recursion. You don't need to calculate tons of terms, but 5 or 6 should suffice (some are easier to calculate than others).
- The next step is to turn the recursion for the sequence below into a functional equation satisfied by the $A(q) = \sum_{n\geq 0} a_n q^n$. These recursions are of a particular sort so they should all involve expressions of the form $A(q), A(q)^2, A'(q)$ and $A''(q)$.
- Next you should try to solve this equation. At this point you might need some help from a book on differential equations to solve your particular formula or just the quadratic equation. There is one other way of getting the answer: cheat. Use a CAS like SAGE or Maple, like magic these tools are often able to solve equations of the type that appear in these problems. The easiest cheat is to use the OLEIS to look up your sequence and if you are lucky someone has entered the generating function as one of the entries in the database. What you should do is show in fact that this generating function satisfies the recursion that you found for it.

(1) The sequence $a_0 = 1$, $a_1 = 3$, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.
(2) The sequence $a_0 = 3$, $a_1 = 1$, $a_n = a_{n-1} - a_{n-2}$ for $n \geq 2$.
(3) The sequence $a_0 = a$ $a_1 = b$ (for a, b some fixed real numbers) and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.
(4) The sequence $a_0 = a$, $a_1 = b$ and $a_n = ca_{n-1} + da_{n-2}$ for $n \geq 2$ (for a, b, c, d some fixed real numbers).
(5) The sequence $a_0 = 1$, $a_n = 2a_{n-1} - 1$ for $n \geq 1$.
(6) The sequence $a_0 = 1$, $a_n = aa_{n-1} - b$ (for a, b some fixed real numbers) for $n \geq 1$.
(7) The sequence $a_0 = 1$, $a_1 = 4$ and $a_n = 2a_{n-1} - 1$ for $n \geq 2$.
(8) The sequence $a_0 = 1$, and $a_n = na_{n-1} - 1$ for $n \geq 1$.
(9) The sequence $a_0 = 1$, $a_1 = 1$ and $a_n = na_{n-1} - a_{n-2}$ for $n \geq 2$.
(10) The sequence $a_0 = 1$, $a_1 = 1$ and $a_n = ta_{n-1} + na_{n-2}$ for $n \geq 2$ where t is some fixed real number.
(11) The sequence $a_0 = 1$, $a_n = n^2 a_{n-1} + 1$ for $n \geq 1$.
(12) The sequence $a_0 = 1$, $a_2 = 1$, $a_n = n^2 a_{n-1} - a_{n-2}$ for $n \geq 2$.
(13) The sequence $a_0 = 1$, $a_1 = 1$ and $a_n = a_{n-1} + n^2 a_{n-2}$ for $n \geq 2$.
(14) The sequence $a_0 = 1$, $a_1 = 1$ and $a_n = -na_{n-1} + n^2 a_{n-2}$ for $n \geq 2$.
(15) The sequence $a_0 = 1$, $a_n = \sum_{i=0}^{n-1} a_{n-1-i} a_i$ for $n \geq 1$.
(16) The sequence $a_0 = 1$, $a_n = \sum_{i=0}^{n-1} a_{n-1-i} a_i + t$ for $n \geq 1$ where t is some fixed real number.
(17) The sequence $a_0 = 1$, $a_n = \sum_{i=0}^{n-1} a_{n-1-i} a_i + a_{n-1}$ for $n \geq 1$.
(18) The sequence $a_0 = 1$, $a_n = \sum_{i=0}^{n-1} a_{n-1-i} a_i + na_{n-1}$ for $n \geq 1$.

(19) The sequence $a_0 = 1$, $a_n = \sum_{i=0}^{n-1} a_{n-1-i}a_i + n^2 a_{n-1}$ for $n \geq 1$.
(20) The sequence $a_0 = 1$, $a_1 = 1$, $a_n = \sum_{i=0}^{n-1} a_{n-1-i}a_i + a_{n-1} - a_{n-2}$ for $n \geq 2$.
(21) The sequence $a_0 = 1$, $a_1 = 1$, $a_n = \sum_{i=0}^{n-1} a_{n-1-i}a_i + na_{n-1} - a_{n-2}$ for $n \geq 2$.
(22) The sequence $a_0 = 1$, $a_1 = 1$, $a_n = \sum_{i=0}^{n-1} a_{n-1-i}a_i + a_{n-1} - na_{n-2}$ for $n \geq 2$.
(23) The sequence $a_0 = 1$, $a_1 = 1$, $a_n = \sum_{i=0}^{n-1} a_{n-1-i}a_i + n^2 a_{n-1} - na_{n-2}$ for $n \geq 2$.