Algebraic expressions and sequences
November 10, 2011

Let $A(q) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots$ and $B(q) = b_0 + b_1 q + b_2 q^2 + b_3 q^3 + \cdots$ where the integers a_i represent a sequence and b_i represent another sequence and q is a variable. That is, the coefficient of q^n in $A(q)$ is a_n and the coefficient of q^n in $B(q)$ is b_n. Recall that $1/(1-q) = 1 + q + q^2 + q^3 + q^4 + \cdots$ and by taking the derivative of this equation we can show that

$$1/(1-q)^2 = \frac{d}{dq}(1/(1-q)) = \frac{d}{dq}(1 + q + q^2 + q^3 + q^4 + \cdots) = 1 + 2q + 3q^2 + 4q^3 + \cdots .$$

What is the coefficient of q^n in the following expressions?

(1) $cA(q)$ (c is a constant here)
(2) $A(q) + B(q)$
(3) $q^7 A(q)$
(4) $q^2 A(q) + q^3 B(q)$
(5) $A'(q)$
(6) $\frac{d^k}{dq^k} A(q)$
(7) $A'(q)B(q)$
(8) $A(q)B(q)$
(9) $A(q)^2$
(10) $A(q^2)$
(11) $A(q)A(q^2)$
(12) $B(q)^2 A(q^2)$
(13) $B(q^2)A(q^2)$
(14) $A(q^2)^2$
(15) $A(q)/(1-q)$
(16) $A(q)/(1-q^2)$
(17) $A(q)/(1-2q)$
(18) $A(q)/(1-2q^2)$
(19) $A(q)/(1-q)^2$
(20) $A(q)/(1-2q^2)^2$
(21) $A'(q)/(1-q)^2$
(22) $A''(q)/(1-q)^3$
(23) $qA'(q)/(1-q)$
(24) $(1+q)^k A(q)$
(25) $(1+q+q^2)A(q)$
(26) $A'(q)(1+q)$
(27) $\frac{1-q^k}{1-q} A(q)$
(28) $\frac{1}{2}(A(q) + A(-q))$
(29) $\frac{1}{2}(A(q) - A(-q))$
(30) $A(q) + qA'(q)$
(31) $A(q) + 3qA'(q) + q^2 A''(q)$
(32) $A(q) + 7qA'(q) + 6q^2 A''(q) + q^3 A'''(q)$
(33) $A'(q) + qA''(q)$
(34) $A(q)^3$
(35) $A(q)^4$
(36) $1/(1 - qA(q))$
(37) $A(qB(q))$