A FEW WORDS ABOUT TELESCOPING SUMS

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Say that you want to prove an identity of the form

$$p(1) + p(2) + p(3) + \cdots + p(n) = q(n)$$

where $p(x)$ and $q(x)$ are expressions and n is a non-negative integer. We must have that $q(0) = 0$ since the left hand side of the equation will be the empty sum when $n = 0$. One way to go about this is to show that

$$q(r) - q(r-1) = p(r)$$

for all $r \geq 0$, and then write down

$$\begin{aligned} q(1) - q(0) &= p(1) \\ q(2) - q(1) &= p(2) \\ q(3) - q(2) &= p(3) \\ &\vdots \\ q(n-1) - q(n-2) &= p(n-1) \\ q(n) - q(n-1) &= p(n) \end{aligned}$$

Now the sum of the expressions on the right hand side of this equation is

$$p(1) + p(2) + p(3) + \cdots + p(n-1) + p(n)$$

and the sum of the expressions on the left hand side of this equation is

$$(q(1)-q(0))+(q(2)-q(1))+(q(3)-q(2))+\cdots+(q(n-1)-q(n-2))+(q(n)-q(n-1)) = q(n)-q(0) = q(n)\ .$$

We conclude therefore that

$$p(1) + p(2) + p(3) + \cdots + p(n-1) + p(n) = q(n)\ .$$

Example There are lots of ways of proving the following identity.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}\ .$$

Since $\frac{r(r+1)}{2} - \frac{(r-1)r}{2} = r$, we have

$$\frac{1 \cdot 2}{2} - \frac{0 \cdot 1}{2} = 1$$

$$\frac{2\cdot 3}{2} - \frac{1\cdot 2}{2} = 2$$
$$\frac{3\cdot 4}{2} - \frac{2\cdot 3}{2} = 3$$
$$\vdots$$
$$\frac{(n-1)\cdot n}{2} - \frac{(n-2)\cdot(n-1)}{2} = n-1$$
$$\frac{n\cdot(n+1)}{2} - \frac{(n-1)\cdot n}{2} = n$$

The sum of the terms on the left hand side of these equations is $\frac{n(n+1)}{2}$ and the sum of the terms on the right hand side of these equation is $1+2+3+\cdots+(n-1)+n$, therefore they are equal.

Example Define the Fibonacci sequence by $F_0 = 1$, $F_1 = 1$, and for $n \geq 0$, $F_{n+2} = F_{n+1} + F_n$. Say that we want to show that

$$F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1},$$

or in words "The sum of the first n Fibonacci numbers indexed by even n is the next Fibonacci number indexed by odd n." So we know that for $r \geq 1$, $F_{2r+1} - F_{2r-1} = F_{2r} + F_{2r-1} - F_{2r-1} = F_{2r}$. Therefore

$$F_3 - F_1 = F_2$$
$$F_5 - F_3 = F_4$$
$$F_7 - F_5 = F_6$$
$$\vdots$$
$$F_{2n-1} - F_{2n-3} = F_{2n-2}$$
$$F_{2n+1} - F_{2n-1} = F_{2n}$$

Since the sum of the left hand side of these equations is $F_{2n+1} - F_1 = F_{2n+1} - F_0$ and the sum of the right hand side of this equation is $F_2 + F_4 + F_6 + \cdots + F_{2n}$, we conclude that

$$F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1}\ .$$

Example Here is a general identity that can be fairly useful:

$$1\cdot 2\cdots k \;+\; 2\cdot 3\cdots(k+1) \;+\; 3\cdot 4\cdots(k+2) \;+\; \cdots \;+\; n\cdot(n+1)\cdots(n+k-1)$$
$$= n\cdot(n+1)\cdots(n+k)/(k+1)$$

Observations: (1) if $k=1$, then this identity reduces to $1+2+3+\cdots+n = n(n+1)/2$. (2) if $k=2$, then this identity reduces to $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + n\cdot(n+1) = n(n+1)(n+2)/3$.

(3) there is shorthand notation that makes this sum easier to work with. Let $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$, then the identity becomes

$$(1)_k + (2)_k + (3)_k + \cdots + (n)_k = (n)_{k+1}/(k+1)$$

We note that

$$\begin{aligned} &r\cdot(r+1)\cdots(r+k)/(k+1) \;-\; (r-1)\cdot r\cdots(r+k-1)/(k+1) \\ &\quad = r\cdot(r+1)\cdots(r+k-1)((r+k) \;-\; (r-1))/(k+1) \\ &\quad = r\cdot(r+1)\cdots(r+k-1)\ . \end{aligned}$$

Therefore we have

$$\begin{gathered} 1\cdot 2\cdots(k+1)/(k+1) - 0\cdot 1\cdots k/(k+1) = 1\cdot 2\cdots k \\ 2\cdot 3\cdots(k+2)/(k+1) - 1\cdot 2\cdots(k+1)/(k+1) = 2\cdot 3\cdots(k+1) \\ 3\cdot 4\cdots(k+3)/(k+1) - 2\cdot 3\cdots(k+2)/(k+1) = 3\cdot 4\cdots(k+2) \\ \vdots \end{gathered}$$

$$(n-1)\cdot n\cdots(n+k-1)/(k+1) - (n-2)\cdot(n-1)\cdots(n+k-2)/(k+1) = (n-1)\cdot n\cdots(n+k-2)$$

$$n\cdot(n+1)\cdots(n+k)/(k+1) - (n-1)\cdot(n-2)\cdots(n+k-1)/(k+1) = n\cdot(n+1)\cdots(n+k-1)$$

The sum of the entries on the left hand side of these equalities is $n\cdot(n+1)\cdots(n+k)/(k+1)$ and the sum of the entries on the right hand side of these equalities is

$$1\cdot 2\cdots k \;+\; 2\cdot 3\cdots(k+1) \;+\; 3\cdot 4\cdots(k+2) \;+\; \cdots \;+\; n\cdot(n+1)\cdots(n+k-1),$$

therefore the two expressions are equal.

One final observation: It is always possible to express n^k as a sum in the notation $(n)_r$. $n^1 = (n)_1$, $n^2 = (n)_2 - (n)_1$, $n^3 = (n)_3 - 3(n)_2 + (n)_1$, $n^4 = (n)_4 - 6(n)_3 + 7(n)_2 - (n)_1$. This can be used to give a sum of $1^k + 2^k + 3^k + \cdots + n^k$. The coefficients in this expansion are known as the Stirling numbers of the second kind.