COMPUTER EXERCISES ON GRÖBNER BASES

You will need to use either Maple or some other computer algebra system. The ones that I recommend for this type of calculation can be downloaded from the web and are free. Google for Macaulay, Macaulay 2, muPad, Cocoa (maybe GAP) and you may even come across other such programs.

Maple (or its arch-rival Mathematica) is available on most university computers, but it is expensive for individuals. muPad or GAP are excellent alternatives which are free but less universal. I will stick with Maple because it is the one I use for most of my calculations however the commands are similar on ANY of these systems. The most important first step in using them is to learn how to use the help system.

(1) Determine the Gröbner bases for the following ideals of $\mathbb{Q}[x, y]$.
   (a) $(x^2 + xy^5 + y^4, xy^6 - xy^3 + y^5 - y^2, xy^5 - xy^2, y^5 - y^2)$ using *lex* order (see p.326-7 ex. 1)
   (b) $(x^2 + xy^5 + y^4, xy^6 - xy^3 + y^5 - y^2, xy^5 - xy^2, y^5 - y^2)$ using *rlex* order.
   (c) $(x^2 + xy^5 + y^4, xy^6 - xy^3 + y^5 - y^2, xy^5 - xy^2, y^5 - y^2)$ using *glex* order.
   (d) $(xy^3 + y^3 + 1, x^3y - x^3 + 1, x + y, y^4 - y^3 - 1)$ using *lex* order.

(2) Show that the ideals $(x + y, y^4 - y^3 - 1)$ and $(x^3y - xy^2 + 1, x^2y^2 - y^3 - 1)$ and $(xy^3 + y^3 + 1, x^3y - x^3 + 1, x + y)$ are all equal. (see p. 327 ex. 2)

(3) Find the intersection of the two ideals $(x^3 - xy, y^3 - y^2 + 1)$ and $(3xy + y^2, x^2 - 2x + 1)$ by the method described on page 330 of the book.

(4) Compute the Gröbner basis of the ideal $(x^2 + xy + y^2 - 1, x^2 + 4y^2 - 4)$ and use it to find the four points of intersection of the ellipse $x^2 + xy + y^2 = 1$ with the ellipse $x^2 + 4y^2 = 4$ in $\mathbb{R}^2$. (# 30 p. 333)

(5) Solve the system of equations $x^2 - yz = 3$, $y^2 - xz = 4$, $z^2 - xy = 5$ over $\mathbb{C}$ by computing a Gröbner basis. (#29 p. 333)

(6) Assume the order is *lex* and the polynomial ring is $\mathbb{Q}[x, y]$. Determine if the following polynomials are in the ideal $(x^2 - yx, y^3 - y^2, x^3 + x^3y)$. If they are not in the ideal then give the minimal representative of the coset that they lie in.
   (a) $(x + y)^2$
   (b) $(x + y)^3$
   (c) $(x + y)^4$
   (d) $(x + y)^5$
   (e) $x^2 + 2y^3$

(7) Using the same ideal as in the last problem, reduce $(x + y)^2$ using reverse lexicographic order. Explain why the answer is not the same.

(8) Determine if the following polynomials are in the algebra $\mathbb{Q}[p_1, p_2, p_3]$ where $p_1 = x + y + z$, $p_2 = x^2 + y^2 + z^2$, $p_3 = x^3 + y^3 + z^3$. If so, find an expression for them as a polynomial in the

p_is. Hint: compute the Gröbner basis of the ideal $(x+y+z-r, x^2+y^2+z^2-s, x^3+y^3+z^3-t)$ with respect to the lexicographic order $x > y > z > r > s > t$ and find the reduced form of the following polynomials

(a) $xy + yz + zx$
(b) $x^3 + yx^2 + x^2z - xy^2 - y^2z - xzy$
(c) $x^3 + 2\,yx^2 + 2\,x^2z + xzy + xz^2 - y^2z$
(d) $x^2y + xy^2 + y^2z + yz^2 + x^2z + xz^2$
(e) xyz
(f) $3\,yx^2 + 2\,xy^2 + 2\,x^2z + 4\,xzy + 2\,y^2z$

(9) For the polynomials in the last problem determine if they are in the ideal $(p_1, p_2, p_3) \subseteq \mathbb{Q}[x, y, z]$. If so write them as a polynomial in the p_1, p_2, p_3 with coefficients in x, y, z.