ON EQUATIONS OF DEGREE $3 * 2^m$

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Introduction

For a Galois covering $\pi : A \to B$ the Schwarz genus $g(\pi)$ is the minimum number for which one can cover the base with open sets U_i so that the covering, restricted to U_i is trivial. This number can also be defined as $d+1$ where d is the minimum dimension of a space X with a Galois covering $Y \to X$ so that $A = p^*(Y)$ is the pull back of some classifying map $\rho : B \to X$. In this language lower bounds for d can be determined by the non vanishing of cohomological obstructions. For a more precise discussion we refer to [DPS], [Sc].

In this paper we continue the study of a special but important example. Let $\mathcal{P}_n$ be the space of monic polynomials of degree n over $\mathbb{C}$ with distinct roots, the complement of the *discriminant hypersurface* and let

$$\Delta := \{(z_1, z_2, \ldots, z_n\} \in \mathbb{C}^n \mid z_i \neq z_j, \ \forall i \neq j\}, \quad \mathcal{P}_n = \mathbb{C}^n - \Delta / S_n$$

(Δ is the big diagonal).

Problem Compute the Schwarz genus $g(n) := g(\pi_n)$ for the covering

$$\pi_n : \mathbb{C}^n - \Delta \to \mathcal{P}_n = \mathbb{C}^n - \Delta / S_n.$$

It is well known and simple to prove that $g(n) \leq n$ and it was known that, if $n = p^k$ is a prime power this is an equality ([Va]).

In a previous paper ([DPS]) it was studied the first unknown case $n = 6$ and showed that $g(6) = 5$.

It seems reasonable to expect that $g(m) = m$ if and only if m is a prime power. In fact in [DPS] we show that a sufficient condition for $g(m) < m$ is the vanishing of the homology group:

$$H_{n-1}(S_n, H^{n-1}(\mathbb{C}^n - \Delta, \mathbb{Z}))$$

and we conjecture that this vanishing holds when n is not a prime power.

In order to understand this conjecture we need to recall some facts about the structure of $H^{n-1}(\mathbb{C}^n - \Delta, \mathbb{Z})$ as an S_n module. We will use its relationship with another standard module $Lie(n)$ the space spanned over $\mathbb{Z}$ by the multilinear Lie monomials in n variables.

It is proved in [FC] that

$$H^{n-1}(\mathbb{C}^n - \Delta, \mathbb{Z}) = hom(Lie(n), \mathbb{Z}) \otimes sign$$

sign denotes the sign representation of the symmetreic group.

In a recent paper ([AD]) Arone and Dwyer have proved the vanishing of the homology $H_i(S_n, hom(Lie(n), \mathbb{Z})), \ i \geq 0$. On the other hand the twisting by sign seems to change drastically the picture and at least we were not able to use their methods.

On the other hand, in a private letter G. Z. Arone informs us that the methods contained in his paper with M. Mahowald (cf. [AM]) allow to prove the conjecture for every n which is not of the form $2 * p^k$ (so almost a complete solution).

In this paper we present a different approach to the computation which we are able to complete only for numbers of type $n = 3 * 2^k,\ k > 0$, for these numbers we show that the required homology vanishes by proving a stronger result which seems to be of independent interest.

The method in principle might be extended to numbers in which odd primes appear in a square free manner but at the moment the final computations become combinatorially very heavy and we have carried them out using a computer program only for $n = 10$ getting again the vanishing required.

REMARK. In this paper all the computations will be made using the description $H^{n-1}(\mathbb{C}^n - \Delta, \mathbb{Z}) = hom(Lie(n), \mathbb{Z}) \otimes sign$.

In the previous paper with Salvetti ([DPS]) we had made the computations using the presentation of $H^{n-1}(\mathbb{C}^n - \Delta, \mathbb{Z})$ which comes directly from the Salvetti complex. Also this presentation leads to some rather interesting combinatorics, quite different from the one of this paper, and we hope in the future to make more explicit the relationship between the two approaches.

1. Some remarkable representations

1.1. **The structure of $Lie(n)$.** Let us simplify the notations and write for short $hom(Lie(n), \mathbb{Z}) \otimes sign := L(n)$.

We need to recall some standard facts on $Lie(n)$.

To make the notations as simple as possible we denote the variables appearing in the formal expressions of $Lie(n)$ simply as numbers $1, 2, \ldots$ instead of the more cumbersome notation $x_1, x_2, \ldots$.

Let us use the notation (defined recursively):

$$\{i, j\} = [i, j], \quad \{i_1, \ldots, i_n\} := [i_1, \{i_2, \ldots, i_n\}].$$

It is well known [Re] that the elements:

$$(i_1, i_2, \ldots, i_{n-1}) := \{i_1, i_2, \ldots, i_{n-1}, n\} = [i_1, [i_2, [\ldots [i_{n-1}, n] \ldots]]$$

as $(i_1, i_2, \ldots, i_{n-1})$ runs on the set of $(n-1)!$ permutations of $1, 2, \ldots, n-1$ form a basis of $Lie(n)$ (formed by the expressions ending with n), which we will call the *normal basis*.

Some of the computations consist in writing a general monomial in *normal form* or as linear combination of the normal basis.

We like to think of the elements $(i_1, i_2, \ldots, i_{n-1})$ as words (in the given alphabet of integers), as such we can do some combinatorial operations on words, like *juxtaposition* $A\ B$ and also reversing the letters in a word, we will use the symbol $H^\vee$ to indicate the word H reversed.

Finally if J is a word and H a *subword*, we will write by abuse of notation $H \subset J$ (we do not require the letters of H to be consecutive in J) we define $J - H$ to be the complementary subword, obtained deleting the elements in H.

We want to describe the action of S_n (and later a hidden action of S_{n+1}) on the space $Lie(n)$ described combinatorially by the basis of words in the elements $1, 2, \dots, n-1$.

The action of S_{n-1} is the obvious one permuting the letters.

We consider now $\tau := (n-1, n)$, we need to determine the action of $(n-1, n)$ and have:

Theorem 1. *Write a word M as $M := I\ n-1\ J$. We get*

$$\tau(M) = \sum_{H \subset J} (-1)^{|H|+1} I\ (J-H)\ n-1\ \check{H}. \tag{1}$$

Proof. We have

$$\tau(I\ n-1\ J) = \tau\{I\ n-1\ J\ n\} = \{I\ n\ J\ n-1\}$$

We proceed by induction on the cardinality of J. If J is empty the statement is clear. Let $J := aK$ it will be enough to rewrite in normal form

$$\{n\ J\ n-1\} = [n, \{J\ n-1\}] = [n, [a, \{K\ n-1\}]] = -[[a, n], \{K\ n-1\}] + [a, [n, \{K\ n-1\}]]$$

we apply induction (think of $[a, n]$ as a variable $\overline{n}$ and remark that $\{I, \overline{n}\} = \{I, a, n\} = (I\ a)$:

$$[[a, n], \{K\ n-1\}] = \sum_{H \subset K} (-1)^{|H|+1} \{\ (K-H)\ n-1\ \check{H}\ a\},$$

$$[a, [n, \{K\ n-1\}]] = \sum_{H \subset K} (-1)^{|H|+1} \{a\ (K-H)\ n-1\ \check{H}\} = \sum_{H \subset K} (-1)^{|H|+1} \{J-H\ n-1\ \check{H}\}$$

also clearly

$$\sum_{H \subset K} (-1)^{|H|+1} \{\ (K-H)\ n-1\ \check{H}\ a\} = - \sum_{M = \{a, H\} \subset J,\ a \in M} (-1)^{|M|+1} \{\ (J-M)\ n-1\ \check{M}\}$$

clearly this separates the sum into two terms, the subsets which contain a and the others.

□

1.2. **The structure of $\tilde{L}(n)$.** We want to study now a closely related object. We start by embedding $Lie[n-1]$ into $Lie[n]$ by the map $j : m \to [n, m]$. If m is an element of the normal basis, $[n, m]$ is not in normal form but this will be useful for our computations. In any case we have by definition that:

$$j(\{i_1, \dots, i_{n-1}\}) = \{n, i_1, \dots, i_{n-1}\}$$

Let $\sigma_h := \{1, 2, 3, \dots, h-1, n-1, h+1, \dots, n-2, n, h\}$ (the cycle $(h, n-1, n)$, if $h \neq n-1, n$, otherwise the transposition $\sigma_{n-1} = (n-1, n)$ and finally $\sigma_n = 1$) that fixes all integers $i < n-1, i \neq h$) , then $\sigma_h S_{n-1} = \{\tau \in S_n \mid \tau(n) = h\}$.

We want to introduce now the induced representation $Ind_{S_{n-1}}^{S_n} Lie(n-1)$ and identify

$$Ind_{S_{n-1}}^{S_n} Lie(n-1) = \oplus_{h=1}^{n} \sigma_h Lie(n-1),$$

of course σ_n is the identity and in this way we identify $\sigma_n Lie(n-1)$ with $Lie(n-1)$.

Use the notation $L_h := \sigma_h Lie(n-1)$ and remark that $L_h = \sigma Lie(n-1)$ if $h = \sigma(n)$, notice that we identify $L_n = Lie(n-1)$ ant that, given a permutation τ we have $\tau(L_h) = L_{\tau(h)}$.

The embedding j of $Lie(n-1)$ into $Lie(n)$ extends to a map

$$\tilde{j} : Ind_{S_{n-1}}^{S_n} Lie(n-1) = \oplus_{\sigma \in S_n/S_{n-1}} \sigma Lie(n-1) = \oplus_h \sigma_h Lie(n-1) \to Lie(n)$$

by $\tilde{j}(\sum \sigma_i m_i) = \sum \sigma_i j(m_i) = \sum \sigma_i([n, m_i])$. Notice that, for $h \neq n$ the image of $\tilde{j}(\sigma_h m)$, where $m = (i_1, \ldots, i_{n-2}) \in Lie(n-1)$ is a normalized monomial, is still a normalized monomial, in fact:

$$\sigma_h(j(i_1, \ldots, i_{n-2})) = \sigma_h([n, (i_1, \ldots, i_{n-2})]) =$$
$$= \sigma_h([n, \{i_1, \ldots, i_{n-2}, n-1\}]) = \sigma_h(\{n, i_1, \ldots, h, \ldots, i_{n-2}, n-1\}) =$$
$$= \{h, i_1, \ldots, n-1, \ldots, i_{n-2}, n\} = (h, i_1, \ldots, n-1, \ldots, i_{n-2})$$

Let us prove that the map $\oplus_{h \neq n} \sigma_h Lie(n-1) \to Lie(n)$ is a linear isomorphism.

From the previous formula the monomials image of $\sigma_h Lie(n-1),\ h \neq n$ are normalized and starting with h, these are linearly independent and of the correct dimension so this is proved.

Thus the kernel of the map $\tilde{j}$, which we will denote by $\tilde{L}(n) := Ker(\tilde{j})$, projects isomorphically to the factor with $\sigma_n = 1$.

Remark that, since $\oplus_{\sigma \in S_n/S_{n-1},\ \sigma(n) \neq n} \sigma Lie(n-1) = \sum_{\sigma \notin S_{n-1}} \sigma Lie(n-1)$, this is an S_{n-1} submodule and thus the projection, which we will denote by ρ, is S_{n-1} equivariant.

On $\tilde{L}(n)$ which we identify to $L(n-1)$ with ρ, we need to identify the action of $(n-1, n)$. One way to procede is to compute explicitely:

$$Lie(n-1) \xrightarrow{\rho^{-1}} \tilde{L}(n) \subset Ind_{S_{n-1}}^{S_n} Lie(n-1) = \oplus_{\sigma \in S_n/S_{n-1}} \sigma Lie(n-1)$$

and then, given a normalized monomial J, compute $\rho((n-1, n)\rho^{-1} J)$.

So reconsider the formula of theorem 2,

$$\tau(M) = \sum_{H \subset J} (-1)^{|H|+1} I\ (J - H)\ n-1\ \check{H}.$$

applied to $J = (i_1, \ldots, i_{n-2}),\ M = n-1\ J$ when $\tau(M) = \{n\ J\ n-1\}$, it means that:

$$0 = \{n\ J\ n-1\} - \sum_{H \subset J} (-1)^{|H|+1}\ (J - H)\ n-1\ \check{H}.$$

In the sum $-\sum_{H \subset J} (-1)^{|H|+1}\ (J-H)\ n-1\ \check{H}$ call b_h the contribution of the summands relative to normalized monomials starting with h (here $h = 1, \ldots, n-1$). By the previous analysis $b_h = \sigma_h(j(c_h)),\ c_h \in Lie(n-1)$.

We may interpret this relation as the element

$$\rho^{-1}(J) = R_J = (J) + \sum_{h<n} \sigma_h c_h \in \tilde{L}(n), \quad c_h = T_h(J).$$

let us look now at the term $T_{n-1}(J)$, it is the signed sum of all the words $(J - H)\ n-1\ \check{H}$ which start with $n-1$, but this happens only when $H = J$, giving the term $-(-1)^{n-2+1} J^\vee = (-1)^n J^\vee$.

When we apply the exchange $(n, n-1)$ to this relation notice that we have:

$$(n-1, n) R_J = R_{T_{n-1}(J)},$$

this gives the formula:

$$(n-1,n)(J) = (-1)^n(J^\vee). \tag{2}$$ □

REMARK Since n is arbitrary we have in fact explicited the *hidden action* of S_{n+1} on $Lie(n)$. Conceivably this coincides with the one that is obtained by the theory of moduli of pointed rational curves (cf. [G]).

Let us finish with some remarks.

Remark. i) $Lie(n)$ restricted to S_{n-1} is free of rank 1.

ii) $\tilde{L}(n)$ restricted to S_{n-2} is free of rank 1.

iii) The same statement is true after dualizing or tensoring with the sign representation.

2. Homology.

2.1. **Some reductions.** Recall that $hom(Lie(n), \mathbb{Z}) \otimes sign := L(n)$ and we want to compute $H_*(S_n, L(n))$.

Theorem 2. *The groups $H_i(S_n, L(n))$ are killed by n, for all $i > 0$.*

Proof. $H_i(S_{n-1}, L(n)) = 0,\ \forall i > 0$ by the freeness property.

We have that multiplication by n factors as:

$$n : H_i(S_n, L(n)) \xrightarrow{res} H_i(S_{n-1}, L(n)) \longrightarrow H_i(S_n, L(n)).$$ □

2. We have the exact sequence:

$$0 \to \tilde{L}(n) \to Ind_{S_{n-1}}^{S_n} Lie(n-1) \to Lie(n) \to 0$$

dualizing and tensoring with sign we get:

$$0 \to L(n) \to Ind_{S_{n-1}}^{S_n} L(n-1) \to hom(\tilde{L}(n), \mathbb{Z}) \otimes sign \to 0$$

By Shapiro's lemma $H_i(S_n, Ind_{S_{n-1}}^{S_n} L(n-1)) = H_i(S_{n-1}, L(n-1))$ is killed by $n-1$ (for $i > 0$) and so the $n-$torsion of $H_i(S_n, Ind_{S_{n-1}}^{S_n} L(n-1)),\ i > 0$ is 0.

By the long exact sequence the $n-$torsion of $H_i(S_n, L(n))$ equals the $n-$torsion of $H_{i+1}(S_n, hom(\tilde{L}(n), \mathbb{Z}) \otimes sign))$.

Let us denote for short $P(n) := hom(\tilde{L}(n), \mathbb{Z}) \otimes sign)$, it is free as S_{n-2} module.

Now look at $n = pm$ with $(p, m) = 1$, notice that $C_p \times S_{n-p}$ has index $I := n!/(p \times (n-p)!)$ in S_n prime with p (with C_p the cyclic group of order p), the composition, is multiplication by I, invertible on p torsion:

$$I : H_i(C_p \times S_{n-p}, P(n)) \to H_i(S_n, P(n)) \to H_i(C_p \times S_{n-p}, P(n)).$$

In order to show that the $p-$torsion of $H_{n-1}(S_n, L(n))$ vanishes it is thus sufficient to show that the $p-$torsion of $H_n(C_p \times S_{n-p}, P(n))$ is 0.

Now consider the Lyndon Serre spectral sequence (in homology):

$$E^2_{i,j} = H_i(C_p, H_j(S_{n-p}, P(n))) = 0, \quad if \quad j > 0.$$

Since $P(n)$ is a free S_{n-p} module, it degenerates and gives that:

$$H_i(C_p \times S_{n-p}, P(n)) = H_i(C_p, H_0(S_{n-p}, P(n)) = H_i(C_p, P(n)_{S_{n-p}})$$

By $P(n)_{S_{n-p}} = H_0(S_{n-p}, P(n))$ we mean the space of *coinvariants*, which for a $G-$module M means M modulo the submodule spanned by the elements $m - g.m,\ g \in G$.

Now we restrict in particular to the case $p=3$, $P(n)$ has dimension $(n-2)!$ and is a free S_{n-2} module of rank 1, so has rank $n-2$ as free S_{n-3} module and the coinvariants of S_{n-3} have rank $n-2$ as $\mathbb{Z}$ module.

We first do our computations in a dual module $P(n)^* = Lie(n-1)\otimes sign$. We want to compute with the coinvariants with respect to S_{n-3}, we will denote by M_{n-2} this space of coinvariants thought of as a $\mathbb{Z}/(3)$ module.

Let us select in the basis the elements $c_h := (1,\dots,h-1,n-2,h,\dots,n-3)$ in which $n-2$ appears in the h position and the others are linearly ordered. Given an element $(i_1,\dots,i_{n-2})$ its image in the coinvariants is ϵc_h, where $i_h = n-2$ and ϵ is the sign of the permutation that reorders in increasing order the elements $i_k,\ k\neq h$.

The two elements $(n-2,n-1),(n-1,n-2)$ commute with S_{n-3}, hence act on the coinvariants. We want to compute their matrices in the basis c_h. We will use the explicit formulas that we have developed. By abuse of notation we will use the word symbols $J,(J)$ also to mean their images in the coinvariants and use the allowed reordering laws.

Start with the exchange $(n,n-1)$ which maps $A\ n-2\ B$ to $(-1)^n B^\vee\ n-2\ A^\vee$, when we apply it to $c_h = (1,2,\dots,h-1,n-2,h,h+1,\dots,n-3)$ we have to reorder the numbers $1,2,\dots,n-3$ which appear in reverse order.

The permutation of S_{n-3} which reverses order has sign $(-1)^{(n-3)(n-4)/2}$ so the sign is $(-1)^{n+(n-3)(n-4)/2} = (-1)^{n(n-1)/2}$, finally:

$$(n,n-1)c_h = (-1)^{\binom{n}{2}} c_{n-1-h} \tag{3}$$

For the action of $(n-2,n-1)$ let us compute first $(n-2,n-1)c_1 = (n-2,n-1)(n-2\ J)$:

$$(n-2,n-1)(n-2\ J) = \sum_{H\subset J} (-1)^{|H|+1}(J-H)\ n-2\ H^\vee, \quad J=(1,2,\dots,n-3);$$

reordering we get

$$(n-1,n-2)c_1 = (n-1,n-2)(n-2\ J) = \sum_{H\subset J}(-1)^{|H|+1}(-1)^{\binom{|H|}{2}}(J-H)\ n-2\ H.$$

If $J-H$ has h elements, the image of the element $(J-H)\ n-2\ H$ in the coinvariants is ϵc_{h+1} where ϵ is the sign of the *shuffle* $(J-H)H$.

The case $c_h = (I\ n-2\ J),\quad |I| = h-1$ can be easily reduced to this case.

$$(n-1,n-2)c_h = (n-1,n-2)(I\ n-2\ J) = \sum_{H\subset J}(-1)^{|H|+1}(-1)^{\binom{|H|}{2}} I\ (J-H)\ n-2\ H.$$

The sum $\sum_{H\subset J}(-1)^{|H|+1}(-1)^{\binom{|H|}{2}} I\ (J-H)\ n-2\ H$, should be grouped according to the terms which map to the same coinvariant up to sign as:

$$\sum_{H\subset J}(-1)^{|H|+1+\binom{|H|}{2}} I\ (J-H)\ n-2\ H = \sum_{k=0}^{n-2-h}(-1)^{k+1+\binom{k}{2}} \sum_{H\subset J,\ |H|=k} I\ (J-H)\ n-2\ H,$$

if we denote by $\epsilon_{(J-H)\ H}$ the sign of this as shuffle permutation we have finally in the coinvariants:

$$\sum_{H\subset J,\ |H|=k} I\ (J-H)\ n-2\ H = \Big(\sum_{H\subset J,\ |H|=k}\epsilon_{(J-H)\ H}\Big)c_{n-2-k} \tag{4}$$

So we need to compute, given two numbers $a<b$ the sum of the signs of the shuffles extracting a elements out of b.

Let us define this number as $\{{b \atop a}\}$ and rewrite our formulas as:

$$\sum_{H\subset J,\ |H|=k} I\ (J-H)\ n-2\ H = \{{n-1-h \atop k}\}c_{n-2-k} \tag{5}$$

$$(n-1,n-2)c_h = -\sum_{k=0}^{n-2-h}(-1)^{\binom{k+1}{2}}\{{n-1-h \atop k}\}c_{n-2-k} \tag{6}$$

To explicit these numbers we have by a simple permutation argument:

$$\{{b \atop a}\} = \{{b-1 \atop a}\} + (-1)^{b-a}\{{b-1 \atop a-1}\}$$

by recursion one gets:

$$\{{2b \atop 2a}\} = \binom{b}{a},\quad \{{2b \atop 2a+1}\} = 0,\quad \{{2b+1 \atop 2a}\} = \binom{b}{a},\quad \{{2b+1 \atop 2a+1}\} = \binom{b}{a}$$

Combining the two formulas (3),(6) we get a formula for the action of the 3 cycle $g := (n-2,n-1,n) = (n-2,n-1)(n-1,n)$.

$$g.c_h = (-1)^{\binom{n}{2}}(n-1,n-2)c_{n-1-h} = -(-1)^{\binom{n}{2}}\sum_{k=0}^{h-1}(-1)^{\binom{k+1}{2}}\{{h-1 \atop k}\}c_{n-2-k}. \tag{7}$$

In order to finish our computations we have to do two things, compute the homology of the cyclic group of order 3 acting with the previously determined matrix and also showing that this equals the homology for the coinvariants of the dual action which is the one we are interested in. Let us first clarify this issue. Since $P(n)$ is dual of $P(n)^* = Lie(n-1)\otimes sign$, and they are both free as S_{n-3} modules we have dual bases also for coinvariants, and since the 3 cycle commutes with S_{n-3} in the dual basis its matrix appears as the inverse transpose. If we change generator of the cyclic group its matrix is just the transpose, so the analysis follows from the next discussion.

Let M be a finitely generated free abelian group and $T : M \to M$ and endomorphism with $T^p = 1$ for a prime p, so M is a C_p module and we want to compute the homologies:

$$H_i(C_p, M),\ \ H_i(C_p, M^*).$$

let $R := \mathbb{Z}[C_p]$. Recall that a free resolution of $\mathbb{Z}$ as R module is given by the sequence $C_p = (g)$:

$$\cdots \to R \xrightarrow{1+g+\cdots+g^{p-1}} R \xrightarrow{1-g} R \xrightarrow{1+g+\cdots+g^{p-1}} R \xrightarrow{1-g} R \longrightarrow \mathbb{Z} \longrightarrow 0$$

thus the homology is the homology of the periodic complex

$$\cdots \to M \xrightarrow{1+T+\cdots+T^{p-1}} M \xrightarrow{1-T} M \xrightarrow{1+T+\cdots+T^{p-1}} M \xrightarrow{1-T} M \longrightarrow 0.$$

This complex, after tensoring with $\mathbb{Q}$ is exact except in degree 0, similarly modulo a prime different from p.

Even homology >0 equals K/I where K is the kernel of $1+T+\cdots+T^{p-1}$ (a direct summand in M), its dimension is the rank of the matrix $1-T$, and I the image of $1-T$. Using elementary divisor theory we get that $\dim_{\mathbb{F}_p}(K/I)$ equals the difference of the ranks of $1-T$ over $\mathbb{C}$ and modulo p.

Similarly for the odd homology we have to use $1+T+\cdots+T^{p-1}$.

Remark that the rank of the homology for M or M^* are the same by the previous discussion.

Now in order to finish our work we have to compute the rank and the rank mod 3 of the two matrices $1-T,\ 1+T+T^2$ where T is the matrix given by the formula (5).

Let us change the notation, set $m := n-2$ and T_m the $m\times m$ matrix (acting on the space M_m) given by (7).

$$T_m.c_h = -(-1)^{\binom{m+2}{2}}\sum_{k=0}^{h-1}(-1)^{\binom{k+1}{2}}\left\{ {h-1 \atop k} \right\}c_{m-k}. \tag{8}$$

3. Computing $H_i(\mathbb{Z}/(3), M_m)$ when $m = 2r$ is even

If we assume that $m = 2r$ is even we have

$$1+\binom{m+2}{2}+\binom{k+1}{2}\equiv\binom{m-k}{2} \mod 2$$

so that $T_m = D_m Z_m$ where D_m is the diagonal matrix given by

$$D_m c_h = (-1)^{\binom{h}{2}}c_h, \quad h = 1,\dots,m \tag{9}$$

and

$$Z_m.c_h = \sum_{k=0}^{h-1}\left\{ {h-1 \atop k} \right\}c_{m-k}. \tag{10}$$

Start from D_2 and Z_2:

$$D_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

acting on $V = \mathbb{Z}^2$ with basis $\{x, y\}$. Consider $S^{r-1}(V)$ with basis $\{x^{r-1}, x^{r-1}y, \dots, y^{r-1}\}$. Then

$$S^{r-1}(D_2)(x^{r-s}y^{s-1}) = (-1)^{s-1}x^{r-s}y^{s-1}$$

and

$$S^{r-1}(Z_2)(x^{r-s}y^{s-1}) = y^{r-s}(x+y)^{s-1} = \sum_{t=0}^{s-1}\binom{s-1}{t}x^t y^{r-1-t}.$$

Take another copy of V with basis $\{\xi, \eta\}$ and consider $S^{r-1}(V)\otimes V$. We want to identify it with M_{2r} setting:

$$c_{2s-1} = x^{m-s}y^{s-1}\otimes\xi, \quad c_{2s} = x^{m-s}y^{s-1}\otimes\eta$$

for $1\le s\le r$. Notice now that

$$\binom{2s-1}{2}\equiv s-1 \mod 2, \qquad \binom{2s}{2}\equiv s \mod 2.$$

It follows that, since

$$S^{r-1}(D_2)\otimes D_2(c_{2s-1}) = (-1)^{s-1}c_{2s-1} \text{ and } S^{r-1}(D_2)\otimes D_2(c_{2s}) = (-1)^{s}c_{2s-1},$$

we have that for each $h=1,\dots m$

$$S^{r-1}(D_2)\otimes D_2(c_h)=(-1)^{\binom{h}{2}}c_h$$

hence $D_m=S^{r-1}(D_2)\otimes D_2$. Also

$$S^{r-1}(Z_2)\otimes Z_2(c_{2s-1})=\sum_{t=0}^{s-1}\binom{s-1}{t}x^ty^{r-1-t}\otimes\eta=\sum_{t=0}^{s-1}\binom{s-1}{t}c_{2(r-t)}=$$

$$=\sum_{k=0}^{2s-2}\{\begin{matrix}2s-2\\k\end{matrix}\}c_{m-k}$$

$$S^{r-1}(Z_2)\otimes Z_2(c_{2s})=\sum_{t=0}^{s-1}\binom{s-1}{t}x^ty^{r-1-t}\otimes(\xi+\eta)=\sum_{t=0}^{s-1}\binom{s-1}{t}(c_{2(r-t)-1}+c_{2(r-t)})=$$

$$=\sum_{k=0}^{2s-1}\{\begin{matrix}2s-1\\k\end{matrix}\}c_{m-k}.$$

So we have that for each $h=1,\dots m$

$$S^{r-1}(Z_2)\otimes Z_2(c_h)=\sum_{k=0}^{h-1}\{\begin{matrix}h-1\\k\end{matrix}\}c_{m-k}$$

hence $Z_m=S^{r-1}(Z_2)\otimes Z_2$. In conclusion we have proved

Proposition 1. *If $m=2r$, $T_m=S^{r-1}(T_2)\otimes T_2$.*

3.1. **Even Homology.** The above result allows us quite easily to compute the rank of $1-T_m$ both in characteristic zero and modulo 3.

If k is a given field, set $I_r^{(k)}$ equal to the dimension of the subspace of vectors fixed by $S^{r-1}(T_2)\otimes T_2$ acting on $S^{r-1}(k^2)\otimes k^2$, we have that:

$$\mathrm{rank}_{\mathbb{C}}(1-T_m)-\mathrm{rank}_{\mathbb{F}_3}(1-T_m)=I_r^{(\mathbb{F}_3)}-I_r^{(\mathbb{C})}$$

In simpler notation, use a generating series $I^{(k)}(t):=\sum_{r=0}^{\infty}I_r^{(k)}t^r$.

So now we proceed to compute the two cases.

In the first case $k=\mathbb{C}$ the matrix T_2 is conjugate to the matrix

$$\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}$$

with ε is a primitive third root of one. Thus we compute first the graded trace of $S(T_2)\otimes T_2$ on $S(V)\otimes V$.

By Molien's formula the trace of $S(T_2)$ on $S(V)$ is $1/(1+t+t^2)$ while the trace T_2 on V is $-t$ (we are in degree 1), so the graded trace required is $-t/(1+t+t^2)$. On the other hand the graded dimension is $2t/(1-t)^2$. In each degree the character is of the form $n(\epsilon+\epsilon^{-1})+m=m-n$ and the dimension is $2n+m$ hence the formula:

$$I^{(\mathbb{C})}(t)=\frac{1}{3}[\frac{2t}{(1-t)^2}-\frac{2t}{1+t+t^2}]=\frac{2t^2}{(1-t)(1-t^3)}$$

Let us consider the case $k=\mathbb{F}_3$ of the matrix modulo 3, changing the basis one has the matrix

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},$$

call a,b the elements of the new basis.

The first remark is that, in the polynomial ring the elements $a, c := b(b-a)(b+a)$ are two invariants (of degree 1,3 respectively) and $1, b, b^2$ is a basis over this ring $A := k[a, c]$ of invariants.

In this basis the matrix of $U := S(T_2)$ is:

$$\begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly treat $S(V) \otimes V$ as free module over A with basis $1 \otimes a, b \otimes a, b^2 \otimes a, 1 \otimes b, b \otimes b, b^2 \otimes b$ in this basis $T := S(T_2) \otimes T_2$ is:

$$T = \begin{pmatrix} 1 & a & a^2 & 1 & a & a^2 \\ 0 & 1 & -a & 0 & 1 & -a \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & a & a^2 \\ 0 & 0 & 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Write an invariant as a vector with 6 coordinates f_i by inspection we get $f_6 = f_5 = f_3 = 0$ and $f_4 = -af_2$.

Thus the invariants of $S(V) \otimes V$ are a free module over the polynomial ring $A = k[a, c]$ with basis $1 \otimes a, b \otimes a - a \otimes b$ of degrees 1,2 respectively. So its generating series of dimension is:

$$\frac{t + t^2}{(1 - t^3)(1 - t)}$$

The difference of the two generating series is:

$$\frac{2t^2}{(1 - t)(1 - t^3)} - \frac{t + t^2}{(1 - t^3)(1 - t)} = \frac{t}{1 - t^3}.$$

We deduce that

Proposition 2. *The even homology $H_{2i}(\mathbb{Z}/(3), M_{2r})$ is F_3 when $r \equiv 1, \mod 3$ and 0 otherwise.*

This is the only computation we need for our main result.

Theorem 3. *When $n = 3 * 2^k,\ k > 0$ we have:*

$$H_{n-1}(S_n, H^{n-1}(\mathbb{C}^n - \Delta, \mathbb{Z})) = 0$$

Proof. We have already remarked that the homology is killed by n so we separately have to show that the 3 and 2 torsion is 0. For 3 we use the work just done and see that all the $H_i(S_n, H^{n-1}(\mathbb{C}^n - \Delta, \mathbb{Z}))$ with i odd have no 3 torsion, since $n \equiv 0$, mod 3 . For the 2 torsion we can use the results of Arone and Dwyer, in fact, when we work modulo 2, the twisting by the sign disappears.

3.2. **Odd homology.** For completeness let us compute odd homology. We easily compute $1 + T + T^2$ getting:

$$1 + T + T^2 = \begin{pmatrix} 0 & 0 & -a^2 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

With the same notation as before, the kernel is formed by the vectors with $f_6 = 0, f_5 = -af_3$ its dimension series is thus:

$$\frac{t^2+2t+t^3}{(1-t)(1-t^3)}$$

Instead the dimension of the kernels over $\mathbb{Q}$ is like the dimension of the image of $1-T$ which from our computations is $\frac{2t+2t^3}{(1-t)(1-t^3)}$ so finally taking the difference we get the series for homology:

$$\frac{t^2}{1-t^3}$$

so that

Proposition 3. *The odd homology $H_{2i+1}(\mathbb{Z}/(3), M_{2r})$ is F_3 when $r \equiv 2, \mod 3$ and 0 otherwise.*

4. Computing $H_i(\mathbb{Z}/(3), M_m)$ when $m = 2r+1$ is odd

If we assume that $m = 2r+1$ is odd we have

$$1 + \binom{m+2}{2} + \binom{k+1}{2} \equiv \binom{m-k+1}{2} + 1 \mod 2$$

Define M_m to be this space of dimension $m = 2r+1$ with the given $\mathbb{Z}/(3)$ action.

We proceed in the following way. Consider the direct sum of all the spaces $M := \oplus_m M_m,\ m = 2r+1$ we want to identify it with a different module. Consider the polynomial ring A in two generators u, v (of degree 1) and the free rank 2 module N over A with basis $1, a$ (of degrees 0,1).

We identify N_r with M_r through the formulas:
$c_{2h+1} = u^{r-h}v^h,\ h = 0, \ldots, r, \quad c_{2h} = u^{r-h}v^{h-1}a.$

Next we need to define the action of an (homogeneous) operator ϕ (or order 3) on N as follows. The action is first defined on A and then on N so that it is semilinear with respect to the action of A

$$\phi(u) = -v, \quad \phi(v) = u - v, \quad \phi(a) = a - v, \quad \phi(1) = 1.$$

Let us check that the isomorphism between M and N preserves the $\mathbb{Z}/(3)$ actions:

$$\phi(c_{2h+1}) = (-1)^{r-h}v^{r-h}(u-v)^h = (-1)^{r-h}(\sum_{k=0}^{h}\binom{h}{k}(-1)^{h-k}u^kv^{r-k} = \sum_{k=0}^{h}\binom{h}{k}(-1)^{r-k}c_{m-2k}.$$

$$\phi(c_{2h}) = (-1)^{r-h}v^{r-h}(u-v)^{h-1}(a-v) = (-1)^{r-h}v^{r-h}\sum_{k=0}^{h-1}\binom{h-1}{k}(-1)^{h-1-k}u^kv^{h-1-k}(a-v) =$$

$$= \sum_{k=0}^{h-1}(-1)^{r-k-1}v^{r-k-1}u^ka + \sum_{k=0}^{h-1}(-1)^{r-k}u^kv^{r-k} =$$

$$= \sum_{k=0}^{h-1}(-1)^{r-k-1}\binom{h-1}{k}c_{2r-k} + \sum_{k=0}^{h-1}(-1)^{r-k}\binom{h-1}{k}c_{2(r-k)+1}$$

$$= \sum_{k=0}^{h-1}(-1)^{r-k-1}\binom{h-1}{k}c_{m-2k-1} + \sum_{k=0}^{h-1}(-1)^{r-k}\binom{h-1}{k}c_{m-2k}$$

With this description of the group action it is not hard to compute the graded dimension of the invariants over $\mathbb{Q}$ and $\mathbb{F}_3$, recall that u, v, a have degree 1 while 1 has degree 0.

$$\phi(u) = -v, \phi(v) = u - v$$

in char. 3 set $z = u + v, w = v$ have:

$$\phi(z) = z, \quad \phi(w) = z + w$$

Then $z, t := w(w-z)(w+z)$ are two invariants and, if $B = \mathbb{F}_3[z,t]$ we have that N is a free module over B with basis $1, w, w^2, a, wa, w^2a$ (of degrees $0,1,2,1,2,3$) in this basis the matrix of ϕ (which is now linear) is:

$$T := \begin{pmatrix} 1 & z & z^2 & 0 & 0 & -t \\ 0 & 1 & -z & -1 & -z & z^2 \\ 0 & 0 & 1 & 0 & -1 & z \\ 0 & 0 & 0 & 1 & z & z^2 \\ 0 & 0 & 0 & 0 & 1 & -z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The invariants are thus a B module and the kernel of the matrix $1 - T$. This kernel is easily computed and is formed by the vectors of coordinates $f_i, i = 1, \dots, 6$ with:

$$f_6 = 0 = f_5, \quad f_4 = -zf_3, f_2 = f_4$$

So the kernel is a free module with one generator of degree 0 and one of degree 2 so its dimension series is $(1+t^2)/((1-t^3)(1-t))$.

Now over $\mathbb{C}$ we compute as follows. We have that $A \subset N$ is a submodule but, in characteristic 0, we have $N = A \oplus N/A$, clearly by the formulas N/A is A shifted in dimension by 1, computing by Molien's formula we get for the dimension series of the invariants $(1+t^3)/((1-t^3)(1-t))$ over $\mathbb{C}$ taking the difference we obtain $t^2/(1-t^3)$.

Proposition 4. *The even homology $H_{2i}(\mathbb{Z}/(3), M_{2r+1})$ is F_3 when $r \equiv 2, \mod 3$ and 0 otherwise.*

For the odd homology we use the matrix $1 + T + T^2$ which mod. 3 is:

$$\begin{pmatrix} 0 & 0 & -z^2 & -z & z^2 & -z^3 \\ 0 & 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & -z^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Its kernel is given by $f_6 = 0, f_4 = z(f_5 - f_3)$ so its dimension series is $(1+t+2t^2)/((1-t^3)(1-t))$

Over $\mathbb{Q}$ we have to take the difference:

$$\frac{1+t}{(1-t)^2} - \frac{1+t^3}{(1-t^3)(1-t)} = \frac{2(t+t^2)}{(1-t^3)(1-t)}$$

finally taking again the difference of the two ranks, in char. 0 and 3 we get:

$$\frac{1+t+2t^2}{(1-t^3)(1-t)} - \frac{2(t+t^2)}{(1-t^3)(1-t)} = \frac{1}{(1-t^3)}$$

Proposition 5. *The odd homology $H_{2i+1}(\mathbb{Z}/(3), M_{2r+1})$ is 0 when $r \equiv 0, \mod 3$ and F_3 otherwise.*

4.1. $n = 10$ **and final comments.** We have written a Mathematica notebook to compute the matrices $T(m,h)$ which give the action of the cyclic group of order h on the coinvariants with respect to S_{m+2-h} acting on $P(m+2)$.

This notebook can be downloaded from the internet location:
web/ricercatori/rogora/lie/LIEn.nb

In particular, the matrices T_m of this paper can be computed as $T(m,3)$ in the notebook. The matrix $T_{8,5}$ can be used to prove that $g(10) < 10$. The details are given in the notebook.

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We thank Fred Cohen for suggesting the reduction approach, R. Salvati Manni and M. Manetti for suggestions on the matrix T_m.

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