NOTES ABOUT SATURATED CHAINS IN THE DYCK PATH POSET

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1. Basic Definitions

Dyck paths are one of the many combinatorial objects enumerated by the Catalan numbers, sequence $A000108$ in [2]: $1, 1, 2, 5, 14, 42, 132, 429, \ldots$. We recall the following definitions:

Definition 1. *A Dyck path is a lattice path in the $n \times n$ square consisting of only north and east steps and such that the path doesn't pass below the line $y = x$ (or main diagonal) in the grid. It starts at $(0, 0)$ and ends at (n, n). A walk of length n along a Dyck path consists of $2n$ steps, with n in the north direction and n in the east direction. By necessity the first step must be north and the final step must be east.*

Comparing paths that have the same length, we say that one path is 'less than' another if it lies below the other when the grids are superimposed. Note that within this definition, the path need not remain strictly below the other; it is possible that they coincide at some points. Using this $\leq$ relation, we can impose a partial order on the set of Dyck paths. We refer to this as ordering by inclusion and, for the rest of this paper, will use the notation D_n to indicate the poset of Dyck paths of length n ordered in this way.

Definition 2. *Given a poset, P and two elements $x, y \in P$, we say that x is comparable to y iff $x \leq y$ or $y \leq x$. Otherwise we say that x and y are incomparable.*

Definition 3. *An element y covers (or is a cover for) an element x in a poset, P, if $x < y$ (i.e. $x \leq y$, but $x \neq y$) and there is no other element, $z \in P$ where $x < z < y$.*

It is common to use a Hasse diagram to provide a pictorial representation of the poset. In the example below showing D_3 and D_4, the vertices are the Dyck paths and the edges illustrate the cover relationships that exist among the paths. Also, note that the elements are grouped into ranks based on the area (or number of complete lattice cells) between each Dyck path and the line $y = x$.

Example 4. *The diagrams below show the Hasse diagrams for D_3 and D_4.*

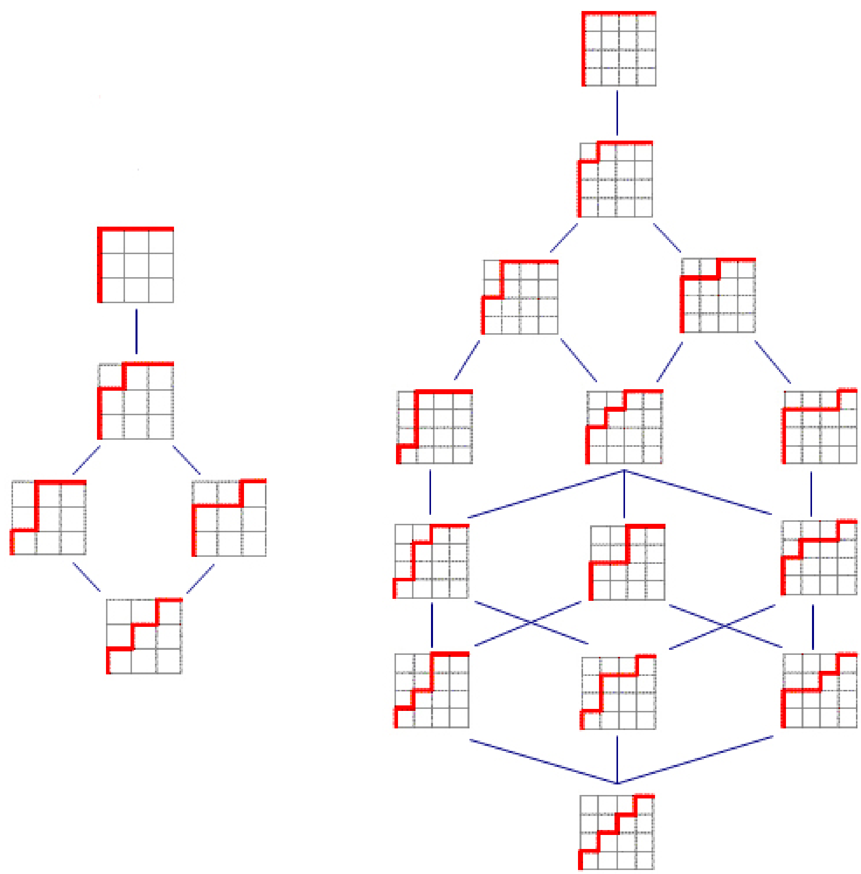

FIGURE 1. Hasse diagrams for D_3 (left) and D_4 (right)

2. Counting Chains

Definition 5. *A chain is a totally ordered subset of a poset. This means that within a chain any two elements are comparable.*

Given that the zeta function is defined as $\zeta(x, y) = 1$ for all $x \leq y$ in D_n and 0 otherwise and that the identity function, denoted δ, is defined such that $\delta(x, y) = 1$ if $x = y$ and 0 otherwise, we established the following theorems in [4].

Theorem 6. *(from* [3]*, p. 115)*
Let $x = x_0 < x_1 < \ldots < x_k = y$ be a chain on the interval $[x, y]$ in a poset, P, of n

elements. Then the total number of chains of length k on the interval $[x, y]$ is equal to $(\zeta - \delta)^k(x, y)$.

Theorem 7. *(from [3], p. 115)*
The total number of chains $x = x_0 < x_1 < \ldots < x_k = y$ from x to y in D_n is equal to $(2\delta - \zeta)^{-1}(x, y)$.

Using these results we determined the total number of chains in the D_n poset for $n = 0$ to 5, included as A143672 in [2]:

n	Number of Chains
0	1
1	2
2	4
3	24
4	816
5	239968

Proposition 8. *I would have voted no.*

Definition 9. *The chain polynomial, $T(P, t) = 1 + \sum_k c_k t^{k+1}$ where c_k is the number of chains (or totally ordered subsets) in a poset P of length k. The exponent $k + 1$ indicates the number of vertices in the chain and the 1 in front of the sum denotes the empty chain.*

In [4] we also used Maple to find the chain polynomial for the D_n poset. Since $(\zeta - \delta)^k$ counts chains of length k, the chain polynomial for D_n is easily determined by choosing values $k = 0$ to l (where l is the length of the longest chain in the poset) for each value of n. The results are summarized in the table:

n	Chain Polynomial	Factored Form
0	1	
1	$1 + t$	$(1 + t)$
2	$1 + 2t + t^2$	$(1 + t)^2$
3	$1 + 5t + 9t^2 + 7t^3 + 2t^4$	$(1 + 2t)(1 + t)^3$
4	$1 + 14t + 70t^2 + 176t^3 + 249t^4 + 202t^5 + 88t^6 + 16t^7$	$(1 + 8t + 8t^2)(1 + 2t)(1 + t)^4$
5	$1 + 42t + 552t^2 + 3573t^3 + 13609t^4 + 33260t^5 + 54430t^6 +$	
	$60517t^7 + 45248t^8 + 21824t^9 + 6144t^{10} + 768t^{11}$	$(1 + 37t + 357t^2 + 1408t^3 + 2624t^4 + 2304t^5 + 768t^6)(1 + t)^5$

In attempting to determine an explicit formula for the number of chains, we have obtained the following result which, although cumbersome, gives us a recursive method for counting chains.

Theorem 10. *The chain polynomial, T_j, in a poset of j Dyck paths is given by*

$$T_j = 1 + \sum_{i=1}^{j} (q_i * \sum_{q<q_j} T)$$

where q_i represents the ith path in the poset.

Proof. Let D be a poset of j Dyck paths. Every chain in D must have a maximal element. Suppose that q_k is the path corresponding to the maximal element for a particular set of chains, C_k, i.e. C_k will be the set of all chains having q_k as their maximal element. But, each of these chains will consist of q_k and some number of paths below q_k in the poset. Since the Dyck path poset is ranked, it is only possible that one path from each rank be included in any particular chain. Consider q_k as having rank r. Applying the multiplication and addition principles, the chains in C_k are:

q_k AND [(chains on paths $< q_k$ whose maximal element has rank $r-1$) OR (chains on paths $< q_k$ whose maximal element has rank $r-2$) OR ... OR (chains on paths $< q_k$ whose maximal element has rank 1) OR (chains on paths $< q_k$ whose maximal element has rank 0)].

Note that rank 0 indicates the empty chain. Since there is one empty chain, we can rewrite the above expression as: $q_k * \Big(1 + \sum_{i=1}^{r-1}$ (chains on paths $< q_k$ whose maximal element has rank i)$\Big)$. But, by applying this argument recursively, we see that the expression $1 + \sum_{i=1}^{r-1}$(chains on paths $< q_k$ whose maximal element has rank i) is just the sum of the chain polynomials, T, for the elements $< q_k$. Therefore we can rewrite this expression as: $q_k * \sum_{q<q_k} T$. Now we can categorize each chain in the poset D into one of the following cases. It is either the empty chain OR has q_1 as the maximal element OR has q_2 as the maximal element OR ... OR has q_k as the maximal element OR ... OR has q_j as the maximal element. Therefore by the addition principle, if q_i represents the ith path in the poset, then the chain polynomial for the poset D is

$$T_j = 1 + \sum_{i=1}^{j} (q_i * \sum_{q<q_j} T)$$

$\square$

Example 11. *Consider D_4. If we number the vertices on the Hasse diagram from bottom to top of the poset with $q_0, q_1, \ldots, q_{14}$, then the chain polynomial is*

$$
\begin{aligned}
&((1 + q_1)\cdot(1 + q_2 + q_3 + q_4 + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4) + q_6\cdot(1 + q_2 + q_4) + q_9\cdot((1 + q_2 + q_3 + q_4) + q_5\cdot(1 + q_2 + q_3)\\
&+ q_7\cdot(1 + q_3 + q_4) + q_6\cdot(1 + q_2 + q_4)) + q_8\cdot((1 + q_2 + q_3) + q_5\cdot(1 + q_2 + q_3)) + q_{10}\cdot((1 + q_3 + q_4) + q_7\cdot(1 + q_3 + q_4)) + q_{11}\cdot(1\\
&+ q_2 + q_3 + q_4 + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4) + q_6\cdot(1 + q_2 + q_4) + q_9\cdot((1 + q_2 + q_3 + q_4) + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3\\
&+ q_4) + q_6\cdot(1 + q_2 + q_4)) + q_8\cdot((1 + q_2 + q_3) + q_5\cdot(1 + q_2 + q_3))) + q_{12}\cdot(1 + q_2 + q_3 + q_4 + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4)\\
&+ q_6\cdot(1 + q_2 + q_4) + q_9\cdot((1 + q_2 + q_3 + q_4) + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4) + q_6\cdot(1 + q_2 + q_4)) + q_{10}\cdot((1 + q_3 + q_4) + q_7\cdot(1\\
&+ q_3 + q_4))) + q_{13}\cdot(1 + q_2 + q_3 + q_4 + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4) + q_6\cdot(1 + q_2 + q_4) + q_9\cdot((1 + q_2 + q_3 + q_4) + q_5\cdot(1 + q_2\\
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&\cdot(1 + q_2 + q_3 + q_4 + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4) + q_6\cdot(1 + q_2 + q_4) + q_9\cdot((1 + q_2 + q_3 + q_4) + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3\\
&+ q_4) + q_6\cdot(1 + q_2 + q_4)) + q_8\cdot((1 + q_2 + q_3) + q_5\cdot(1 + q_2 + q_3))) + q_{12}\cdot(1 + q_2 + q_3 + q_4 + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4)\\
&+ q_6\cdot(1 + q_2 + q_4) + q_9\cdot((1 + q_2 + q_3 + q_4) + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4) + q_6\cdot(1 + q_2 + q_4)) + q_{10}\cdot((1 + q_3 + q_4) + q_7\cdot(1\\
&+ q_3 + q_4)))) + q_{14}\cdot(1 + q_2 + q_3 + q_4 + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4) + q_6\cdot(1 + q_2 + q_4) + q_9\cdot((1 + q_2 + q_3 + q_4) + q_5\cdot(1 + q_2\\
&+ q_3) + q_7\cdot(1 + q_3 + q_4) + q_6\cdot(1 + q_2 + q_4)) + q_8\cdot((1 + q_2 + q_3) + q_5\cdot(1 + q_2 + q_3)) + q_{10}\cdot((1 + q_3 + q_4) + q_7\cdot(1 + q_3 + q_4)) + q_{11}\\
&\cdot(1 + q_2 + q_3 + q_4 + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4) + q_6\cdot(1 + q_2 + q_4) + q_9\cdot((1 + q_2 + q_3 + q_4) + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3\\
&+ q_4) + q_6\cdot(1 + q_2 + q_4)) + q_8\cdot((1 + q_2 + q_3) + q_5\cdot(1 + q_2 + q_3))) + q_{12}\cdot(1 + q_2 + q_3 + q_4 + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4)\\
&+ q_6\cdot(1 + q_2 + q_4) + q_9\cdot((1 + q_2 + q_3 + q_4) + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4) + q_6\cdot(1 + q_2 + q_4)) + q_{10}\cdot((1 + q_3 + q_4) + q_7\cdot(1\\
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&+ q_3) + q_7\cdot(1 + q_3 + q_4) + q_6\cdot(1 + q_2 + q_4)) + q_8\cdot((1 + q_2 + q_3) + q_5\cdot(1 + q_2 + q_3)) + q_{10}\cdot((1 + q_3 + q_4) + q_7\cdot(1 + q_3 + q_4)) + q_{11}\\
&\cdot(1 + q_2 + q_3 + q_4 + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4) + q_6\cdot(1 + q_2 + q_4) + q_9\cdot((1 + q_2 + q_3 + q_4) + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3\\
&+ q_4) + q_6\cdot(1 + q_2 + q_4)) + q_8\cdot((1 + q_2 + q_3) + q_5\cdot(1 + q_2 + q_3))) + q_{12}\cdot(1 + q_2 + q_3 + q_4 + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4)\\
&+ q_6\cdot(1 + q_2 + q_4) + q_9\cdot((1 + q_2 + q_3 + q_4) + q_5\cdot(1 + q_2 + q_3) + q_7\cdot(1 + q_3 + q_4) + q_6\cdot(1 + q_2 + q_4)) + q_{10}\cdot((1 + q_3 + q_4) + q_7\cdot(1\\
&+ q_3 + q_4)))))));
\end{aligned}
$$

with the first appearance of each of the q_i indicated in red.

More efficiently, this is equivalent to recursively assigning the value $1+$ (label sum of ALL covering paths) on a Hasse diagram as we move down the poset. Alternatively, we can move up from the bottom and label the vertices using $1+$ (label sum of ALL paths covered). The totals will (of course) be identical, and by adding one to the total (for the empty chain) we have found a handy way to count total chains using only a Hasse diagram and some simple addition. This is a huge advantage over the $(2\delta - \zeta)^{-1}$ method or the Maple recursion listed above since we avoid having to enter progressively larger matrices or expressions into computer software.

Example 12. *Once again using D_4, here are the labels assigned using the above process.*

Note that, for the total chains in the poset, it suffices to find the vertex label for either the maximal (or minimal) element in the poset and then double that value. It corresponds to a Dyck path that is 'above' (or respectively, 'below') all other paths in the poset and its label is therefore one more than the sum of all the other labels. Hence doubling the maximal (or minimal) element's label is equivalent to summing all of the labels and including one more for the empty chain. For example, in figure 2 we see that $408 \times 2 = 816$, which we know to be the total number of chains for D_4.

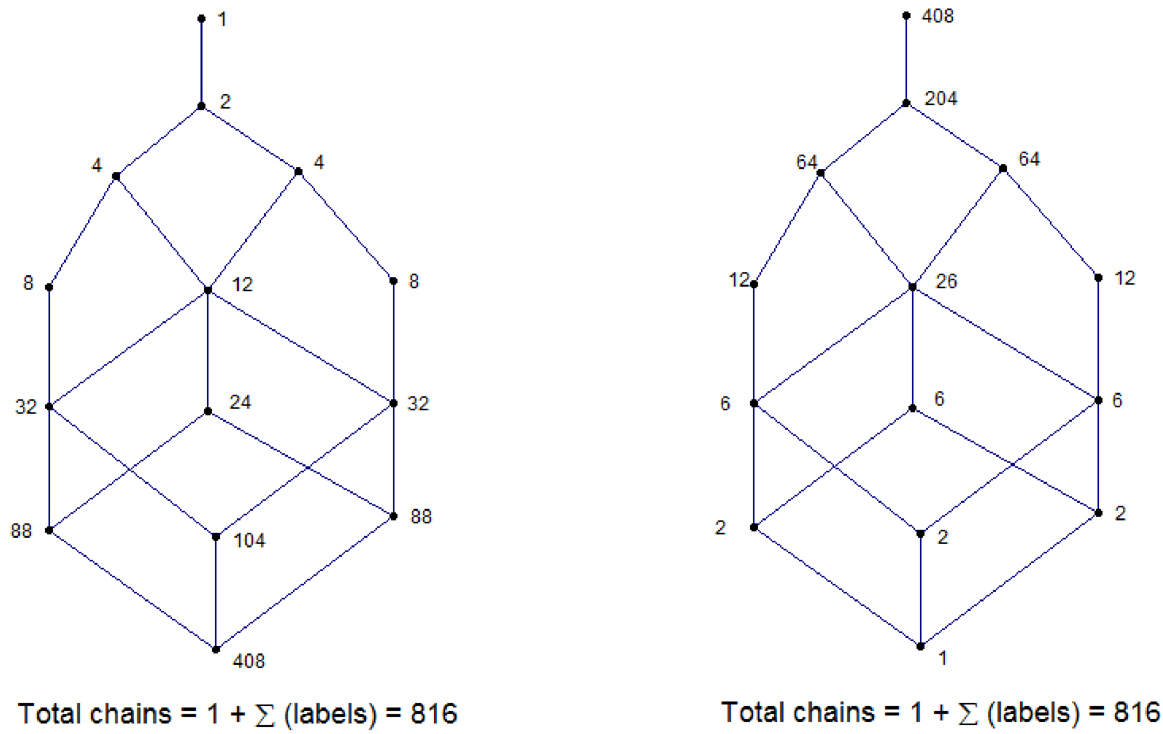

Figure 2. Counting total chains down from the top (left) and up from the bottom (right)

Since we have discovered recursive formulas for counting the chains and also note that the Catalan numbers themselves can also easily be defined using a recursion, it seems sensible to next consider how D_n might be constructed from D_{n-1}. The method described below is known as ECO construction in the literature [1]. This is not because it's environmentally friendly, but instead because these are the initials for 'Enumerating Combinatorial Objects'.

Begin with a Dyck path, P of length $2n$ - this is the 'father'. Suppose that there are k East steps following the final North step in the path (this is analogous to having k steps in the last descent if the path is oriented horizontally as in MuPAD). Then we can construct $k+1$ Dyck paths of length $2n+2$ starting from P - these are the 'sons'. We do this by inserting a pair of steps, travelling first North then East, at each node along the final k East steps in P (i.e. inserting a peak at each point of the last descent).

Example 13. *This figure shows a Dyck path with $n = 7$ (the father) which ends with two East steps (on 3 nodes). The sons ($n = 8$) are created by inserting a North step at each of the nodes and then travelling East to finish the path.*

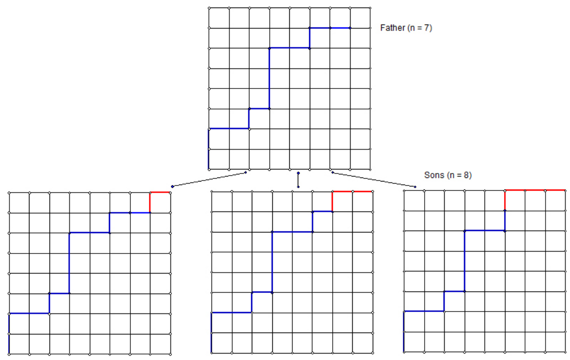

Figure 3. Father and Sons

If you do this on all Dyck paths of length $2n$ you will produce every Dyck path of length $2n+2$ exactly once. Furthermore, every Dyck path which ends in k East steps produces $k+1$ sons which end with $1, 2, \cdots, k, k+1$ East steps respectively. This process seems recursive. For those that enjoy graph theory and accompanying diagrams, we include this (lovely) tree. The indices refer to the number of East steps following the final North step.

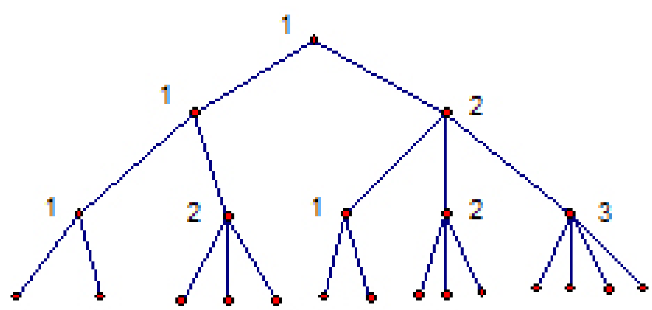

Figure 4. Dyck path tree diagram

If we count the number of nodes (not the numbers!) across each row of the tree diagram, we once again recover the Catalan numbers! So the number of Dyck paths of length $2n$ is equal to the number of nodes at level n of this tree.

Now notice that this ECO construction provides us with a natural partition into sets of Dyck paths for any given length. We can separate the Dyck paths of length $2n$ into equivalence classes by grouping together those that have a common father. This leads to some interesting results (see [1] for applicable proofs).

1) The set of sons of a Dyck path of length $2n-2$ forms a saturated chain in D_n (saturated

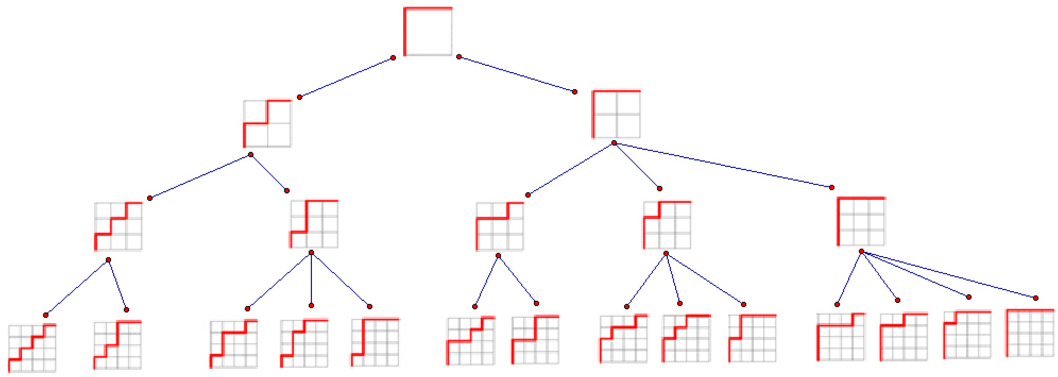

Figure 5. Dyck path family tree

means a chain such that for any two elements x, y in it, $x < y \implies x \preceq y$).

2) Staircase paths form the endpoints for these saturated chains, when the equivalence class contains staircase paths.

3) If we connect these chains using solid lines on a Hasse diagram, the remaining cover relations (dotted lines on the figures below) are copies of the D_{n-1} poset.

4) Corresponding vertices in the copies of D_{n-1} are linked together by the saturated chains.

Here are D_3 and D_4 with the equivalence class chains indicated by solid lines and the remaining cover relationships drawn using dotted lines. Staircase paths are indicated with red vertices. Notice these form the endpoints of all but one of these chains (and it doesn't contain a staircase).

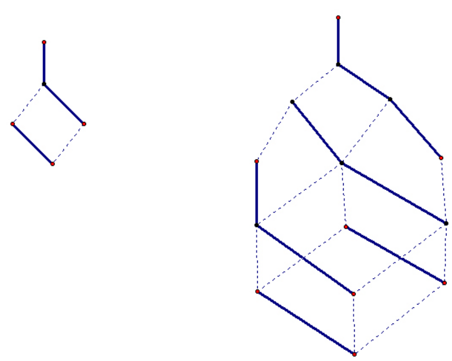

Figure 6. Saturated chain partitions of D_3 and D_4

To better see how copies of D_3 are used to make D_4, we distort the Hasse diagram for D_4 by translating the vertices (diagram is from [1]).

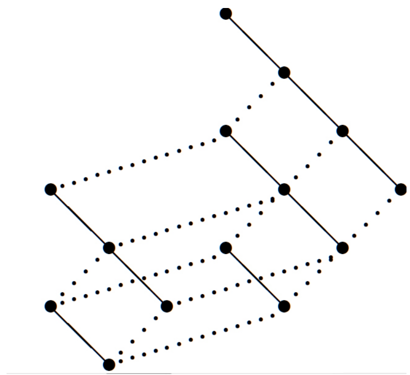

FIGURE 7. D_4 - A different viewpoint

3. Counting Saturated Chains

3.1. **Definitions, examples and previous results.**
Given the importance of saturated chains in building up D_n, it seems logical to focus on this interesting subset of the chains. Perhaps by examining the saturated chains more closely we will gain insight into the overall chain structure. Discrepancies exist in the literature regarding the definitions of maximal and saturated chains. For example, some suggest that these terms are interchangeable. We claim otherwise; as in [4], we follow Stanley's approach. In [3] he writes: A chain C of poset P is saturated (or unrefinable) if $\nexists\, z \in P - C$ st $x < z < y$ for some $x, y \in C$ and st $C \cup \{z\}$ is a chain. In other words, no element can be added *between* two of its elements without losing the property of being totally ordered.

Example 14. *Here is the Hasse diagram of the* D_3 *poset. It contains* 17 *saturated chains, broken down as follows: There is the empty chain, 5 chains on one vertex (A, B, C, D, E), 5 chains on two vertices (AB, AC, BD, CD, DE), 4 chains on three vertices (ABD, ACD, BDE, CDE), and the 2 maximal chains on 4 vertices (ABDE, ACDE).*

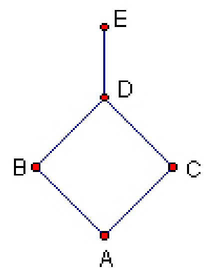

FIGURE 8. D_3 poset

In contrast to this, a maximal chain is one which cannot be extended, i.e. no element can be added (either within or on the end) without losing the chain property. We used this definition in determining that the number of maximal chains in the poset of Dyck paths ordered by inclusion is given by OEIS sequence A005118. Thus, while it is true that all maximal chains are saturated, the converse does not necessarily hold. Maximal chains are particularly important in the Dyck path poset since they contain one element of each rank. From [3] again: If every maximal chain in a poset P has the same length n then we say P is graded of rank n. The Dyck paths we looked at were ranked by area. Recall that we found the total number of chains and number of maximal chains to be:

n	Number of Chains	Number of Maximal Chains
0	1	1
1	2	1
2	4	1
3	24	2
4	816	16
5	239968	768

Also recall (from [4]) the following theorem and explicit formula for the maximal chains using a bijection with Standard Young Tableaux (SYT).

Theorem 15. *The number of maximal chains in D_n is equal to the number of standard Young tableaux for the staircase partition of size $\binom{n}{2}$ above the minimal Dyck path in D_n and therefore are counted using the formula*

$$\frac{\binom{n}{2}!}{\prod_{i=1}^{n-1}(2i-1)^{n-i}}$$

Conveniently, the number of standard Young tableaux for a partition is straightforward to calculate using the hook length formula [5].

Theorem 16. *(Robinson-Frame-Thrall Theorem)*
The hook length of a cell, x, which lies in a Young diagram for a partition, λ, can be calculated by summing the number of cells in λ that lie to the right of x and the number of cells in λ that lie below x and then adding one for the cell itself. If a partition, λ, consists of n cells, then:

$$\textit{the number of standard Young tableaux in } \lambda = \frac{n!}{\prod_{x\in\lambda} hook(x)}$$

A proof of this theorem is provided in [5].

Since we are dealing only with staircase partitions, we can specialize this formula. For D_n, staircase partitions above the minimal Dyck path always consist of $\binom{n}{2}$ cells, so we use $\binom{n}{2}!$ for the numerator. The hook lengths for our diagrams always consist of $(n-1)$ ones, $(n-2)$ threes, $(n-3)$ fives, and so on. Therefore the product of the hook lengths in the denominator will be $\prod_{i=1}^{n-1}(2i-1)^{n-i}$. We use the specialized formula in the solution for the example below.

Example 17. *In the diagram below (the staircase partition for D_5), the cells have been labelled with their hook lengths.*

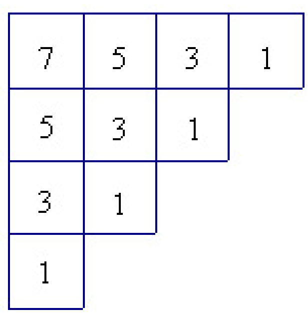

FIGURE 9. Hook lengths for D_5 staircase partition

Then, by the hook length formula, the number of standard Young tableaux corresponding to *this diagram is equal to*

$$\begin{aligned}\frac{\binom{n}{2}!}{\prod_{i=1}^{n-1}(2i-1)^{n-i}} &= \frac{\binom{5}{2}!}{\prod_{i=1}^{4}(2i-1)^{5-i}}\\ &= \frac{10!}{(1^4)\cdot(3^3)\cdot(5^2)\cdot(7^1)}\\ &= \frac{3628800}{4725}\\ &= 768\end{aligned}$$

which is equal to the number of maximal chains in D_5*.*

3.2. **Saturated chain data (Maple) and recursive formula.**
Using Stembridge's Maple posets package to list all the chains, and a program written by Zabrocki to filter out the 'unsaturated' ones, we obtain these size breakdowns and totals for the saturated chains:

n	Size Breakdown	Number of Saturated Chains
1	1	1
2	2,1	3
3	5,5,4,2	16
4	14,21,30,38,40,32,16	191
5	42,84,168,322,578,952,1408,1808,1920,1536,768	9586

Note that the first value in each breakdown, corresponding to saturated chains on 1 vertex, is C_n and that the last number in the breakdowns matches the maximal chain values previously listed. Also, according to Richard Stanley (personal communication), the empty chain should NOT, in general, be included in the saturated chain count.

By applying the addition and multiplication principles and using the cover relationships from the Hasse diagram as we did in determining the recursive formula for the total number of chains, we are able to find similar parentheses-filled expressions for the saturated chains. We illustrate this for the D_3 poset from above.

The saturated chains can be written

$$q_1+q_2+q_3+q_4+q_5+q_1q_2+q_1q_3+q_2q_4+q_3q_4+q_4q_5+q_1q_2q_4+q_1q_3q_4+q_2q_4q_5+q_3q_4q_5+q_1q_2q_4q_5+q_1q_3q_4q_5$$
$$= q_1(1+q_2(1+q_4(1+q_5))+q_3(1+q_4(1+q_5)))+q_2(1+q_4(1+q_5))+q_3(1+q_4(1+q_5))+q_4(1+q_5)+q_5$$

and if we set $q_i = 1$ for all i, then this expression produces the answer 16.

Given a Hasse diagram of the poset, it is also easy to recursively determine the number of saturated chains. This can be done by assigning each vertex a label using the following algorithm (similar to the one described above for the total chains). Start with the uppermost vertex and label this with a 1. Proceed down through the diagram, labelling each vertex, i, with the value $= 1 + \sum$(labels of the vertices immediately covering i). When all vertices have been labelled in this manner, simply sum the labels to obtain the total number of saturated chains. Examples for D_3 and D_4 are shown.

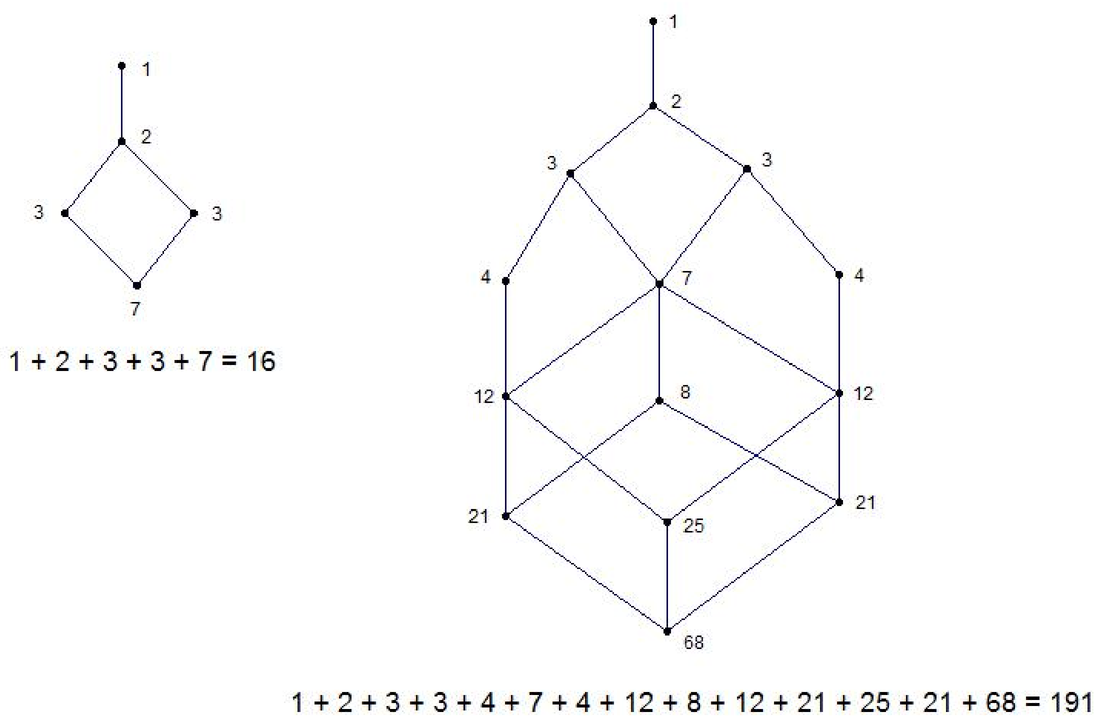

Figure 10. Counting saturated chains in D_3 and D_4

3.3. **And back to partitions.**

Our goal is to find an explicit formula for the saturated chains, perhaps analogous to the hook length formula. Of course there are other methods for counting the number of SYT and hence, the number of maximal chains. Consider again the partitions that lie above the Dyck paths and within the grid. For example, D_3 would look like this:

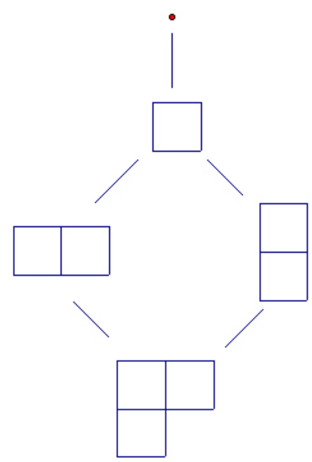

FIGURE 11. Partitions above D_3 paths

The two maximal chains are obvious. Because of the bijection with SYT we can count these maximal chains using any method for enumerating the SYT of $\lambda = (2, 1)$. Although previously mentioned, we include the hook length formula again on this example for completion.

1) Listing all possibilities.

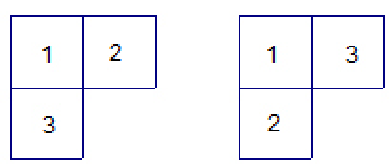

FIGURE 12. Standard Young Tableaux $\lambda = (2, 1)$

2) Hook length formula (Robinson, Frame, Thrall) as described above.

This formula states that

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}$$

where f^λ is the number of SYT in partition λ and $h_(i, j)$ is the hook length of a cell (i, j) in the partition.

For our example $\lambda = (2, 1)$, the hook lengths are:

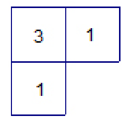

FIGURE 13. Hook lengths for $\lambda = (2, 1)$

and therefore,

$$f^{(2,1)} = \frac{3!}{3 \cdot 1 \cdot 1} = 2.$$

3) Determinantal formula (Frobenius and Young??).

In this much older formula, $1/r! = 0$ if $r < 0$. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell) \vdash n$, then

$$f^\lambda = n! \cdot det[1/(\lambda_i - i + j)!].$$

In the denominators of the determinant entries, the partition parts are written along the main diagonal. For the other entries, increase or decrease the number inside the factorial by one for every step to the right or left, respectively. Using this formula for $\lambda = (2, 1)$,

$$f^{(2,1)} = 3! \begin{vmatrix} 1/2! & 1/3! \\ 1/0! & 1/1! \end{vmatrix} = 3! \begin{vmatrix} 1/2 & 1/6 \\ 1 & 1 \end{vmatrix} = 3!(1/3) = 2.$$

Seems this formula would not be easy to work with for partitions with many rows!

I hoped that by understanding these formulas, perhaps I could figure out how to count the subsequences of partitions that go into building the staircase (there's a bijection between these and the saturated chains). Here are examples of a 2-element partition subsequence and a 3-element one. Maybe there is some other partition formula that counts subsets???

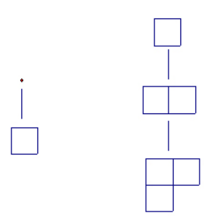

FIGURE 14. Examples of partition subsequences

3.4. **Examining partitions in between pairs of Dyck paths.**

Perhaps the answer lies in returning to Dyck paths and considering the partitions which lie between superimposed Dyck paths for each pair in the poset.

Conjecture 18. *For $D_i, D_j \in D_n$,*

$$\textit{number of saturated chains} = \sum_{D_i \le D_j} \textit{(SYT for } D_j \backslash D_i \textit{ partition)}$$

This assumes that for $D_i = D_j$ (the difference partition is empty), there is one SYT. If we want to avoid this assumption, we can modify the right side of the equation:

$$C_n + \sum_{D_i < D_j} (\text{SYT for } D_j \backslash D_i \text{ partition}).$$

I'm fairly certain that the conjecture is true and I have verified it up to D_4 by drawing out the difference partitions (70 of them for D_4!) by hand and counting the tableaux containing integers 1 to n increasing across rows and down columns. I may be totally misusing vocabulary here since some of the shapes created with $D_j \backslash D_i$ are not left-justified Young diagrams, but rather skew shapes. This is also the reason why I'm not writing the hook length formula in the equation - it only worked for the left-justified diagrams. I'm sure there are formulas for counting the tableaux for skew shapes, but I just counted them by writing out all of the possibilities.

Example 19. *Figure 15 shows the diagrams for D_3. D_j is in red, D_i in blue. The numbers written under each diagram are the number of SYT. I have not shown diagrams for the cases where $D_i = D_j$. Figure 16 is a tally by 'cell' of the contents of each difference partition. When these tallies are multiplied by the number of SYT and added, we obtain saturated chain totals. Notice that the number of cells in the partition corresponds to the chain length, so for example, $a_{12}a_{21}, a_{11}a_{21}, a_{11}a_{12}$ are chains of length 2 (i.e. on 3 vertices) and there are $2 + 1 + 1 = 4$ of them in D_3.*

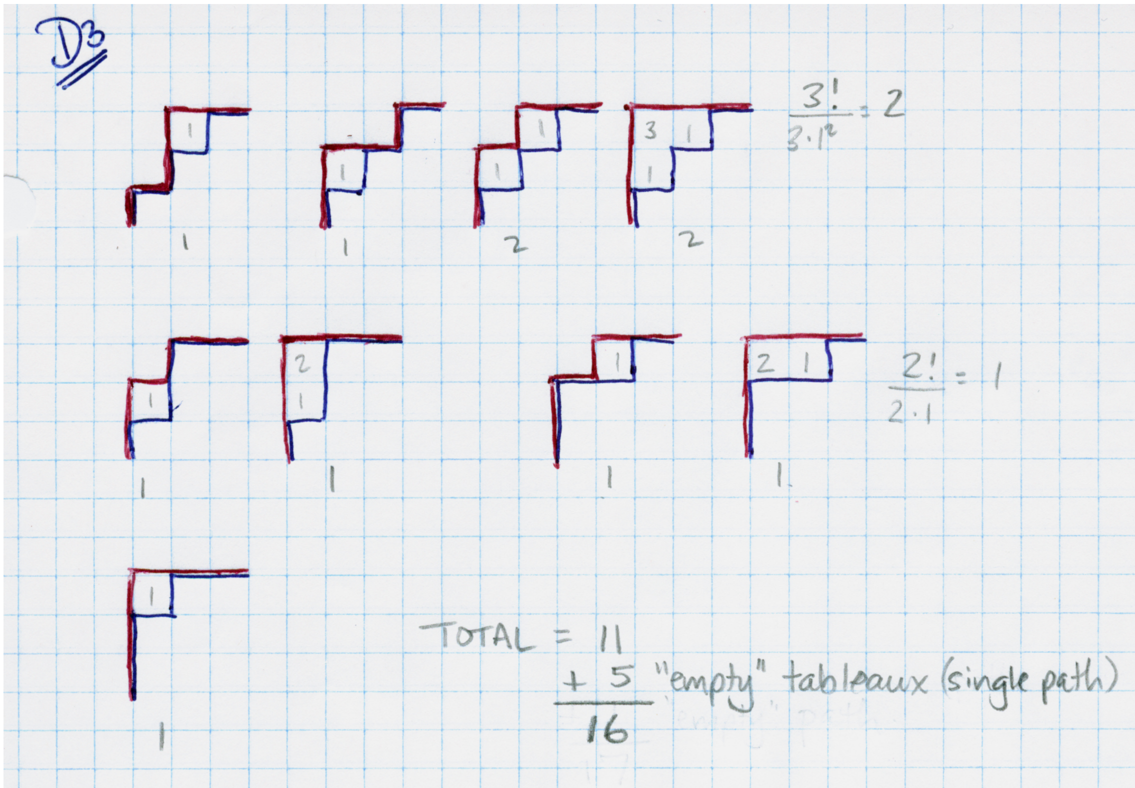

FIGURE 15. Difference partitions for D_3

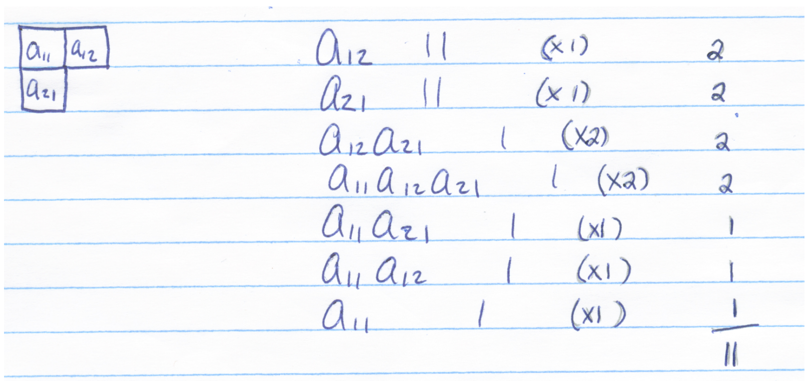

FIGURE 16. D_3 partition tally

3.5. **Saturated chains as Young's lattice walks.**
In considering the connection between saturated chains and SYT, it becomes evident that our problem is equivalent to counting walks of a particular type within Young's lattice. Young's lattice is a partially ordered set composed of integer partitions whose Young diagrams have been ordered by inclusion. Notice that the Hasse diagrams of DP posets (ordered by inclusion) are clearly visible within Young's lattice, although they are upside-down.

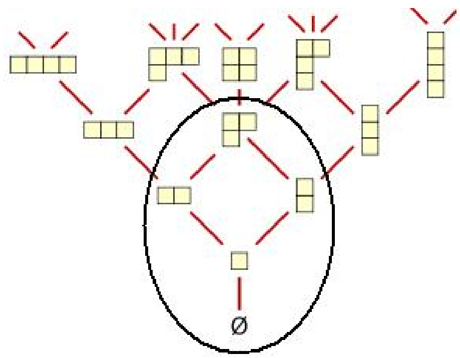

FIGURE 17. Young's lattice with D_3 circled

Walks beginning at $\emptyset$, consisting only of n up-steps and ending at a particular partition are just saturated chains: $\emptyset = \lambda^0 < \lambda^1 < \lambda^2 < \ldots < \lambda^n$ from $\emptyset$ to the partition. If we encode the steps in this walk by taking the partition diagram and inserting numbers in each cell to represent the stage at which the cell was 'added' in the sequence, we obtain a standard Young tableau. So, we see that saturated chains from $\emptyset$ to a partition, p are in a natural bijection with SYT of the shape p. For example, here is a 'walk' sequence of partition diagrams and the corresponding SYT.

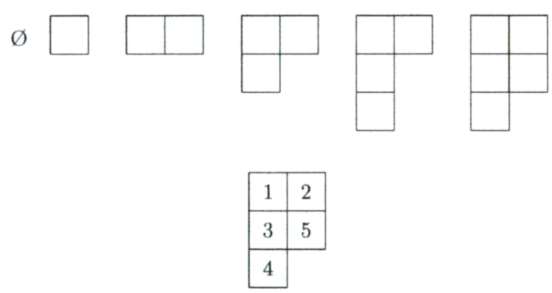

FIGURE 18. Creating a SYT from a Young's lattice walk

The number of possible walks from $\emptyset$ to a given partition then, is equal to the number of possible SYT for that diagram and as we know these can be counted using the hook length formula. Perhaps there are additional formulas that count SYT between partitions that don't start at $\emptyset$ or walks between lattice points. Seems where back where we started again.

3.6. **Divide and conquer method.**

For another strategy aimed at determining an explicit formula for the saturated chains, we revisit the data breakdown according to chain size. Recall the following table:

n	Size Breakdown by rank	Number of Saturated Chains
1	1	1
2	2,1	3
3	5,5,4,2	16
4	14,21,30,38,40,32,16	191
5	42,84,168,322,578,952,1408,1808,1920,1536,768	9586

Clearly, for each n there are C_n chains of length 0 on 1 vertex: $1, 2, 5, 14, 42, \ldots$.

Conjecture 20. *The number of saturated chains of length* 1 *(those consisting of two vertices) = the number of edges in the Hasse diagram is given by*

$$\binom{2n-1}{n-2}.$$

According to the data we've collected for $n = 2$ to 5 the sequence is: $1, 5, 21, 84$. This matches OEIS A002054 so far, and if the conjecture is true there should be bijections between these chains and several other Dyck path/Catalan objects that are listed in the OEIS entry. We note that this would correspond to the number of peaks not along the diagonal in the Dyck path diagram if the diagram is drawn as a mountain range, rather than in an $n \times n$ square. With a saturated chain of length 1, the pair of Dyck paths differ by a single square, hence the peak would not lie on the diagonal.

Let r denote the number of ranks in the poset. At the other end of the size breakdown lists we have the maximal chains. In case you missed it the first couple of times,

Proposition 21. *The number of saturated chains of length $r - 1$ (maximal chains) in the poset can be found using the hook length formula to count the number of SYT in a partition, λ,*

$$\frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}$$

where $h_(i, j)$ is the hook length of a cell (i, j) in the partition.

As we have previously seen, these saturated chains for $n \geq 1$ form the OEIS sequence A005118: $1, 1, 2, 16, 768, \ldots$.

However, we can also establish the following new propositions.

Proposition 22. *The number of saturated chains of length $r-2$ in the poset can be found by doubling the result for the maximal chains $= 2\times$ (number of staircase standard tableaux).*

Proof. (Zabrocki) Every saturated chain of length $r-2 = \binom{n}{2}-1$ either finishes at the Dyck path of area $\binom{n}{2}-1$ and starts at the Dyck path of area 0 or it finishes at the Dyck path of area $\binom{n}{2}$ and starts at a Dyck path of area 1. There are $f_{(n-1,n-2,\ldots,1)}$ chains of the first type. This is because every chain of the first type is in bijection with the number of standard tableaux of skew shape $(n-1, n-2, \ldots, 1)/(1)$ and by filling in the missing cell we obtain a standard tableaux of shape $(n-1, n-2, \ldots, 1)$. There are also $f_{(n-1,n-2,\ldots,1)}$ chains of the second type because every saturated chain in this set corresponds to a standard tableaux of shape $(n-1, n-2, \ldots, 1) - (0 \ldots 010 \ldots 0)$. Each of these standard tableaux can uniquely be completed to (and hence are in bijection with) a standard tableau of shape $(n-2, n-1, \ldots, 1)$ by filling in $\binom{n}{2}$ in the 'missing' cell. □

Proposition 23. *The number of saturated chains of length $r-3$ in the poset can be found by taking 2.5 times the result for the maximal chains $= 5/2\times$ (number of staircase standard tableaux).*

Proof. The proof follows an analogous argument to the previous one. Essentially we break the possibilities into three disjoint cases and set up bijections with the maximal chains. In the first case, the saturated chains start from the Dyck path of area 0 and finish with a Dyck path of area $\binom{n}{2}-2$. This can be done in two possible ways (either the $(2,0)$ partition or the $(1,1)$ partition is 'missing'). Thus, when these possibilities are combined, there is a bijection between this group of saturated chains and the number of standard tableaux. In the second case, the saturated chain begins from a Dyck path of area 1 and finishes with a Dyck path of area $\binom{n}{2}-1$. Summed over the possible paths of area 1, there is again a bijection with the number of standard tableaux. The final case is that the chain starts from a Dyck path of area 2 and finishes with a Dyck path of area $\binom{n}{2}$. The number of these is equal to $1/2$ the number of maximal chains. Imagine filling in the first $k-2$ cells in a tableau of $k = \binom{n}{2}$ cells. Then there are two different ways of filling in the final 2 cells, determined by which you choose to label $k-1$ (the final one is forced to be k). These two options effectively act to double the number of maximal chains, compared to the number of maximal chains obtained by filling in the tableau up to the $k-2$ cell. So stopping at the $k-2$ cell will result in a count of $1/2$ the maximal chains for the entire k cell tableau. Therefore the total number of saturated chains of length $r-3 =$ number of maximal chains$+$number of maximal chains$+$ $1/2 \times$ number of maximal chains $= 5/2 \times$ number of maximal chains. □

3.7. **The connection with symmetric functions.**

The observation above regarding the difficulty in applying the hook length formula to skew shapes led to an investigation of symmetric functions as a possible means for obtaining an explicit formula for the saturated chains. While we still haven't found the desired end result, it did serve as yet another confirmation that the sequence for saturated chains is correct and allowed the sequence to be extended.

Recall that symmetric functions can be written using the Schur function basis, s_λ, for a partition, λ, where

$$s_\lambda = \det|h_{\lambda_i+i-j}|.$$

Equivalently, since the complete homogeneous basis, h_λ, is related to the elementary basis, e_λ, through conjugate partitions, one can write

$$s_\lambda = \det|e_{\lambda'_i+i-j}|.$$

These formulas are called the Jacobi-Trudi formula and dual Jacobi-Trudi formula respectively.

Example 24.

$$s_{(3311)} = \begin{vmatrix} h_3 & h_4 & h_5 & h_6 \\ h_2 & h_3 & h_4 & h_5 \\ 0 & 1 & h_1 & h_2 \\ 0 & 0 & 1 & h_1 \end{vmatrix}$$

Note that the parts of the partition are written on the diagonal, then the subscripts increase when moving to the right and decrease when moving to the left, as they did in the determinantal formula for SYT. When this determinant is evaluated using Maple, we obtain

$$s_{(3311)} = h_3^2h_1^2 - h_3^2h_2 - h_3h_4h_1 + h_3h_5 - h_2h_4h_1^2 + h_4h_2^2 + h_2h_1h_5 - h_2h_6.$$

In accordance with the comment above, we see that s_λ for the conjugate partition evaluated in the elementary basis gives

$$s_{(422)} = e_3^2e_1^2 - e_3^2e_2 - e_3e_4e_1 + e_3e_5 - e_2e_4e_1^2 + e_4e_2^2 + e_2e_1e_5 - e_2e_6.$$

It is useful to consider the product of two Schur functions, $s_\lambda s_\mu$. The Littlewood-Richardson rule determines the expansion if we expand in terms of the Schur basis:

$$s_\lambda^\perp s_\mu = \sum_{\nu\vdash|\mu|-|\lambda|} c^\mu_{\lambda\nu} s_\nu$$

where $c^\mu_{\lambda\nu}$, called the Littlewood-Richardson coefficient, equals the number of a special kind of tableaux, called Littlewood-Richardson tableaux.

We can specialize the Littlewood-Richardson rule for the case where μ is a partition of length 1. In these cases, we can apply a much simpler result, known as the Pieri rule. The Pieri rule states that for any partition, λ, and any integer n,

$$s_\lambda s_n = s_\lambda h_n = \sum_{\mu/\lambda \text{ n-horiz. strip}} s_\mu$$

where the sum is over all partitions μ such that μ/λ is a horizontal strip of size n.

Example 25.

$$s_1 \cdot s_{(3311)} = s_{(33111)} + s_{(3321)} + s_{(4311)}$$

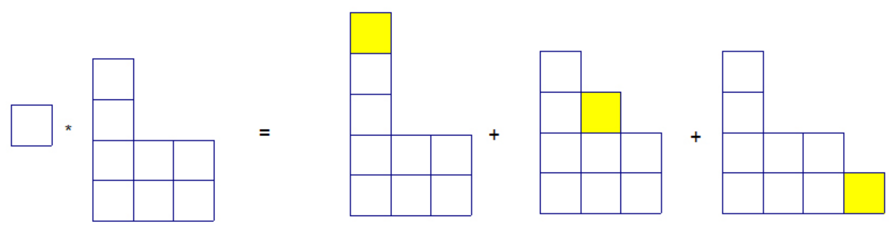

FIGURE 19. Expansion of $s_1 s_{(3311)}$ by the Pieri rule

We note that there is a dual version of the Pieri rule,

$$s_\lambda s_{1^n} = s_\lambda e_n = \sum_{\mu/\lambda \text{ n-vert. strip}} s_\mu$$

where s_{1^n} denotes a single column of size n and thus the sum is over all partitions μ such that μ/λ is a vertical strip of size n.

Next, we should discuss skew Schur functions. We consider the partition μ/λ to be that formed when the cells of partition λ are removed from partition μ.

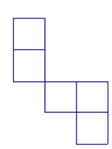

FIGURE 20. The skew shape 3311/21

Analogous to the determinantal formula given above for Schur functions, we have the following for the skew Schur functions:

$$s_{\mu/\lambda} = |h_{\lambda_i - \mu_j + i - j}|.$$

Now, define the skew Schur functions, $skew(s_n, s_\lambda)$, to be the sum over all μ obtained from λ by removing horizontal strips. So, for example, with Stembridge's Symmetric Functions package, the Maple command $skew(s[2], s[3, 2, 1])$ removes horizontal strips totalling 2 cells from the 321 partition. Expressing the answer as a sum in the Schur basis with this example we see that,

$$tos(skew(s[2], s[3, 2, 1])) = s_{(3,1)} + s_{(2,2)} + s_{(2,1,1)}.$$

The coefficients in this expansion are also determined by the Littlewood-Richardson rule mentioned above.

Importantly, the expression $skew(s_\lambda, s_\mu)$ also encodes all of the chains in Young's lattice between the partitions λ and μ. Recall that the poset of Dyck paths ordered by inclusion forms a subset of Young's lattice and that the partitions corresponding to elements in a saturated chain differ by one cell moving up or down the chain. This means that if we sum the s_λ over all partitions λ that correspond to Dyck paths (call this sum AA) and then apply $skew(AA, AA)$, we can encode all of the saturated chains in the Dyck path poset.

In order to count the number of saturated chains, we need a few more formulas,

$$h1^n = \sum_{\lambda \vdash n} f_\lambda s_\lambda$$

where f_λ is the number of standard tableaux of shape λ or the number of maximal chains in Young's lattice from $\emptyset$ to λ, and

$$scalar(s_\lambda, h1^n) = \sum_\mu scalar(s_\lambda, f_\mu s_\mu) = \sum_\mu f_\mu scalar(s_\lambda, s_\mu) = f_\lambda.$$

If n is the degree of the symmetric function f, then the Maple command $scalar(skew(s_\lambda, s_\mu), h1^n)$ will count the number of saturated chains between any two partitions, λ and μ. Therefore we can count the number of standard tableaux for a skew partition μ/λ by considering it to be the difference between two partitions. Consider the example $skew(s[2], s[3, 2, 1])$ from above:

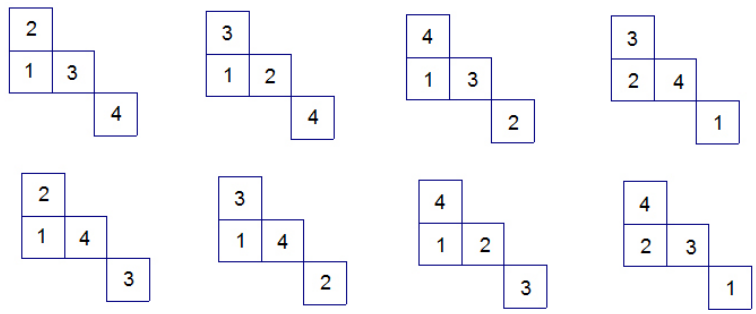

FIGURE 21. The 8 standard tableaux for skew(s[2],s[3,2,1])

Using Maple, we retrieve this answer 8 using the command, $scalar((skew(s[2], s[3, 2, 1]), h1^4)$ since there are 4 cells in the difference partition and in each of the s_λ in the Schur expansion, $skew(s[2], s[3, 2, 1]) = s_{(3,1)} + s_{(2,2)} + s_{(2,1,1)}$.

Putting it all together, we determine the number of saturated chains in the Dyck path poset. For example, here are the Maple commands and output for D_3.

```
> AA := add(s[op(la)], la = subPar([2, 1])); tos(skew(AA, AA));
```

$$AA := s_{2,1} + s_2 + s_{1,1} + s_1 + s_{[\,]}$$

$$s_{2,1} + 2\,s_2 + 2\,s_{1,1} + 5\,s_1 + 5\,s_{[\,]}$$

```
> scalar(%, add(h1^r, r = 0..4));
```

$$16$$

FIGURE 22. Maple results for D_3

Moreover, these Maple commands easily allow us to extend the sequence for $n > 5$:

n	Number of Saturated Chains
6	3621062
7	13539455808
8	596242050871827

That's as far as Maple can go with the memory on my laptop.

The results up to D_5 are summarized in the following chart:

n	Schur Basis Expansion	Number of Saturated Chains
1	$s_{([\,])}$	1
2	$s_{(1)} + 2s_{([\,])}$	3
3	$s_{(2,1)} + 2s_{(2)} + 2s_{(1,1)} + 5s_{(1)} + 5s_{([\,])}$	16
4	$s_{(3,2,1)} + 2s_{(3,2)} + 2s_{(3,1,1)} + 5s_{(3,1)} + 5s_{(3)} + 2s_{(2,2,1)} + 5s_{(2,2)}$	191
	$+5s_{(2,1,1)} + 14s_{(2,1)} + 15s_{(2)} + 5s_{(1,1,1)} + 15s_{(1,1)} + 21s_{(1)} + 14s_{[\,]}$	
5	$84s_{(1)} + 5s_{(3,3,2)} + 5s_{(4,2,1,1)} + 5s(4,3,1) + 84s_{(2)} + 42s_{[\,]} + 84s_{(1,1)} + 49s_{(2,2,1)} + 49s_{(3,1,1)}$	9586
	$+49s_{(3,2)} + 49s_{(3)} + 43s_{(3,2,1)} + 49s_{(1,1,1)} + s_{(4,3,2,1)} + 2s_{(4,3,2)} + 5s(4,3) + 2s(4,2,2,1)$	
	$+14s_{(4,2,1)} + 5s([4,1,1,1) + 21s(4,1) + 14s(4) + 14s_{(3,3,1)} + 14s_{(3,2,2)} + 5s_{(2,2,2,1)} + 21s_{(2,1,1,1)}$	
	$+14s_{(1,1,1,1)} + 5s_{(3,2,2,1)} + 2s_{([3,3,2,1)} + 5s(3,3,1,1) + 15s(2,2,1,1) + 15s_{(3,3)} + 15s_{(2,2,2)} + 15s_{(3,1,1,1)}$	
	$+15s_{(4,1,1)} + 15s_{(4,2)} + 59s_{(2,2)} + 72s_{(3,1)} + 112s_{(2,1)} + 72s(2,1,1) + 5s_{(4,2,2)} + 2s_{(4,3,1,1)} + 14s_{(3,2,1,1)}$	

Finally, we note that one can also compute the number of saturated chains for D_n given the Schur basis expansion and multiplying each of the coefficients by the number of standard tableaux for the associated partition. For example, given the D_3 expansion, $s_{(2,1)} + 2s_{(2)} + 2s_{(1,1)} + 5s_{(1)} + 5s_{([\,])}$, we take:

$$(1 \cdot \text{SYT for } (2,1)\ ) + (2 \cdot \text{SYT for } (2)\ ) + (2 \cdot\ \text{SYT for } (1,1)\ ) + (5 \cdot\ \text{SYT for } (1)\ ) + (5 \cdot\ \text{SYT for } [\,]\ )$$
$$= 1(2) + 2(1) + 2(1) + 5(1) + 5(1)$$
$$= 16$$

Since we have the hook length formula to count the number of SYT for a given partition, *all* we need is a nice, explicit formula for obtaining the coefficients, then we could combine them together and sum over all subpartitions for each staircase to get a formula that directly counts saturated chains. Any ideas?

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