Student: Nikolai Cook
Supervisor: Mike Zabrocki

1. Introduction and Terminology

We need to begin with terminology, so that we can present concepts without ambiguity.

Dimension. In physics and mathematics, the dimension of a space or object is informally defined as the minimum number of coordinates needed to specify each point within it. Thus a line has a dimension of one because only one coordinate is needed to specify a point on it. A surface such as a plane or the surface of a cylinder or sphere has a dimension of two because two coordinates are needed to specify a point on it.[1] More formally, As we know, if Dim R = (n) then there are n vectors in the (orthogonal or not) basis of the space.

Codimension. Codimension is a *relative* concept: it is only defined for one object *inside* another. There is no "codimension of a vector space (in isolation)", only the codimension of a vector *sub*space.[2] As an example, if we were to look at a line which has a dimension of 1, within a cube, its codimension will be 3 - 1 = 2. Whereas a line within a square (of dimension 2) will have codimension 2 - 1 = 1.

Hyperplanes. A generalization of planes into a different number of dimensions than the intuitive 3 space. A hyperplane is determined by a point on the plane and a vector perpendicular to it (its normal). A hyperplane of an *n*-dimensional space is a subset with dimension *n* – 1. By its nature, it separates the space into two half spaces[3]. Hyperplanes are characterized also by the property of having a codimension of 1. They are either parallel, intersect in a subspace of codimension 2. We can describe these ideas more clearly with a diagram from Borovik 2010:

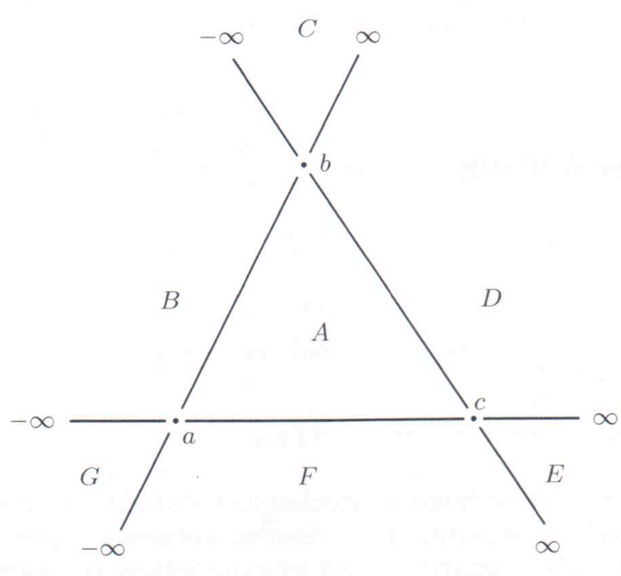


Fig 1: Hyperplanes in 2 Space. In our familiar 2-space, we have three infinite hyperplanes 'cutting' the space into seven chambers, labeled here from A through G. Note that there are 3 chambers with 2 faces, and 4 chambers with three faces. There is only one finite chamber, A.

[1] Wikipedia "Dimension" http://en.wikipedia.org/wiki/Dimension

[2] Wikipedia "Co Dimension" http://en.wikipedia.org/wiki/Codimension

[3] Wikipedia "Hyperplane" http://en.wikipedia.org/wiki/Hyperplane

2. Reflections and Mirrors

Reflections. A **reflection** (also spelled reflexion) is a mapping from a Euclidean space to itself that is an isometry (preserves distance) with a hyperplane as set of fixed points; this set is called the axis (in dimension 2) or plane (in dimension 3) of reflection. The image of a figure by a reflection is its mirror image in the axis or plane of reflection. For example the mirror image of the small Latin letter **p** (if drawn on a blackboard) for a reflection with respect to a vertical axis would look like **q**. Its image by reflection in a horizontal axis would look like **b**. A reflection is an involution: when applied twice in succession, every point returns to its original location, and every geometrical object is restored to its original state.[4] The following diagram generated in Google Sketchup™ (.sku) should help to make this a little clearer.

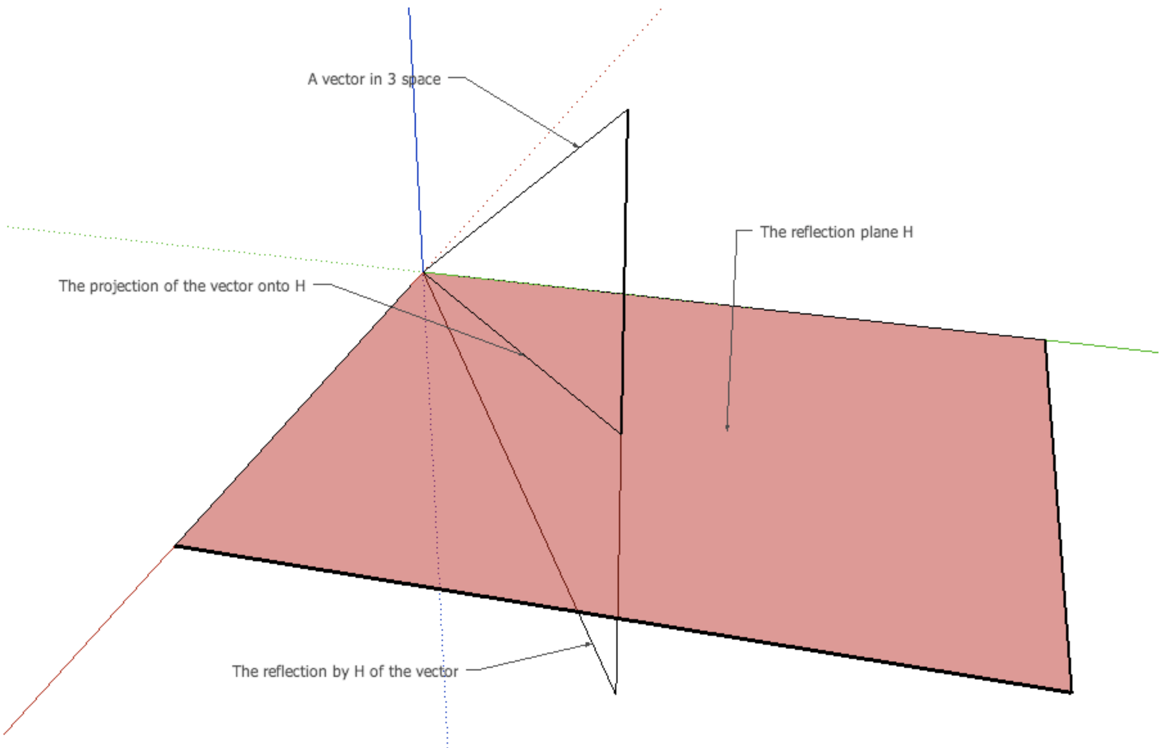


Fig 2: Reflection through a (hyper)plane. In 3 space, reflection through a hyperplane is an isometry (preserves distance) that is, the reflection of a vector has the same magnitude as the original vector.

Reflection through a hyperplane in n dimensions has the following (vector) formula. If the above reflection is considered then if α is orthogonal to the hyperplane, and the vector to be reflected is v, then

(Read reflection of v in α is...)

[4] Wikipedia "Reflection - Mathematics"

$$\mathrm{Ref}_a(v) = v - 2\frac{v \cdot a}{a \cdot a}a$$ [5]

If we now consider hyperplanes as mirrors, in the intuitive sense, we can create systems of them (like in a kaleidoscope). For our purposes, we want closed systems of mirrors that will reflect a solid (such as a square in 2-space) back upon itself. Please see the diagram, again from Borovik.[6]

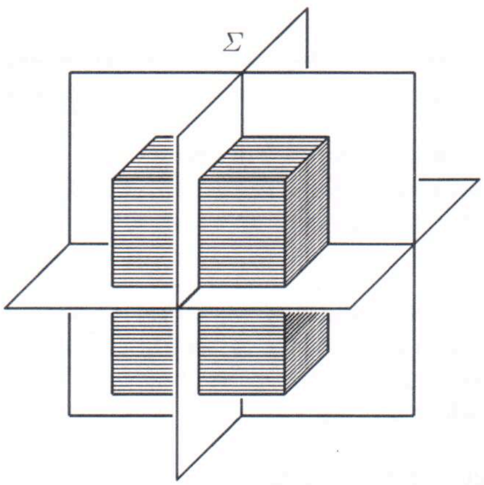


Fig 3: A closed system of mirrors Σ is a finite set of hyperplanes that will reflect a shape unto itself. Note that all hyperplanes will intersect at a single point if the shape is bounded and convex. (i.e. e.g. not a donut). Note that the system of mirrors Σ pictured is not complete, more mirrors could be added, for this polytope.

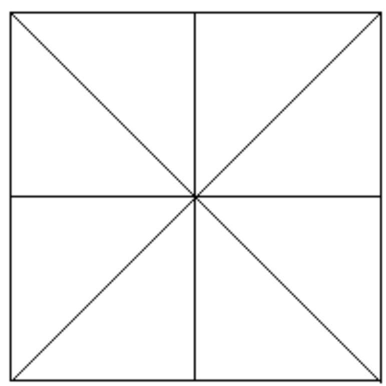


Fig 3.1: A closed system of 4 mirrors for the square. Notice the difference between 2 dimensional and 3 dimensional, and similarities.

We now have enough terminology to move to root systems, where the study of reflections and mirrors began historically.

[5] Wikipedia "Reflections - Mathematics"

[6] Borovik, Borovik. Mirrors and Reflections. 2010. Springer.

3. Root Systems, Especially A2

Root System. In mathematics, a root system is a configuration of vectors in a Euclidean space satisfying certain geometrical properties.[7] As defined on wikipedia we have the following technical but clear definition:

Let V be a finite-dimensional Euclidean vector space, with the standard Euclidean inner product denoted by $(\cdot,\cdot)$. A root system in V is a finite set Φ of non-zero vectors (called roots) that satisfy the following properties:

1. The roots span V.
2. The only scalar multiples of a root $\alpha \in \Phi$ that belong to Φ are α itself and –α.
3. For every root $\alpha \in \Phi$, the set Φ is closed under reflection through the hyperplane perpendicular to α. That is, for any two roots α and β, the set Φ contains the reflection of *β*,
$$\sigma_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \Phi.$$
4. (*Integrality condition*) If *α* and *β* are roots in Φ, then the projection of *β* onto the line through *α* is a half-integral multiple of *α*. That is,
$$\langle \beta, \alpha \rangle = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}.$$

In less technical terms, we have a root system if we have n non-parallel vectors, carefully chosen about the origin, such that closure, scalars and reflections are preserved. In a root system, all angles between two vectors are equal. For example, we can create the root system A2.

[7] Wikipedia “Root System - Mathematics Disambiguation”

Take two vectors, α1 and α2 in 2 space, with a corresponding angle of 2π/3, as pictured below.

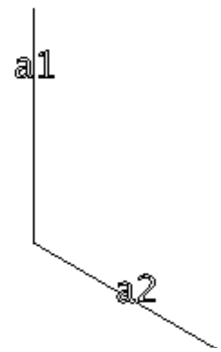


Fig 4.1: Stage 1, α1 and α2 in 2 space, the angle between is 2π/3 and WLOG we consider only vectors of unit length.

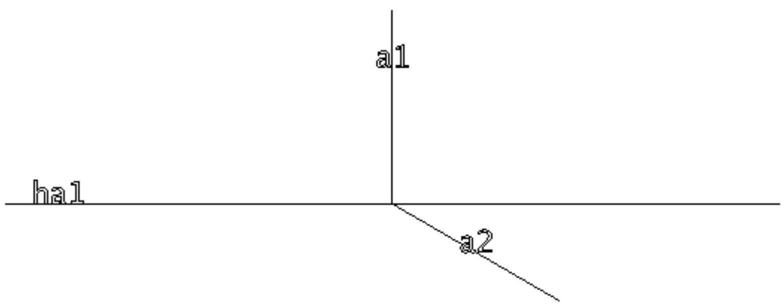


Fig 4.2 : Add in the hyperplane orthogonal to a1.

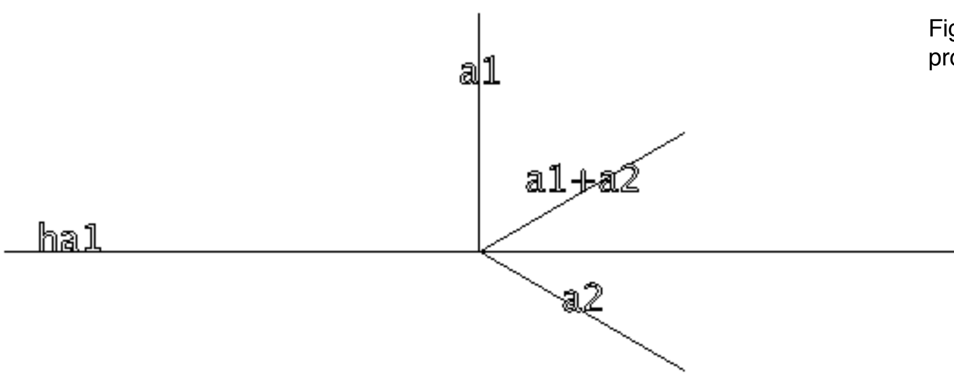


Fig 4.3 : Reflection of a2 through ha1, through the projection equation we find is equal to a1+a2.

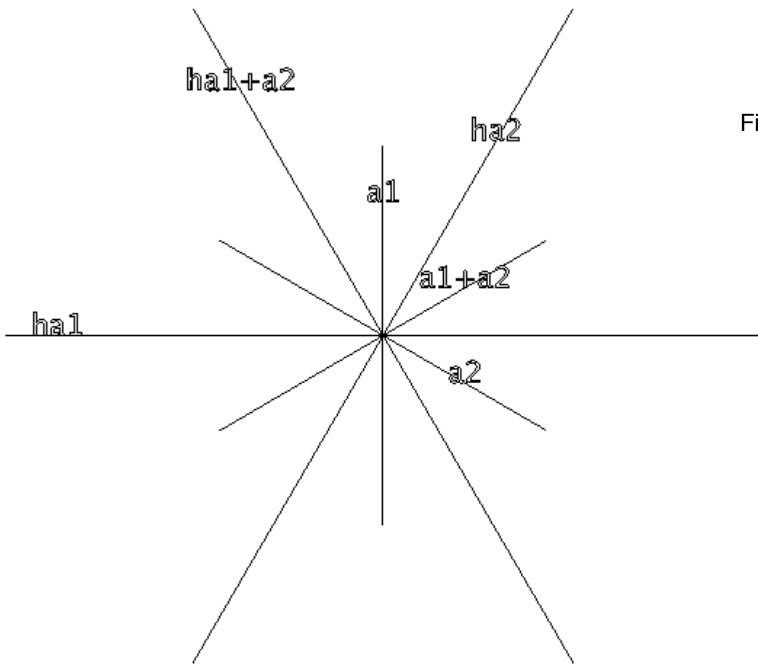


Fig 4.4: Filling in the rest of the reflections.

And from here we add in the rest; reflections and mirrors until we close the system.
Note the consistent angle between all 6 vectors in Φ, and all hyperplanes of reflection.

Looking beyond the geometry of A2, we can (and should have for more rigor) derive A2 from algebraic structures.

From wikipedia, we have the following realization for the construction of An:

Let *V* be the subspace of $\mathbf{R}^{n+1}$ for which the coordinates sum to 0, and let Φ be the set of vectors in *V* of length √2 and which are *integer vectors,* i.e. have integer coordinates in $\mathbf{R}^{n+1}$. Such a vector must have all but two coordinates equal to 0, one coordinate equal to 1, and one equal to –1, so there are $n^2 + n$ roots in all. One choice of simple roots expressed in the standard basis is: $\boldsymbol{\alpha}_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, for $1 \le i \le n$.

The reflection σ_i through the hyperplane perpendicular to $\boldsymbol{\alpha}_i$ is the same as permutation of the adjacent ***i***-th and (***i* + 1**)-th coordinates. Such transpositions generate the full permutation group. For adjacent simple roots, $\sigma_i(\boldsymbol{\alpha}_{i+1}) = \boldsymbol{\alpha}_{i+1} + \boldsymbol{\alpha}_i = \sigma_{i+1}(\boldsymbol{\alpha}_i) = \boldsymbol{\alpha}_i + \boldsymbol{\alpha}_{i+1}$, that is, reflection is equivalent to adding a multiple of 1; but reflection of a simple root perpendicular to a nonadjacent simple root leaves it unchanged, differing by a multiple of 0.[8]

So, begin with the standard orthogonal basis of R3,

$\varepsilon 1 = (1,0,0)$
$\varepsilon 2 = (0,1,0)$
$\varepsilon 3 = (0,0,1)$

and from here generate the two vectors, α1 and α2, that will characterize A2,

$\alpha 1 = \varepsilon 1 - \varepsilon 2 = (1,-1,0)$
$\alpha 2 = \varepsilon 2 - \varepsilon 3 = (0,1,-1)$

When these vectors are plotted in the three space that they should be, along with their reflections in orthogonal hyperplanes, we have the following “tilted” hexagon.

[8] Wikipedia “Root systems, Mathematics, Explicit Construction of the Irreducible Root Systems”

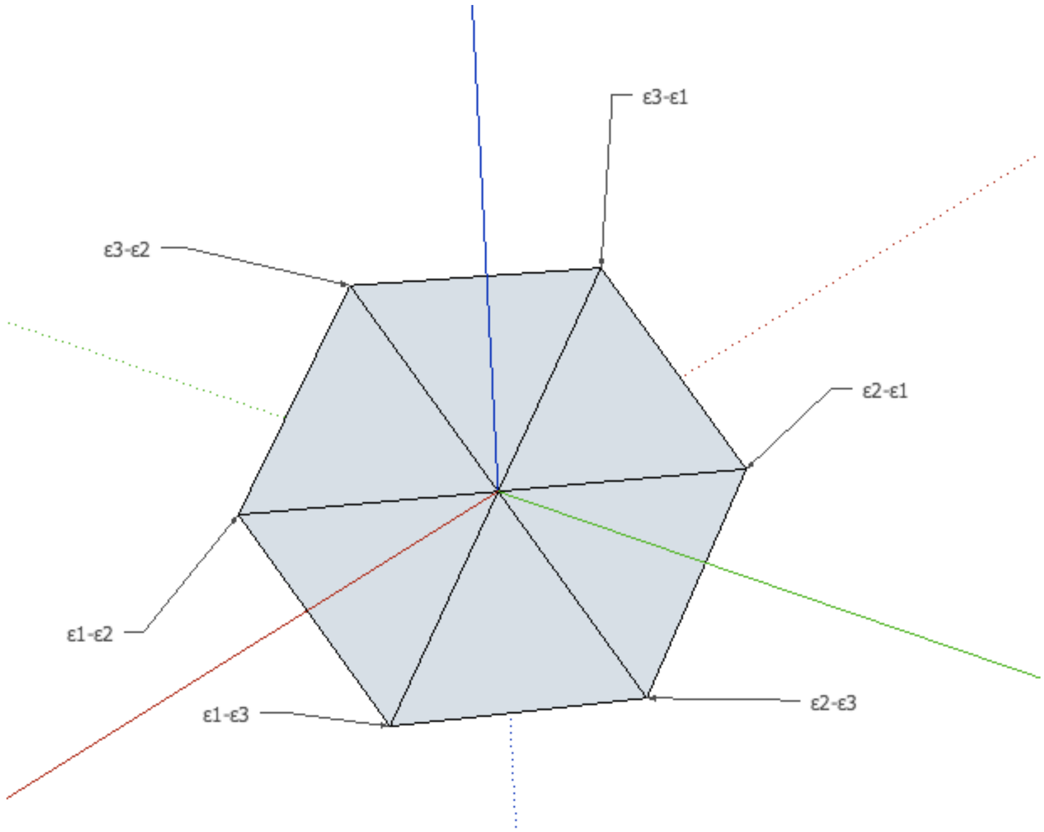


Fig 5: Tilted Hexagon. Note that each of the vectors is one of 6 ways to choose two objects from a set of three with ordering. Each point (vector terminal) corresponds to a linear combination (or subtraction) of two orthogonal basis vectors.

These vectors will play an important role when we discuss the combinatoric properties of the root system in the next section.

Note that the hexagon in 3 space as depicted above and below... is really a permutahedron of order 3. See below, from wikipedia.

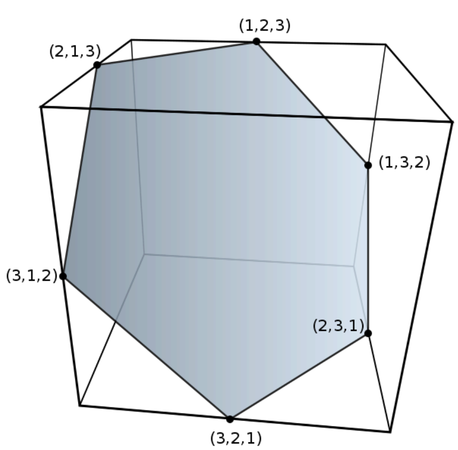


Fig 5.1: The permutohedron of order 3 is a hexagon, filling the cross section of a 2×2×2 cube.

4. Connection to Combinatorics

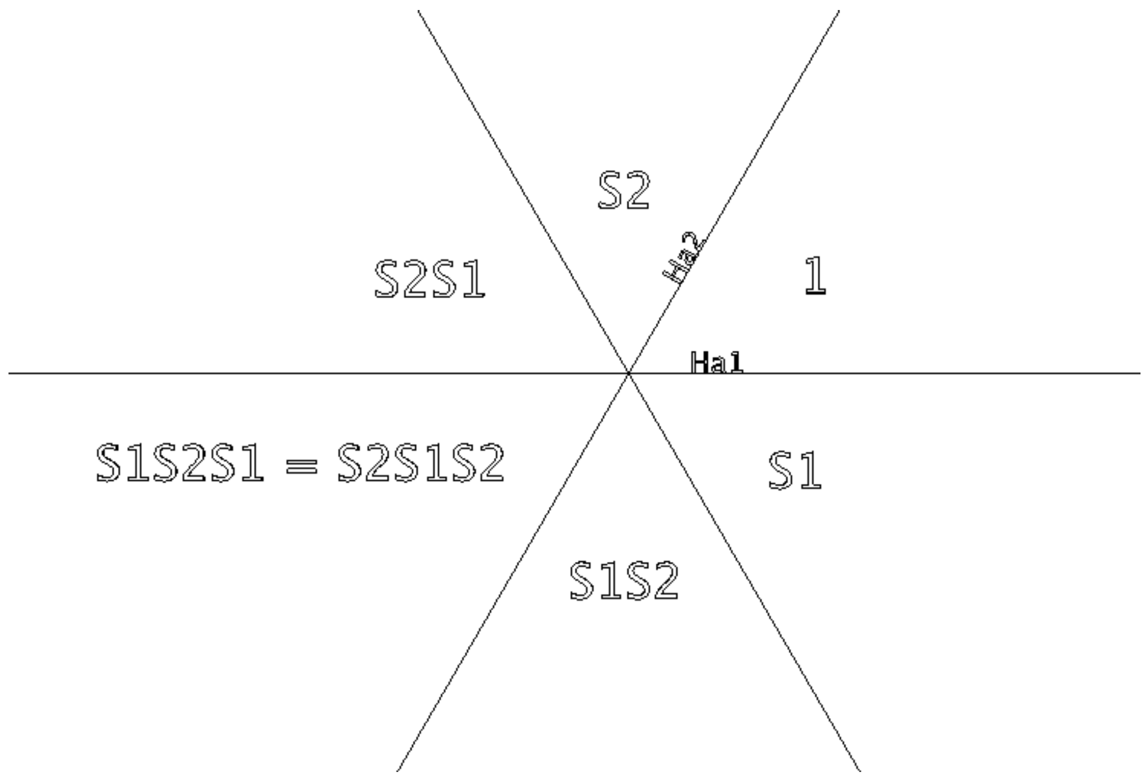


Fig 6 illustrates the three infinite hyperplanes that correspond to the closed system of mirrors (a.k.a. root system) that is generated by two vectors 2π/3 apart in 2-space. The space is partitioned into six chambers, and we can name one of them, WLOG, the identity (or fundamental) chamber. From here we find some very interesting properties. Consider the reflections themselves, through the hyperplanes, and consider them as Si, where i: 1 or 2. Note that, from above, reflection are involutions, that is,
(s1)(s1) = 1 = (s2)(s2). Reflection in a plane twice results in the original.
(s1)(s2)(s1) = (s2)(s1)(s2) and
(s1)(s2) =/= (s2)(s1).

Now consider any combination of reflections and any resulting element will be in one of the six chambers. Further, note the following property of the movements themselves.

sαβ = β - (2β · α)(α · α)^-1 · α (reflection formula)

Sα1(a,b,c) = (a,b,c) - 2((a,b,c) · α)(α · α)^-1 · α

Sα1(a,b,c) = (a,b,c) - 2((a,b,c) · (1,-1,0))((1,-1,0) · (1,-1,0))^-1 · (1,-1,0)

Sα1(a,b,c) = (a,b,c) - 2(a-b)(1,-1,0)

$S\alpha 1(a,b,c) = (a,b,c) + (b-a,a-b,0)$
$S\alpha 1(a,b,c) = (b,a,c)$

That is, reflection in $H\alpha 1$ is a permutation of the coordinates. If we continue analyzing the rest we get the following;

$S\alpha 1(a,b,c) = (b,a,c)$
$S\alpha 2(a,b,c) = (a,c,b)$
$S\alpha 1 S\alpha 2(a,b,c) = (c,a,b)$
$S\alpha 2 S\alpha 1(a,b,c) = (b,c,a)$
$S\alpha 1 S\alpha 2 S\alpha 1(a,b,c) = (c,b,a)$
$S\alpha 1 S\alpha 1(a,b,c) = (a,b,c)$

Which is simply the group of permutations of three elements.

When we extend ourselves up one dimension, to 4, and begin to discuss A3, we would move from the 6 permutations of 3 elements, to the 24 permutations of 4 elements.

5. How to Generate A3

Remember, to generate A3 we are working in 4-space. Instead of using the geometrical approach we had before, we use the mechanical algebra to help us, guided by geometry's intuition. We simply follow the steps as before:

If we take the orthogonal basis of R4, and label them

Basis	x	y	z	w
ε1	1	0	0	0
ε2	0	1	0	0
ε3	0	0	1	0
ε4	0	0	0	1

Then to generate A3 we take only the linear monic combinations of the εi. Labeling them as

Basis	Labels	x1	x2	x3	x4
e1-e2	a	1	-1	0	0
e1-e3	b	1	0	-1	0
e1-e4	c	1	0	0	-1
e2-e3	d	0	1	-1	0
e2-e4	e	0	1	0	-1
e3-e4	f	0	0	1	-1

These 6 'vectors' will be the positive roots and those along with the negatives of these is A3. From here we can plot the vectors in 3 space, but we need to find the angles in between them to plot them (this is how to move from 4 space to 3 space). We will make use of the dot product, where

$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$[9] will give us our θ.

[9] Wikipedia "Dot Product"

	x · y	x · y / 2	α (radians)	α (degrees)
ab	1	0.5	pi/3	60
ac	1	0.5	pi/3	60
ad	-1	-0.5	2pi/3	120
ae	-1	-0.5	2pi/3	120
af	0	0	pi/2	90
bc	1	0.5	pi/3	60
bd	1	0.5	pi/3	60
be	0	0	pi/2	90
bf	-1	-0.5	2pi/3	120
cd	0	0	pi/2	90
ce	1	0.5	pi/3	60
cf	1	0.5	pi/3	60
de	1	0.5	pi/3	60
df	-1	-0.5	2pi/3	120
ef	1	0.5	pi/3	60

These are the relations between the 6 vectors in A3, and now we can construct A3 in 3 dimensions instead of 4. We may choose our start point arbitrarily,
a = (0,0,1). We then know that -a = (0,0,-1).

From here we know that a · f = 0. Then, again WLOG we can choose
f = (1,0,0) and -f = (-1,0,0).

Now we know that a · c = f · c = 1. The vector that satisfies this condition (as well as monic length) is
c = (1/2, √2/2, 1/2). Then -c = (-1/2, -√2/2, -1/2).
I also know that a•b = 1/2 and f•b = -1/2. And thus have
b= (-1/2, √2/2, 1/2) and -b = (1/2, -√2/2, -1/2).
Further, a•d = f•d = -1. Then we have
d = (-1/2, √2/2, -1/2) and -d = (1/2, -√2/2, 1/2)
Lastly we have e, where d•e = c•e = 1/2. Giving us a final vector of
e = (1/2, √2/2, -1/2) and -e = (-1/2, -√2/2, 1/2).
Here's the table.

	x1	x2	x3
a	0	0	1
b	-1/2	√2/2	1/2
c	1/2	√2/2	1/2
d	-1/2	√2/2	-1/2
e	1/2	√2/2	-1/2
f	1	0	0

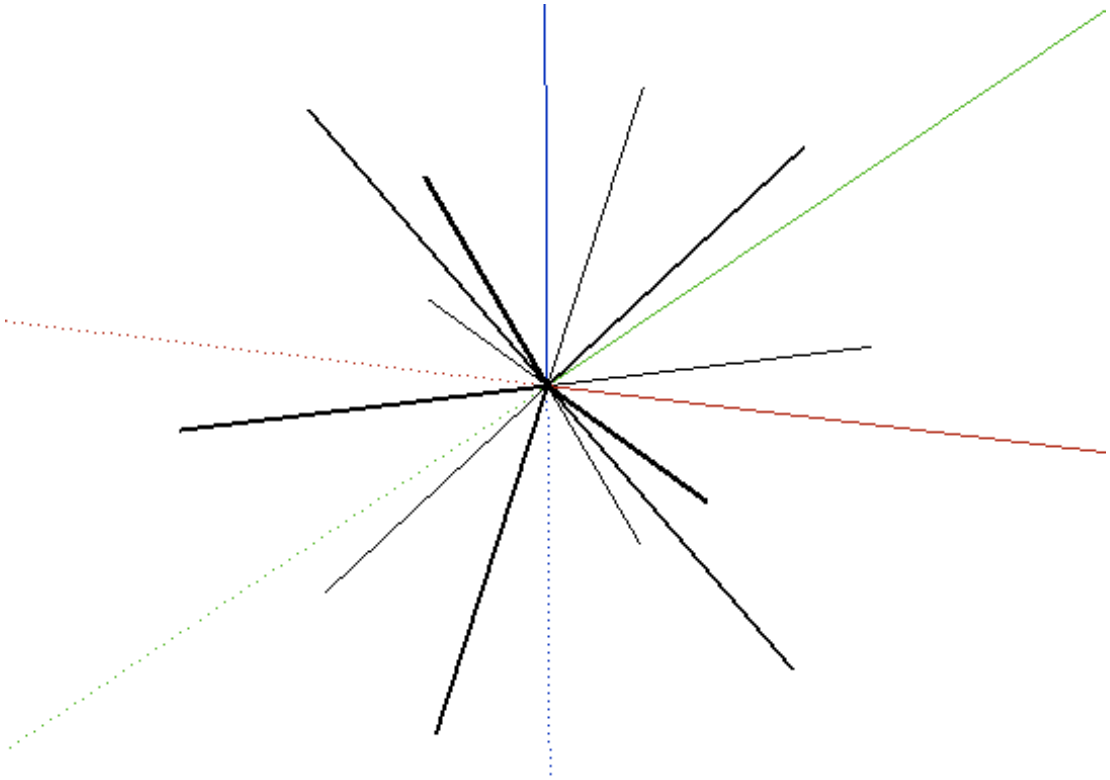


Fig 7.1: The Root System A3, as constructed by the angles derived earlier.

6. The A3 Hyperplane Arrangement

From here, we cut the 3-space into halves using the 6 planes that are uniquely determined by these vectors. Here is the table

Pa	$z=0$
Pb	$-\frac{1}{2}x+\frac{(2^{0.5})}{2}y+\frac{1}{2}z=0$
Pc	$\frac{1}{2}x+\frac{(2^{0.5})}{2}y+\frac{1}{2}z=0$
Pd	$\frac{1}{2}x-\frac{(2^{0.5})}{2}y+\frac{1}{2}z=0$
Pe	$\frac{1}{2}x+\frac{(2^{0.5})}{2}y-\frac{1}{2}z=0$
Pf	$x=0$

And when the planes are mapped out, the resulting R3 space looks like this (done in Grapher, available on macintosh):

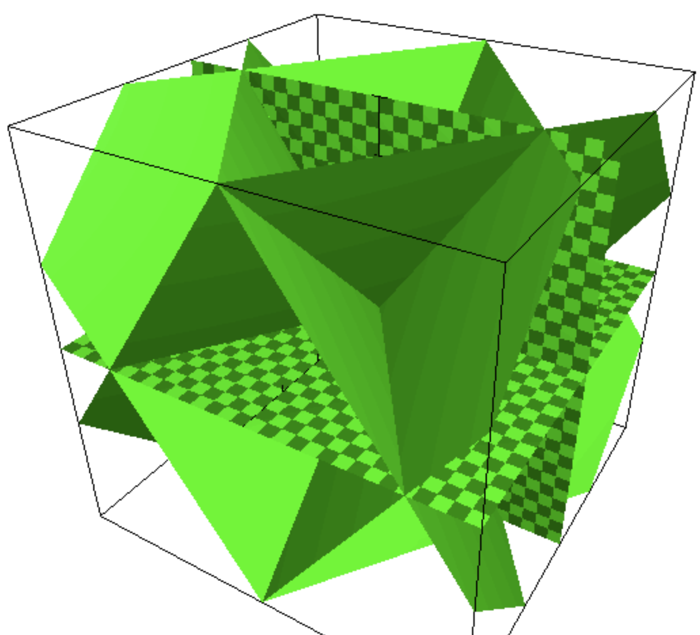


Fig 7.2: Coxeter Complex for A3. The 6 planes partition 3 space into halves. Notice that the space is partitioned into 24 chambers!

The A3 complex is analogous to Fig 6, where A3 partitions 3-space into 24 chambers, A2 partitioned 2-space into 6 chambers. Note the chambers are infinite. Note the chambers, as a product of the mirrors, all meet at the origin. When we restrict ourselves solely to a single chamber, we get what we will call an alcove. If we were to model this into Sketchup, we would find a shape that looks like this:

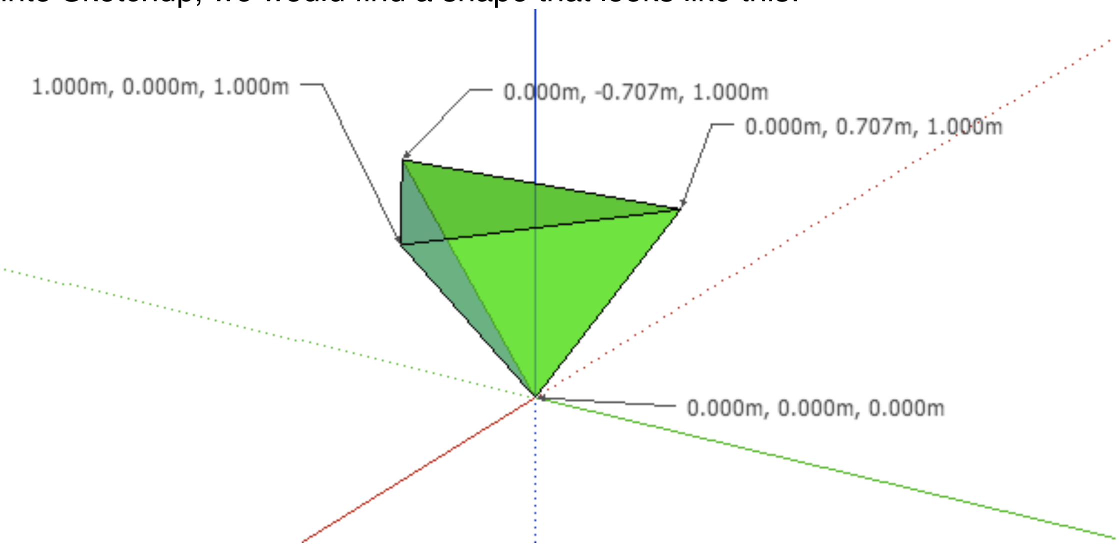


Fig 8: An A3 alcove, one of 24, the coordinates are displayed to help other modelers, and provide intuition about where in space these vertices lie.

One can see where this shape will ‘fit’ into the hollow of the planes. But why are we allowed to bound the alcove from an infinite chamber into a finite alcove? We discuss this in the next section, tessellation.

7. Tessellation

Starting with definitions, A tessellation or tiling of the plane is a pattern of plane figures that fills the plane with no overlaps and no gaps. [10] It may perhaps be best to illustrate this with illustrations. We will begin as before with A2 and graduate to A3. We see that A2 partitions the infinite euclidean space into 6 chambers, using three planes of reflection.

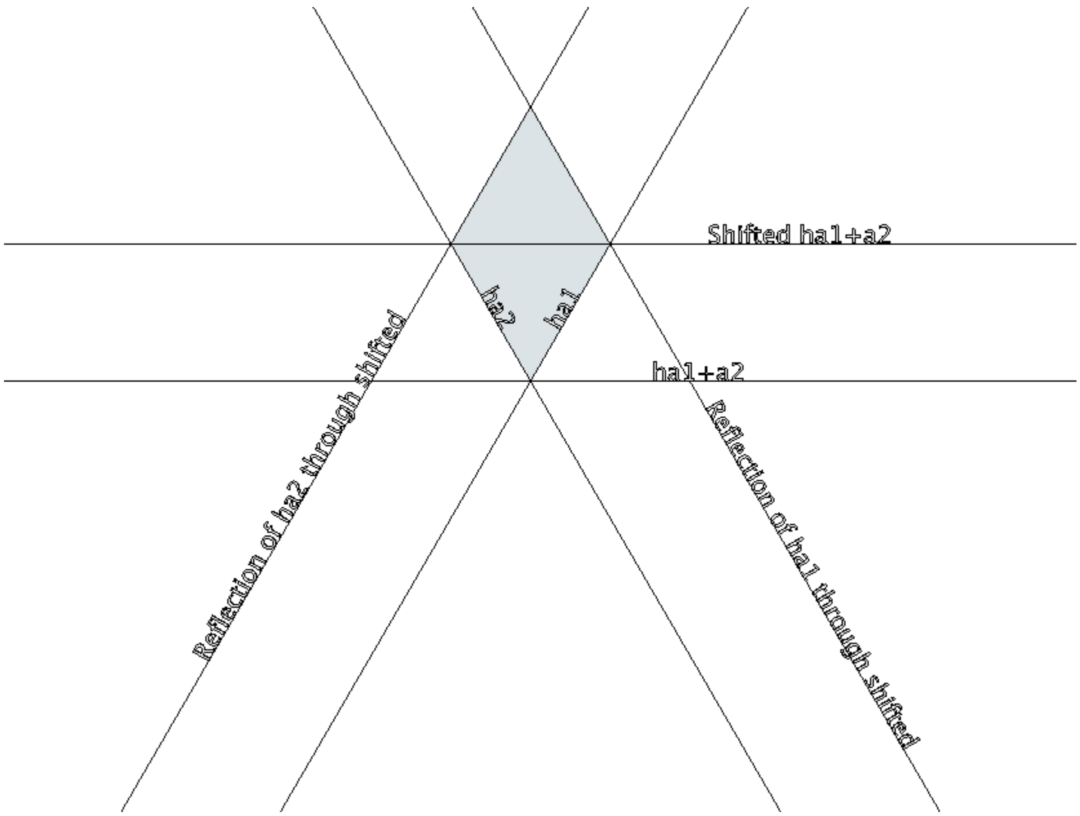


Fig 9: A2 partitions the Euclidean space into 6 chambers, using three hyperplanes, labelled here as s0, s1 and s2.

Now, imagine if we were to 'copy' one of the hyperplanes, and 'paste' it one unit north. But when you do that, you inevitably copy s1 and s2 northwards as well. If we restrict ourselves to viewing only one chamber we see the following result of our copy and paste:

[10] Wikipedia "Tessellation" Non-Disambiguation Page

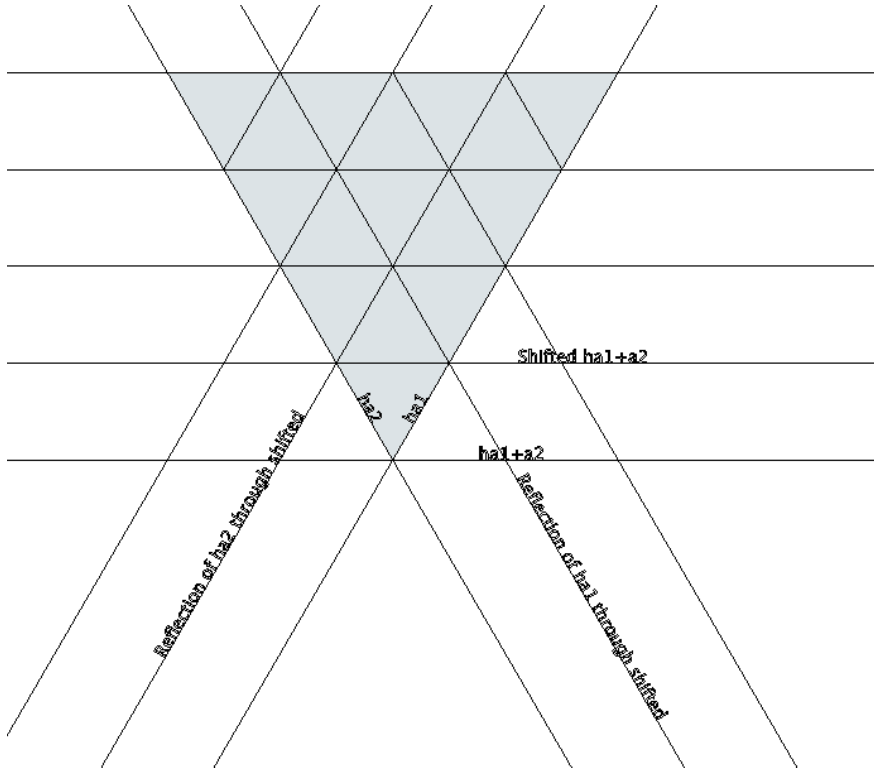


Fig 10: Tessellation of R2 using A2. Also known as tessellation by equilateral triangles (note the pi/3 angles from A2). This is how we can restrict ourselves to an alcove instead of a chamber, as mentioned earlier.

From here, we are prepared to move to A3... using the same method as before, we will arrive at a final tessellation of 3-space that looks like this:

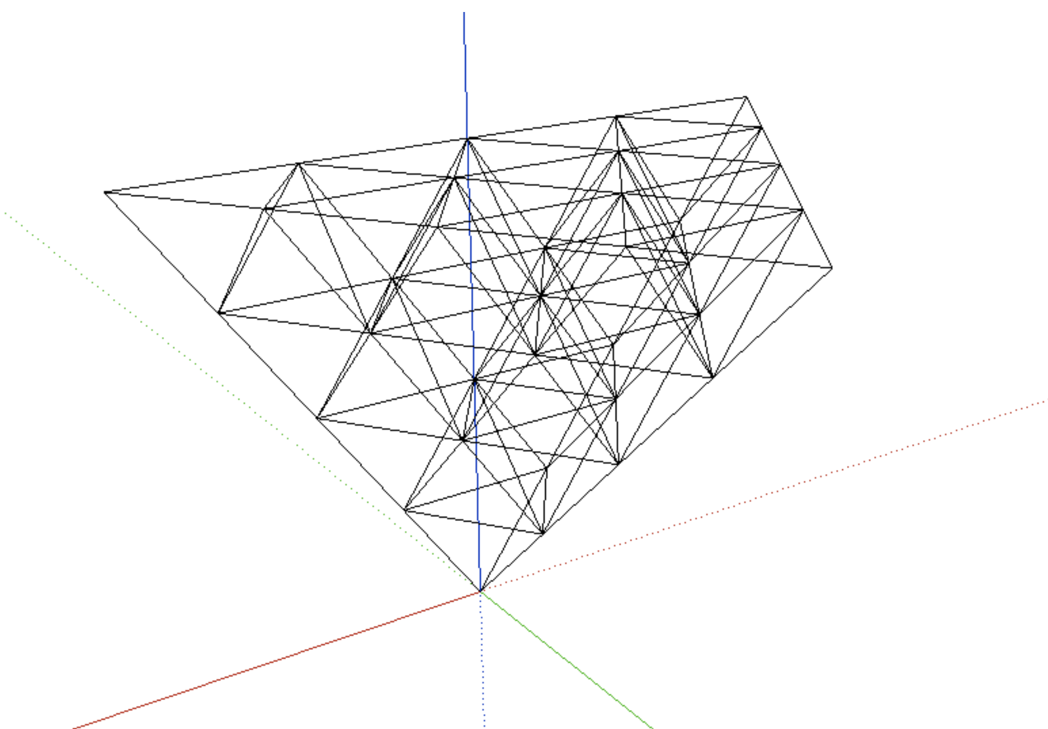


Fig 11: Tessellation of a chamber of A3 by A3 alcove. Four layers are presented here. The alcoves (and their reverses) are dense in space.

An interesting property to note about the A3 tessellation is that when ‘modded’ we can retrieve the tessellation of R2 by A2. Geometrically, we need only view the tessellation from the right angle...

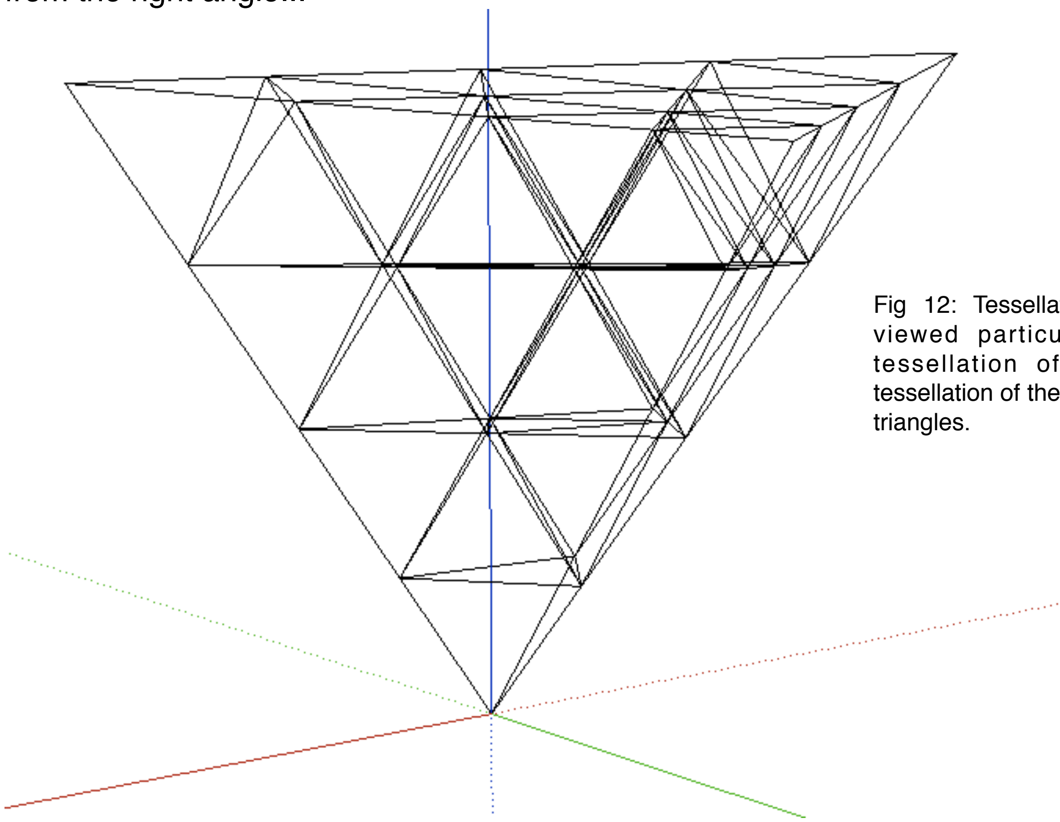


Fig 12: Tessellation of R3 by A3, viewed particularly, shows the tessellation of R2 by A2, i.e. tessellation of the plane by equilateral triangles.

This is not always the case however, as viewed from the final appropriate angle, we get the following image...

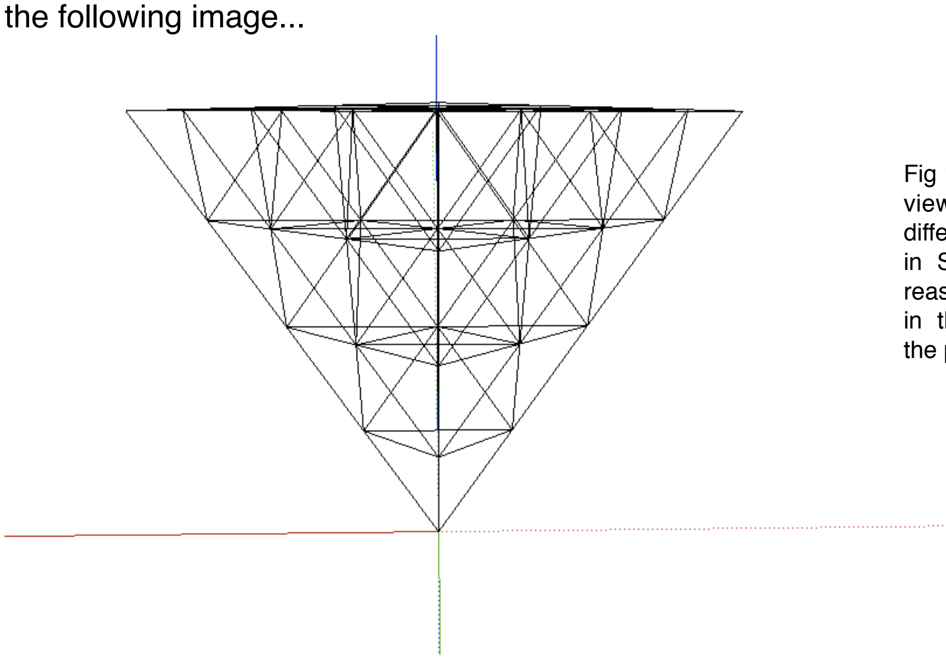


Fig 13: Tessellation of R3 by A3, viewed particularly, shows a different result once modded not in S1, or S3, but in S2. The reasoning will become apparent in the combinatoric section of the permutahedron.

8. Introducing the Permutahedron

Let's take a step back and deal with some shapes for awhile. From wikipedia we have that

"In mathematics, the permutohedron of order n (also spelled permutahedron) is an (n – 1)-dimensional polytope embedded in an n-dimensional space, the vertices of which are formed by permuting the coordinates of the vector (1, 2, 3, ..., n)."[11]

The permutahedron has 24 vertices and 12 faces. Interesting for the purposes of graphing the permutahedron is that it is uniquely determined by the 6 square faces. That is every vertex is a part of exactly one square. Using simple algebra and geometry then, we may construct a permutahedron defined by its squares.

(The points are scaled to result in a permutahedron with all lengths of one.)

The resulting shape will look like this:

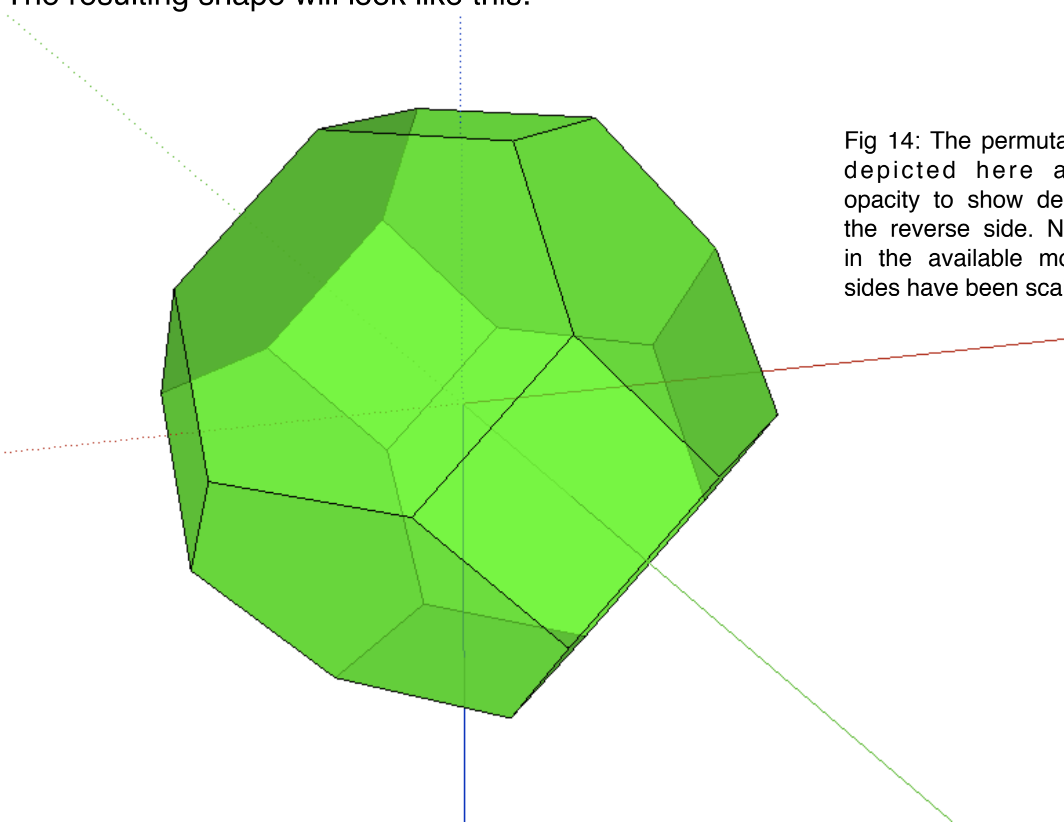


Fig 14: The permutahedron, depicted here at 50% opacity to show depth and the reverse side. Note that in the available model, all sides have been scaled to 1.

Furthermore, when the tessellation from above is combined with the permutahedron...

[11] Wikipedia "Permutahedron"

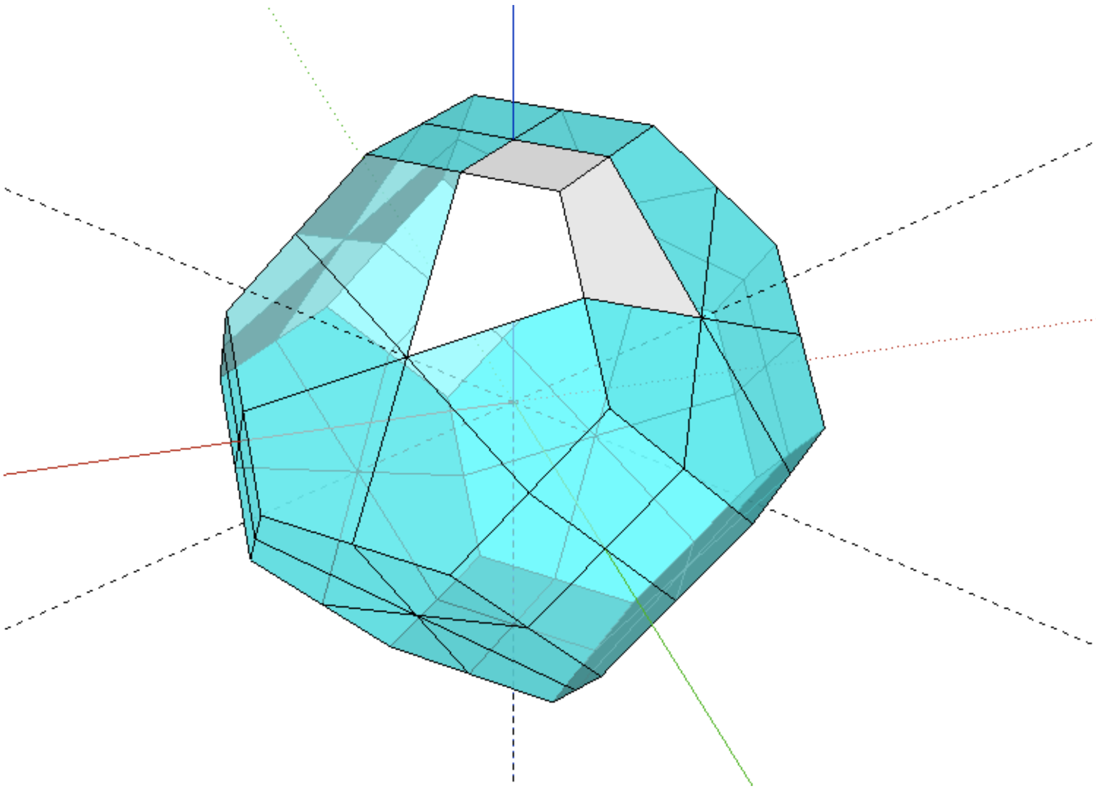


Fig 14.2: The permutahedron combined with the tessellation chamber from earlier in R3, reveals the fact that the chambers in R3, and the mirrors that generated them, is a closed system of mirrors for the permutahedron. They are the mirrors of symmetry, and by modding by all 3 mirrors of symmetry, a single point (the trivial subgroup) can be obtained.

9. Combinatorics and the Permutahedron

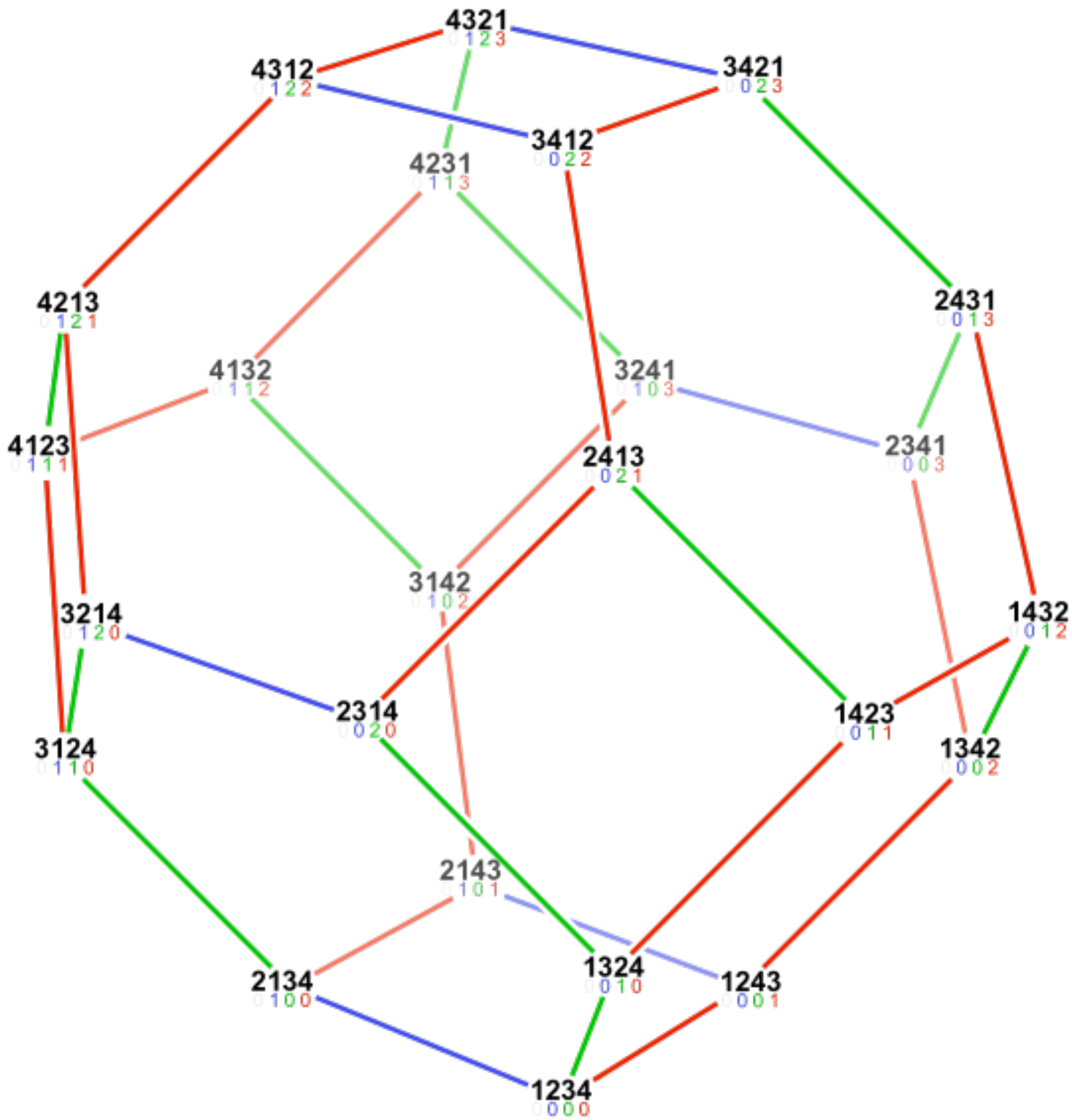


Fig 15: The permutahedron.

The permutahedron as mentioned earlier has some combinatoric properties, mirroring those of the hexagon that we saw earlier in the case of A2. Firstly, there exists three ‘notations’ for the points on the permutahedron, corresponding to one line permutation notation, cycle notation and generator notation. The table below, combined with the map above, characterizes the locations and labels required of a permutahedron with all 3 notations engraved.

One Line	Cyclic	Two Cycles	Breakdown	Generators	Generators Unordered
1234	( )	( )	( )	0	000
2134	(12)	(12)	(12)	1	100
3214	(13)	(13)	(23)(12)(23)	212	120
4231	(14)	(14)	(34)(12)(23)(34)(12)	31231	113
1324	(23)	(23)	(23)	2	010
1432	(24)	(24)	(23)(34)(23)	232	012
1243	(34)	(34)	(34)	3	001
3124	(123)	(13)(12)	2121	2121	110
2314	(132)	(12)(13)	1212	1212	020
1423	(234)	(24)(23)	2322	23	011
1342	(243)	(23)(24)	2232	32	002
4213	(134)	(14)(13)	31231212	31231212	121
3241	(143)	(13)(14)	21231231	21231231	023
4132	(124)	(14)(12)	312311	3123	112
2431	(142)	(12)(14)	131231	131231	013
2143	(12)(34)	(12)(34)	13	13	101
3412	(13)(24)	(13)(24)	212232	2132	022
4321	(14)(23)	(14)(23)	312312	312312	123
4123	(1234)	(14)(13)(12)	312312121	312312121	111
2341	(1432)	(12)(13)(14)	121231231	121231231	003
3142	(1243)	(13)(14)(12)	212312311	2123123	102
2413	(1342)	(12)(14)(13)	131231212	131231212	021
3421	(1423)	(13)(12)(14)	212131231	212131231	023
4312	(1324)	(14)(12)(13)	312311212	3123212	122

Where s1 = (12) s2 = (23) and s3 = (34).

Note that s1, s2 and s3 will generate the group S4.

When combined with the original permutahedron in Fig 15, we would be able to build a translation permutahedron... as depicted here and available on the google sketchup warehouse.

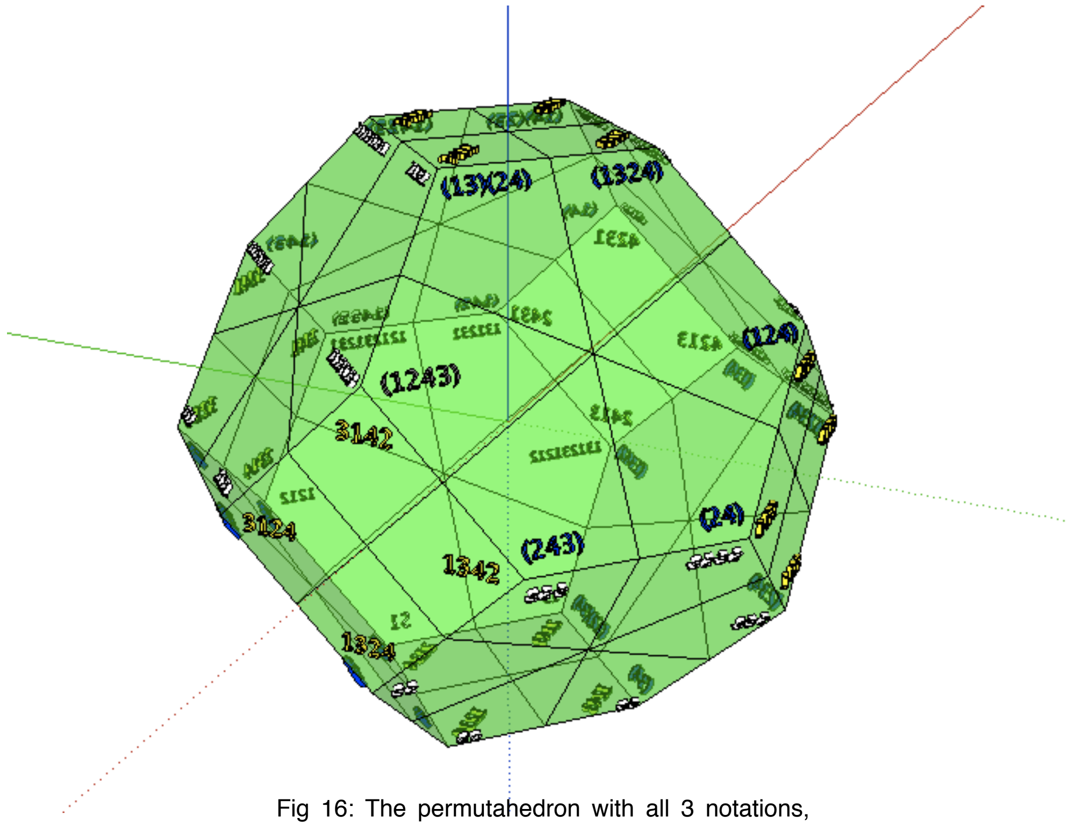


Fig 16: The permutahedron with all 3 notations, together with the intersections of the mirrors of symmetry. This is effectively group division by the entire group. That is, each element is its own representative.

The permutahedron can be 'cut' along these axes in the same way that the group S4 can be 'modded' by removing certain generators or pairs of generators. The following diagrams and captions will display the most entertaining ways this can occur.

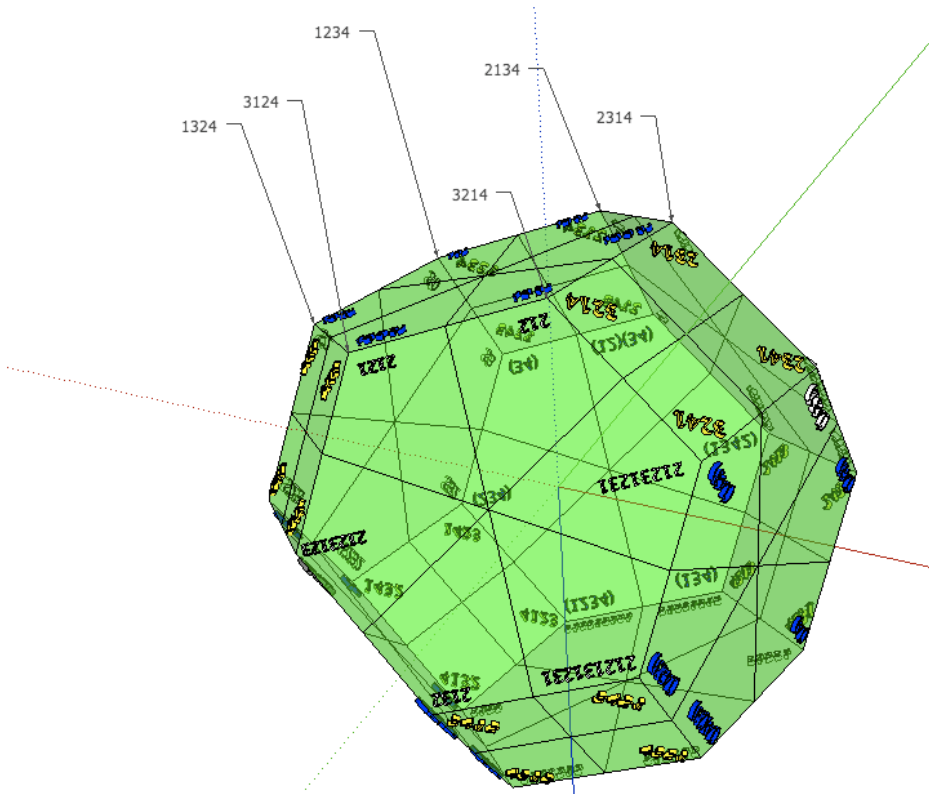


Fig 17: Cutting the shape into equivalence classes modulo S2. Six permutations remain, with S3 properties.

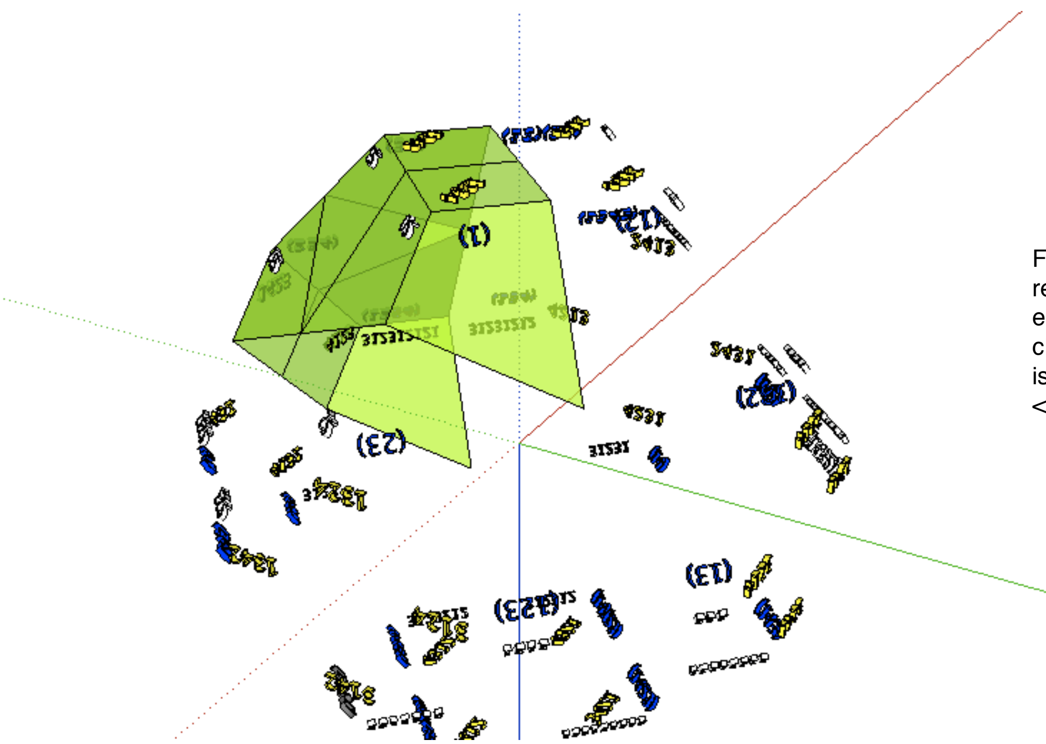
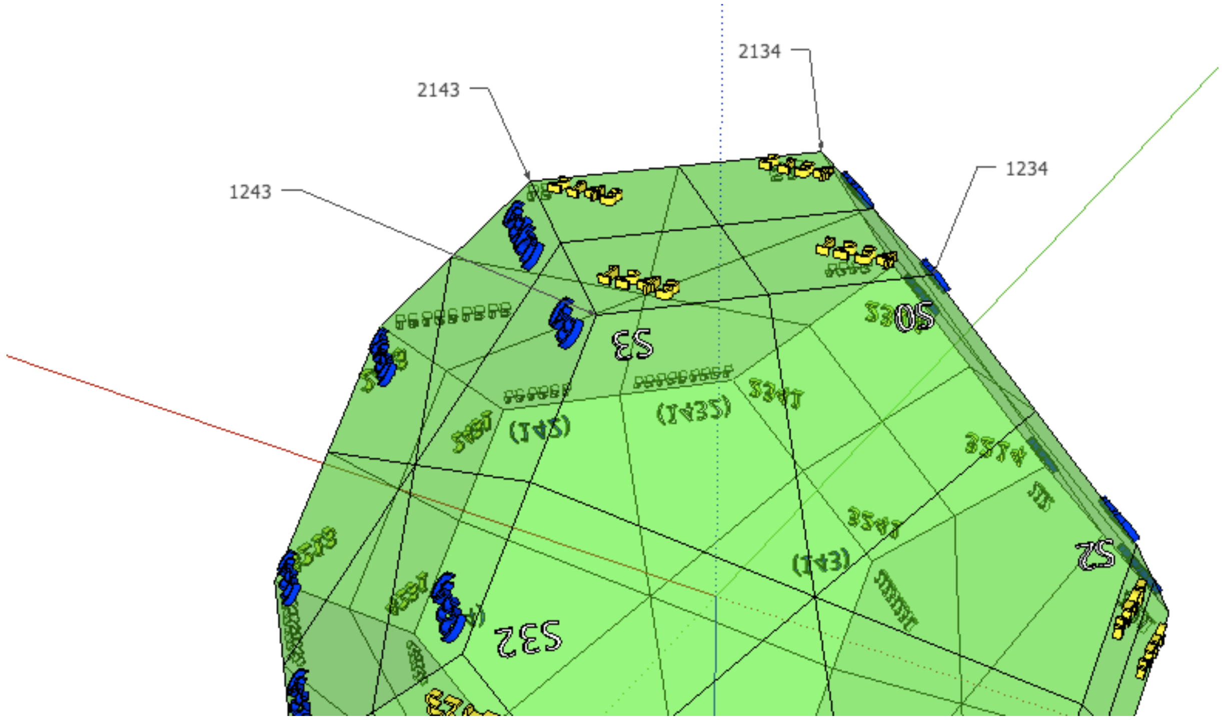


Fig 18: The equivalence class represented by the identity. Four elements comprise this equivalence class modulo S2. This is or is isomorphic to S4 / <S1,S2> and S4 / <S2,S3>



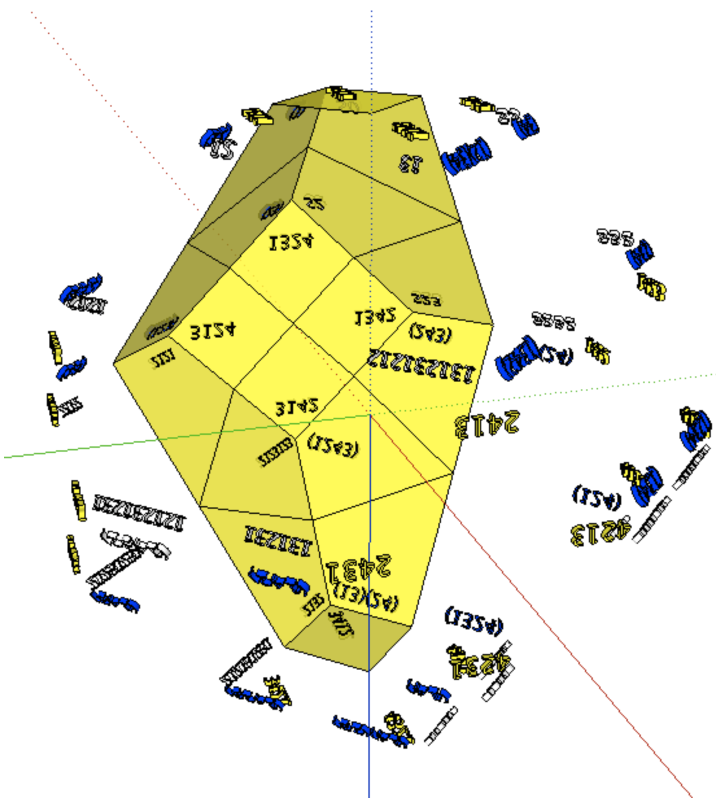
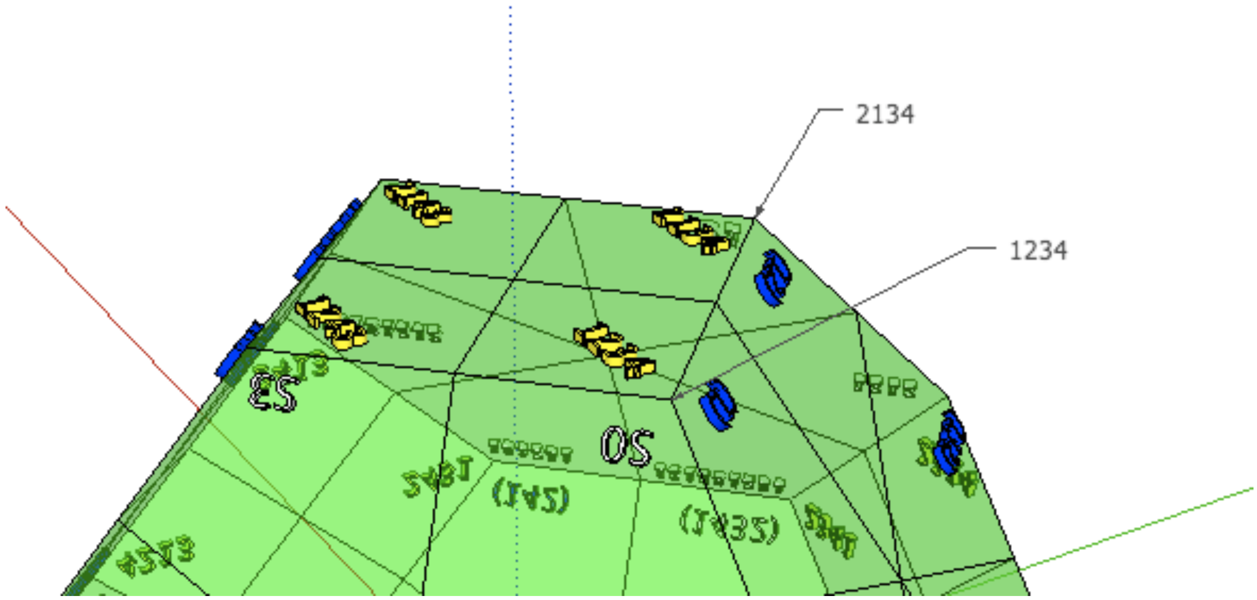


Fig 19 & 20: The representatives of S4 after modulo s2, the second generator. There are four representatives, corresponding to four equivalence classes of six elements each. Pictured left is the equivalence class represented by the identity element.

S4 / <S1,S3>



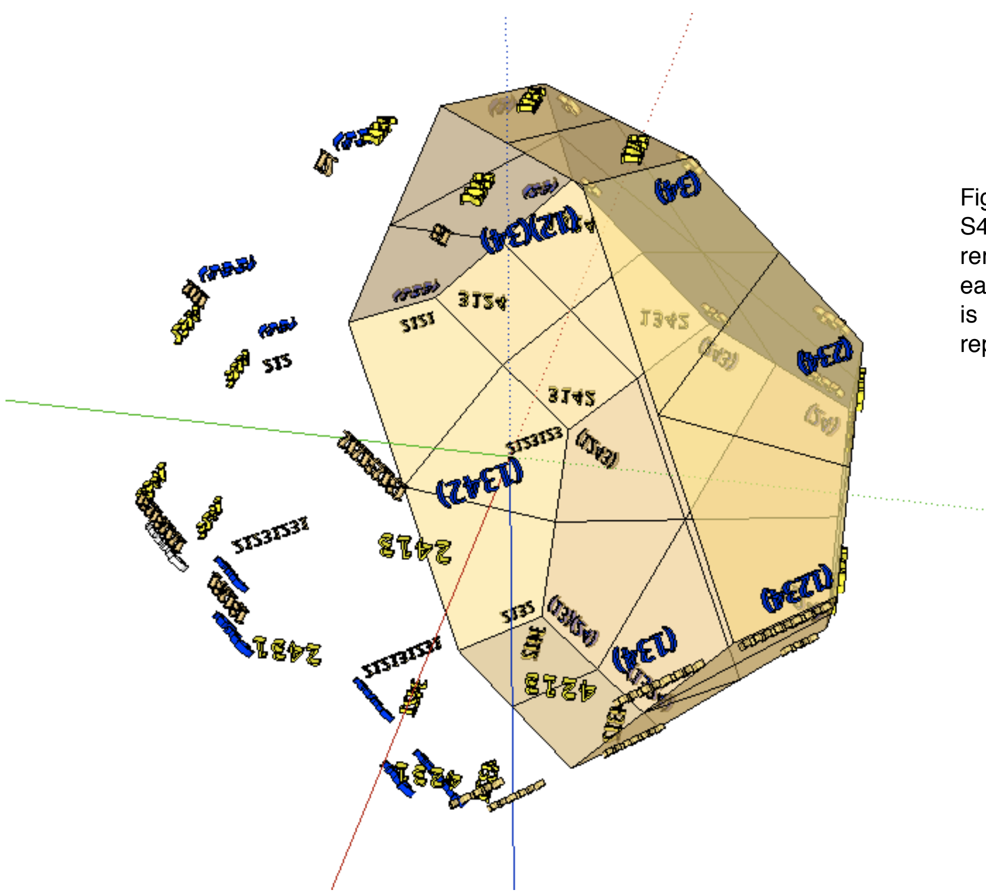


Fig 20 & 21: The representatives of S4 after modulo s2 and s3. There remain only two representatives, each with 12 elements. Pictured left is the equivalence class that is represented by the identity element.

S4 / <S1> OR
S4 / <S2> OR
S4 / <S3>

10. Dual Shapes

From wikipedia, we have the textbook definition of dua-shapes. In geometry, polyhedra are associated into pairs called duals, where the vertices of one correspond to the faces of the other. The dual of the dual is the original polyhedron. The dual of a polyhedron with equivalent vertices is one with equivalent faces, and of one with equivalent edges is another with equivalent edges. So the regular polyhedra — the Platonic solids and Kepler-Poinsot polyhedra — are arranged into dual pairs, with the exception of the regular tetrahedron which is self-dual.

Duality is also sometimes called reciprocity or polarity.[12]

Duality is something I have preferred to be illustrated.

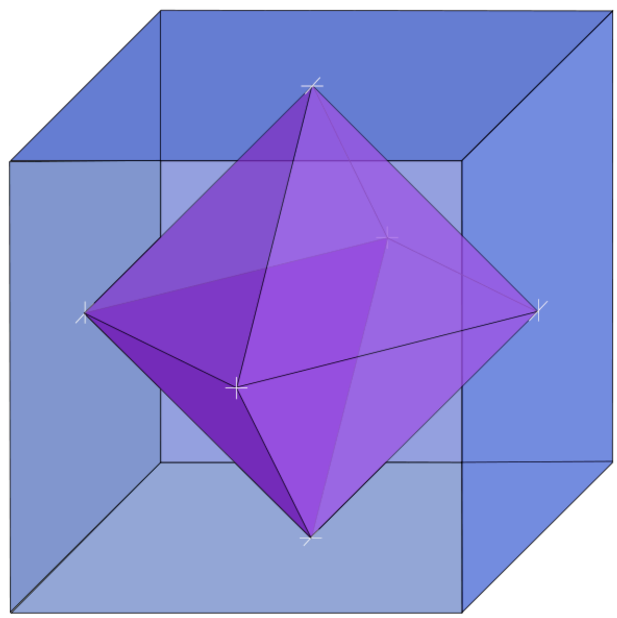


Fig 22: Octahedron inscribed within the cube. The octahedron and cube are duals of one another; the vertices of one correspond to the midpoint of each face of the other.

Most interestingly, we have that dual shapes can be derived geometrically by a process of 'cutting corners' please see below.

[12] http://en.wikipedia.org/wiki/Dual_polyhedron

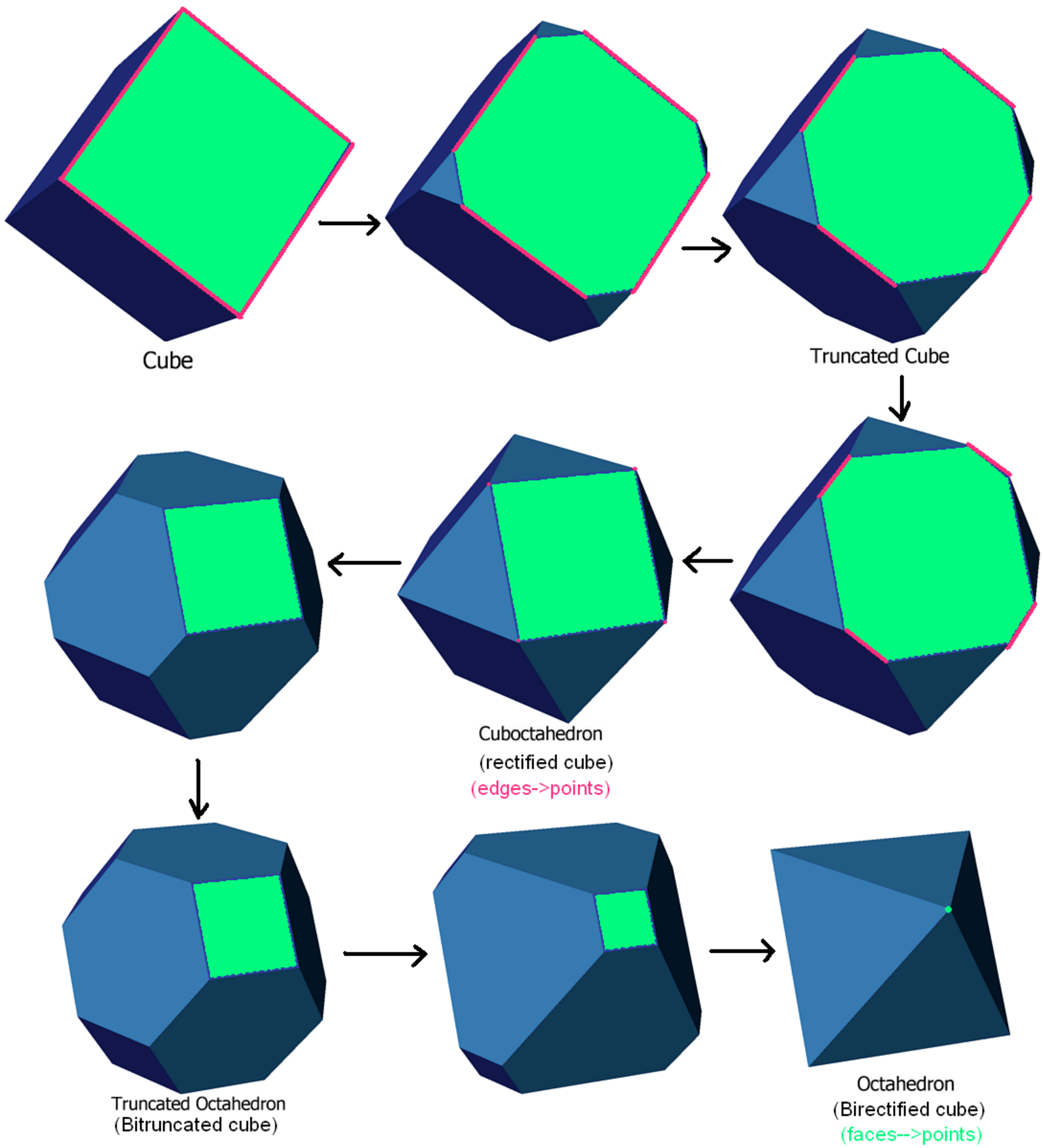
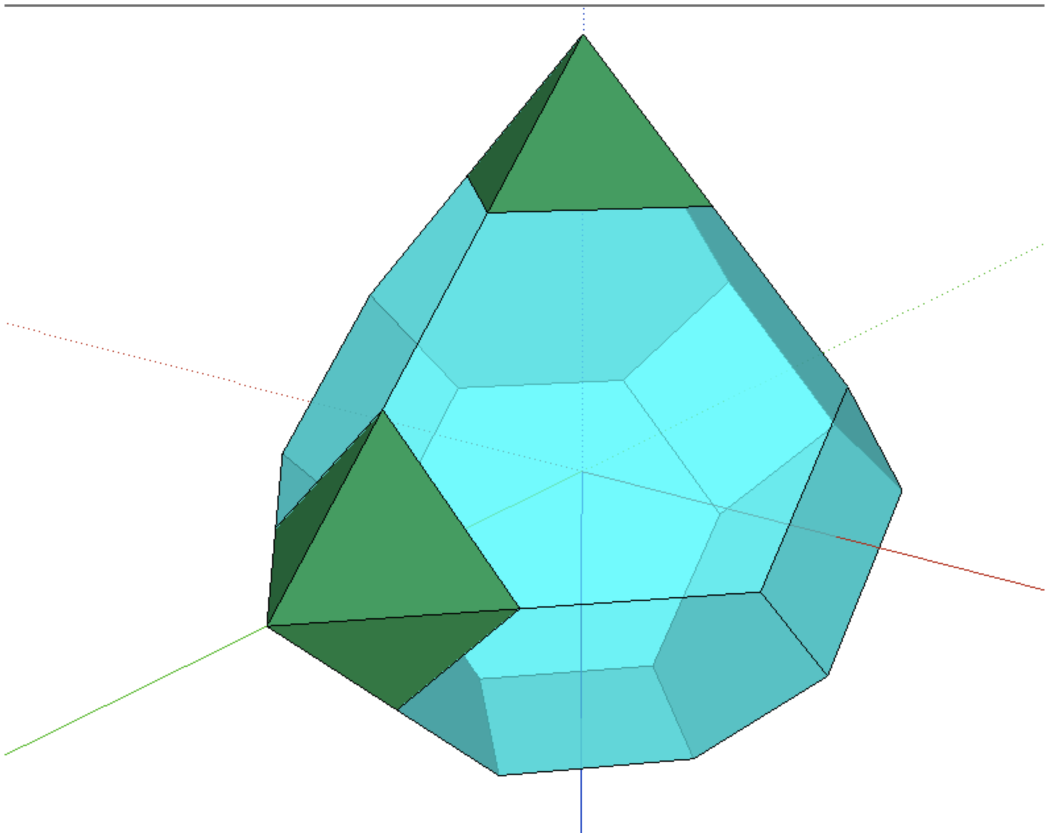
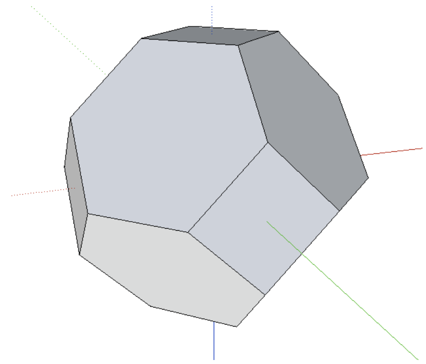


Fig 23 & 24: Truncation sequence of the cube to the octahedron reveals an intuitive way of transforming one into the other. Notice that the permutahedron is in this system... as it is essentially a truncated octahedron, see figure below. Above credit to wikipedia (http://en.wikipedia.org/wiki/Dual_polyhedron).

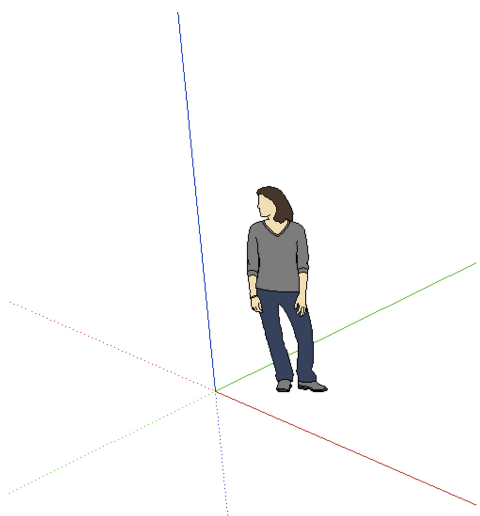


Our familiar permutahedron from earlier, is, when looked at properly very easy to picture, as it is a truncated octahedron. From this point, I will explain exactly how to create these shapes in sketchup, but with the right instructions, these methods can be applied to any AUTOCAD program.

11. How to Model the Permutahedron



Firstly, navigate google sketchup's opening menus (choosing metric engineering for this guide) until we reach this point.



From here, make sure the **select** tool is being used (its the cursor in the top left) and highlight the dummy, once fully highlighted, delete (hit delete) the ambisexual model and we now have a clean slate to work with.

The permutahedron is an easy shape to model, once we notice two facts about it. Firstly, that it is merely a truncated (pointy ends cut off) octahedron. This shape consists of 8 hexagonal faces, and 6 squares. Further, the permutahedron is uniquely determined by its square faces, that is, every vertex of the shape is, lucky for us, part of one and only one square face. From here we know that we need only model the squares of the shape, and allow sketchup to do the rest of the work.

For purposes of generality, our permutahedron will have all edges of unit length. Further, we will rotate the construction so as to keep the coordinates intuitive.

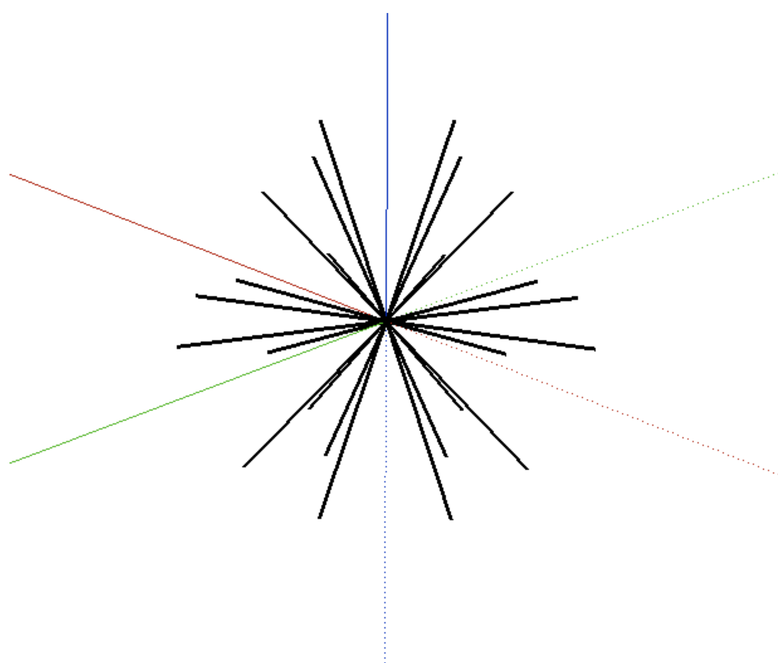
Call the six squares of the permutahedron by their positions relative to the red(solid axis). If red solid is the 'front' square, F, then we can label the squares Back, Left, Right, Up and Down.

A square of edge length one in R2 with midpoint at the origin, and points on the axes (this is important later) will have coordinates (±√2/2,0) and (0, ±√2/2).

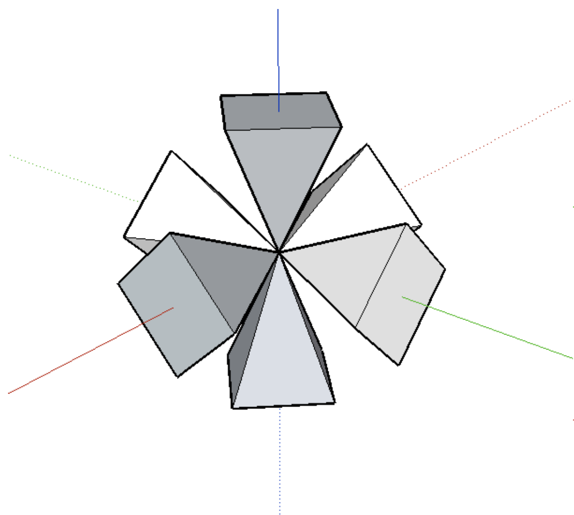
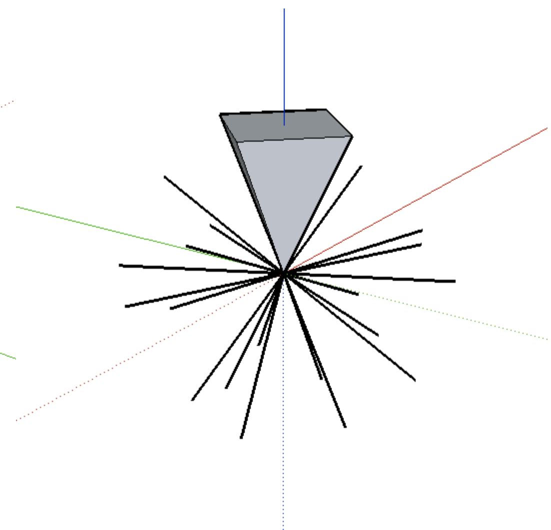
When we extend this to our 3d example we have our front square having coordinates (±√2/2,y,0) and (0,y, ±√2/2). We now need to determine the y value that will assure our hexagons a unit length as well.

Our answer comes from similar triangles, and we find the y value to be √2. To generalize, we now take the general coordinates method to be ±√2/2, ±0 and ±√2. (Where take all possible combinations and permutations of these values (without repetition) and it will determine the 24 vertices of the permutahedron, corresponding to the 24 vertices of the 6 square faces.)

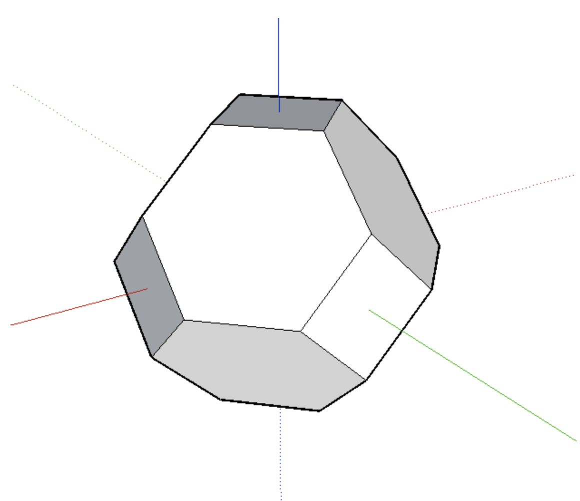
To draw this in sketchup, we will build a spider-frame and connect the terminal points. In sketchup, click the draw line tool (the pencil) and click the origin. This sets the origin as the initial point of the first **vector** you are about to draw. In the bottom right where it says 'measurements' you are now free to type [0.707,1.414,0] (corresponding to [√2/2,√2,0]) for example. (Sketchup seems to prefer decimals, luckily, this will not have any planar effects). Creating the full spider frame, you will end up with this spiky thing:



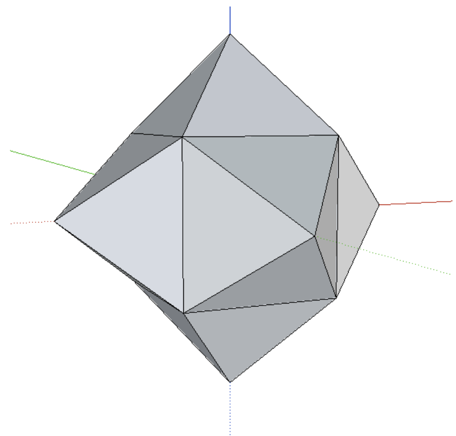
Now sketchup is going to do most of the work from here on. Take a moment to navigate the object. To pitch and yaw, use the button depicted by blue arrows curved around each other (next to the hand button.) As you orient around the object, try to visualize where the six squares will go. Hold down shift in this mode if you want to shift the camera, as if you were walking left or right. After finding a comfortable angle, click the draw tool, and click the terminal point of one of the vectors you have drawn. Now draw a line connecting that point with another of the same square, preferably not the diagonal. Continue this until you get this shape. (If you have any doubts about the vectors you are connecting, the measurements text field in the bottom right will ALWAYS read 1m when you are doing it right.) Continue this process until you have completed all six squares. It should look like this:

Now connect the vertices of each square to each other, being careful to only use unit length edges. (This isnt necessary, but for cleanliness of the final product)

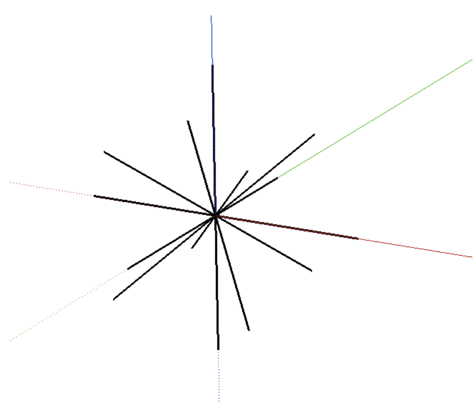


12. How to Model the Permutahedron Dual



Dual shapes have some very interesting properties that will make our job of modeling the dual permutahedron very easy. As an idea of the shape, the dual of an object has vertices for faces and faces for vertices of its partner. That is, if there is a cube, which has 6 faces and 8 vertices, then its dual is the octahedron, which has 8 faces, and six vertices. Then, the permutahedron's dual (the dual) has 14 vertices and 24 faces. Furthermore, the nature of the face is preserved under duality. A hexagon corresponds to a six edged vertex. A square will correspond to a four edged vertex. Further, the midpoint of every face of the permutahedron corresponds to a vertex of the dual, therefore the dual is very easy to map once we have mapped the permutahedron. Just using the midpoint formula gives us the coordinates:
$(\pm\sqrt{2},0,0)(0,\pm\sqrt{2},0)(0,0,\pm\sqrt{2})$ and $(\pm 3\sqrt{2}/2, \pm 3\sqrt{2}/2, \pm 3\sqrt{2}/2)$.

Mapping these out in similar fashion as the permutahedron gives us our spider frame:



And connecting as before, with the intuition that the shape resembles pyramids.