Non-Commutative Symmetric Functions I: A zoo of Hopf algebras

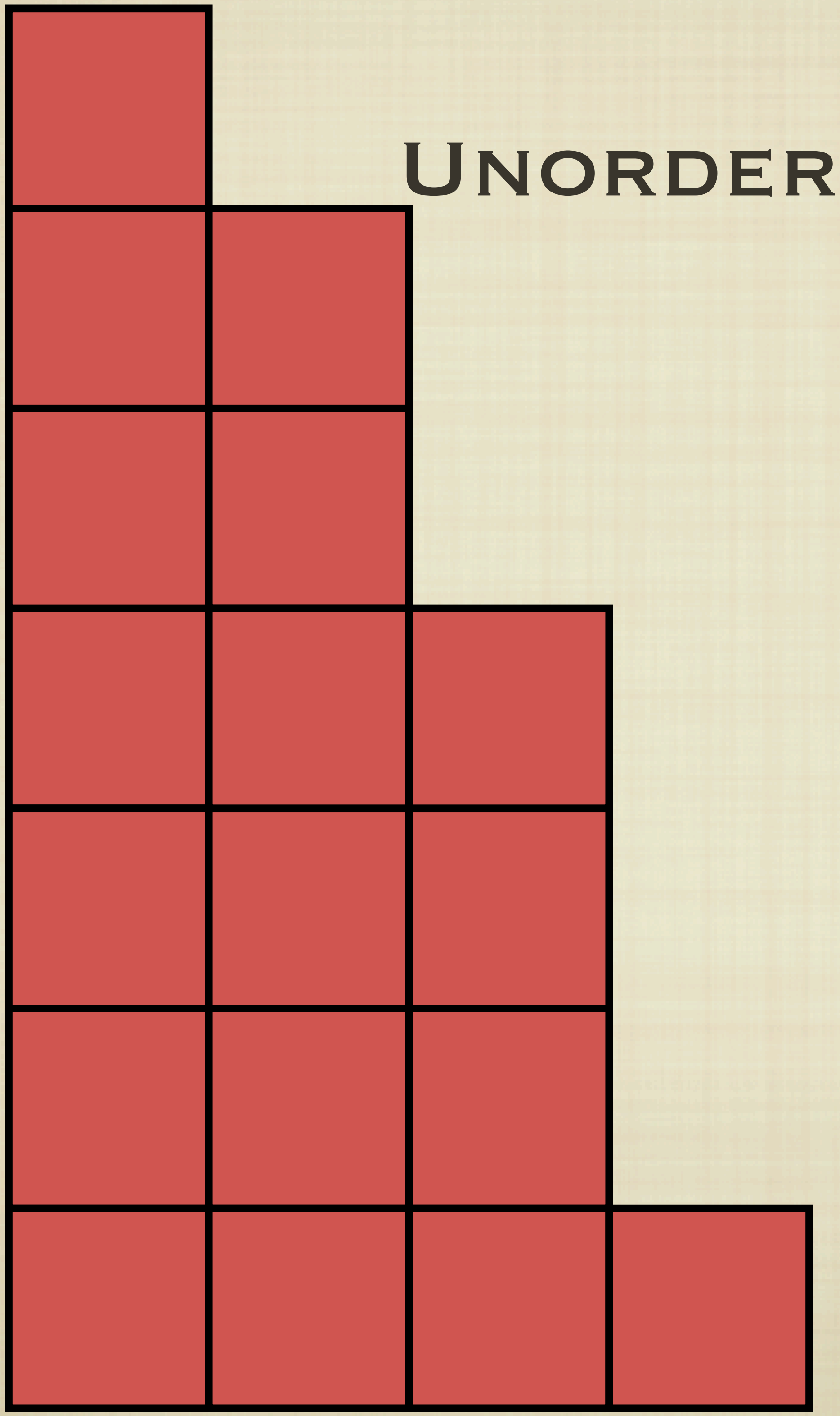
Mike Zabrocki
York University

I will present an overview of what Combinatorial Hopf Algebras (CHAs) are about by introducing definitions and examples. I will try to show how examples of CHAs are related to each other and where they can appear in the literature before they are recognized as CHAs. I will also give some examples where these objects appear in other areas of mathematics.

Partitions

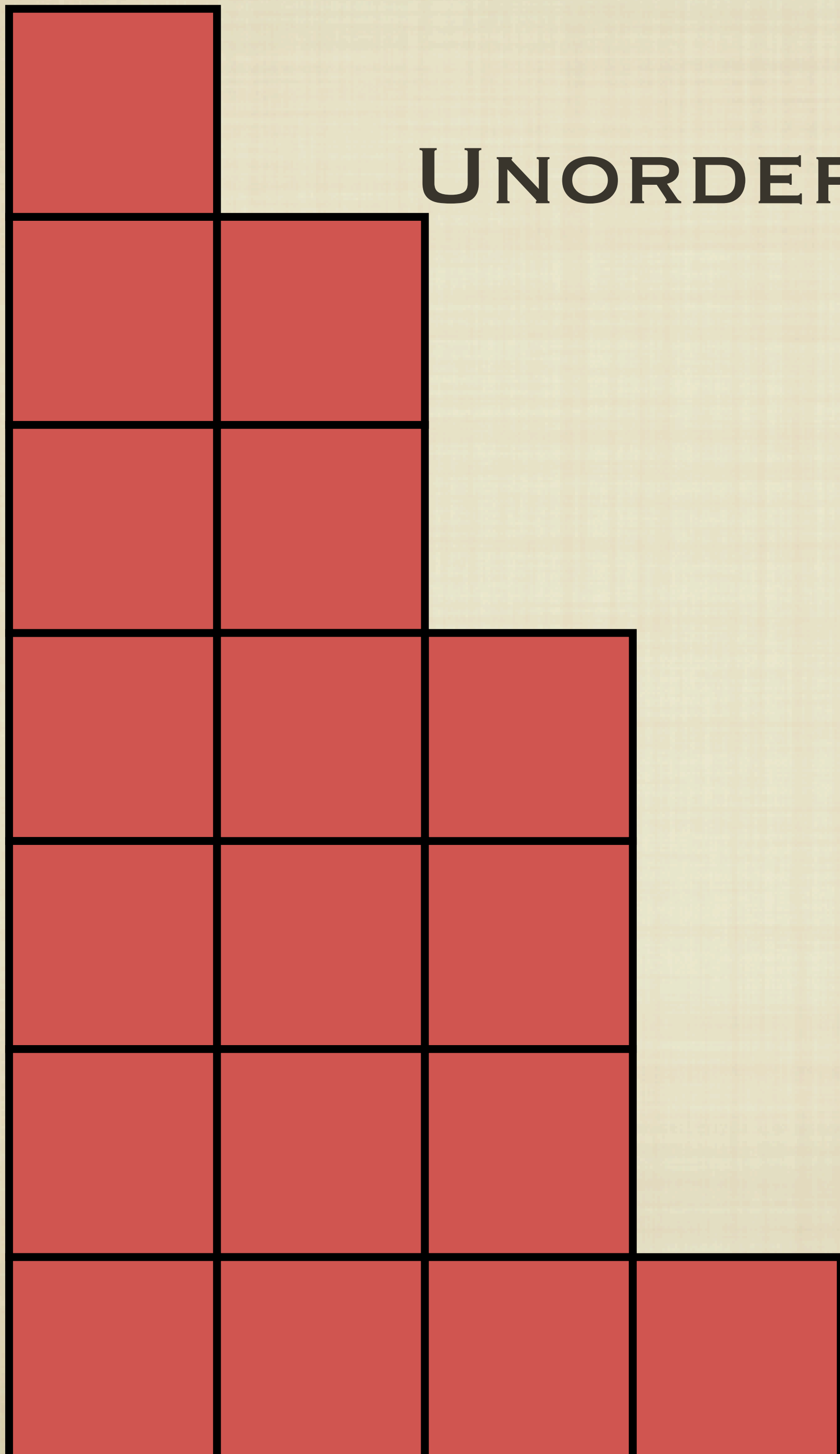
Unordered lists of non-negative integers

Partitions



Unordered lists of non-negative integers

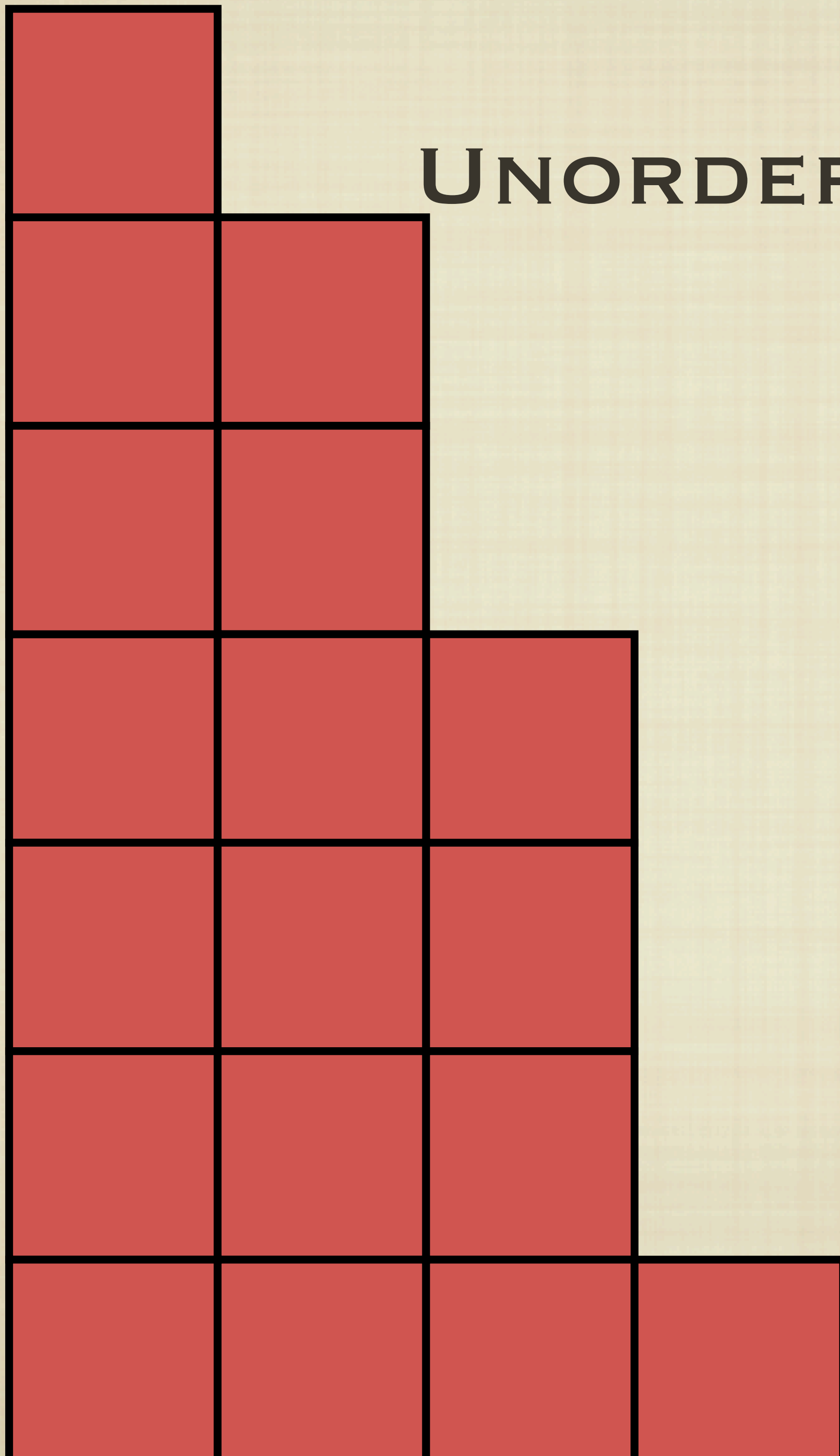
Partitions



Unordered lists of non-negative integers

$$(4, 3, 3, 3, 2, 2, 1)$$

Partitions



Unordered lists of non-negative integers

$$(4, 3, 3, 3, 2, 2, 1)$$

$$4 + 3 + 3 + 3 + 2 + 2 + 1$$

Create an algebra

Create an algebra

- linearly spanned by partitions

Create an algebra

- linearly spanned by partitions
- Commutative product μ

Create an algebra

- linearly spanned by partitions
- Commutative product μ
- Graded by size of partitions

Create an algebra

- linearly spanned by partitions
- Commutative product μ
- Graded by size of partitions

$$\Lambda = \bigoplus_i \Lambda_i$$

Create an algebra

- linearly spanned by partitions
- Commutative product μ
- Graded by size of partitions

$$\Lambda = \bigoplus_i \Lambda_i$$

Λ_i linear span of partitions of size i

Create an algebra

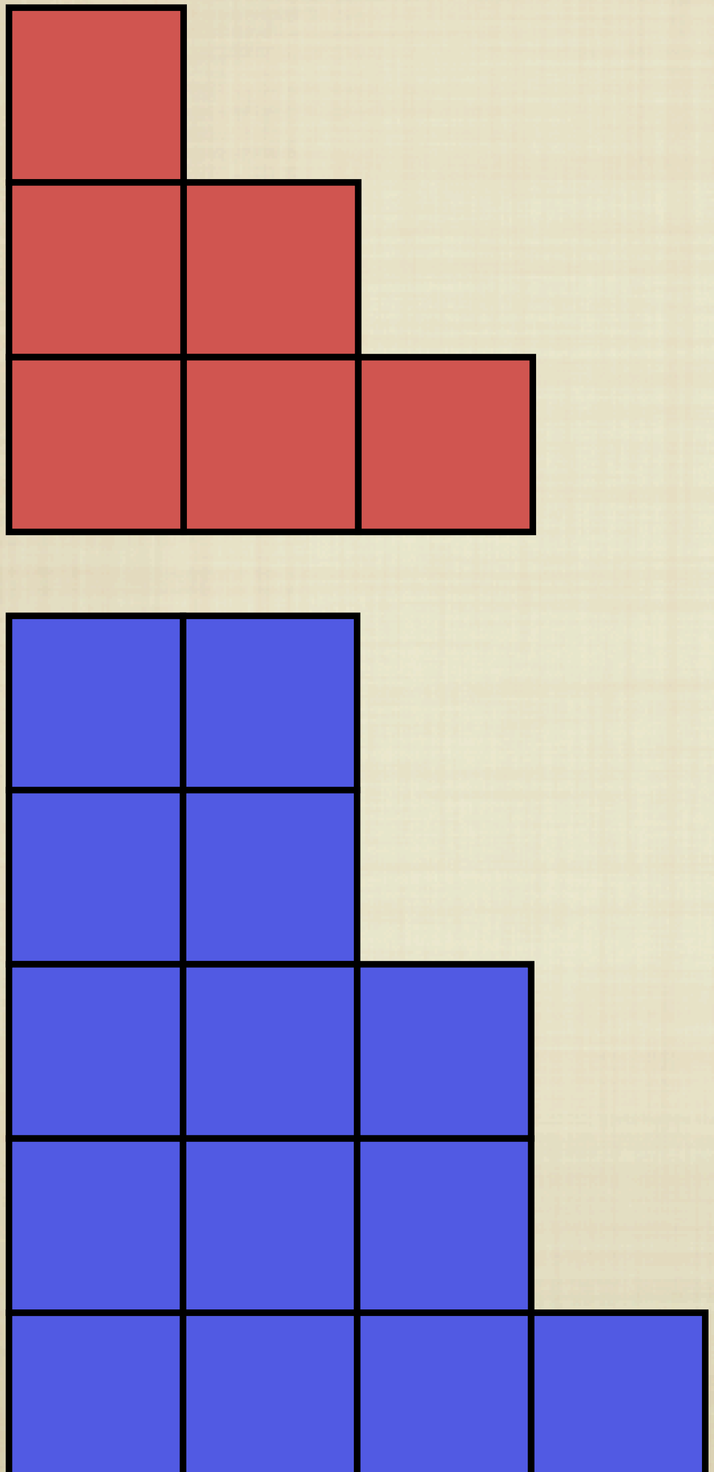
- linearly spanned by partitions
- Commutative product μ
- Graded by size of partitions

$$\Lambda = \bigoplus_i \Lambda_i$$

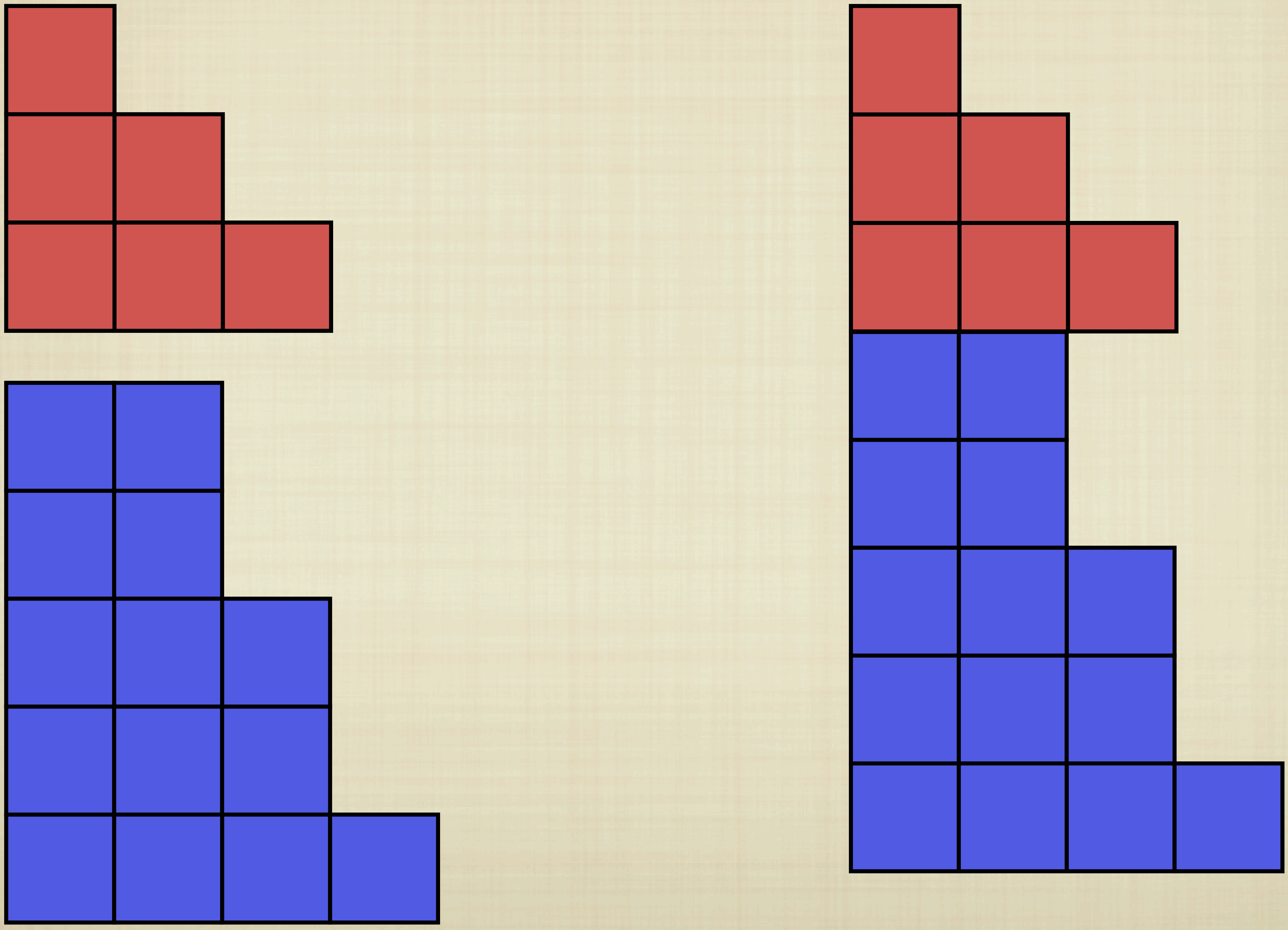
Λ_i linear span of partitions of size i

$$\mu : \Lambda_i \otimes \Lambda_j \rightarrow \Lambda_{i+j}$$

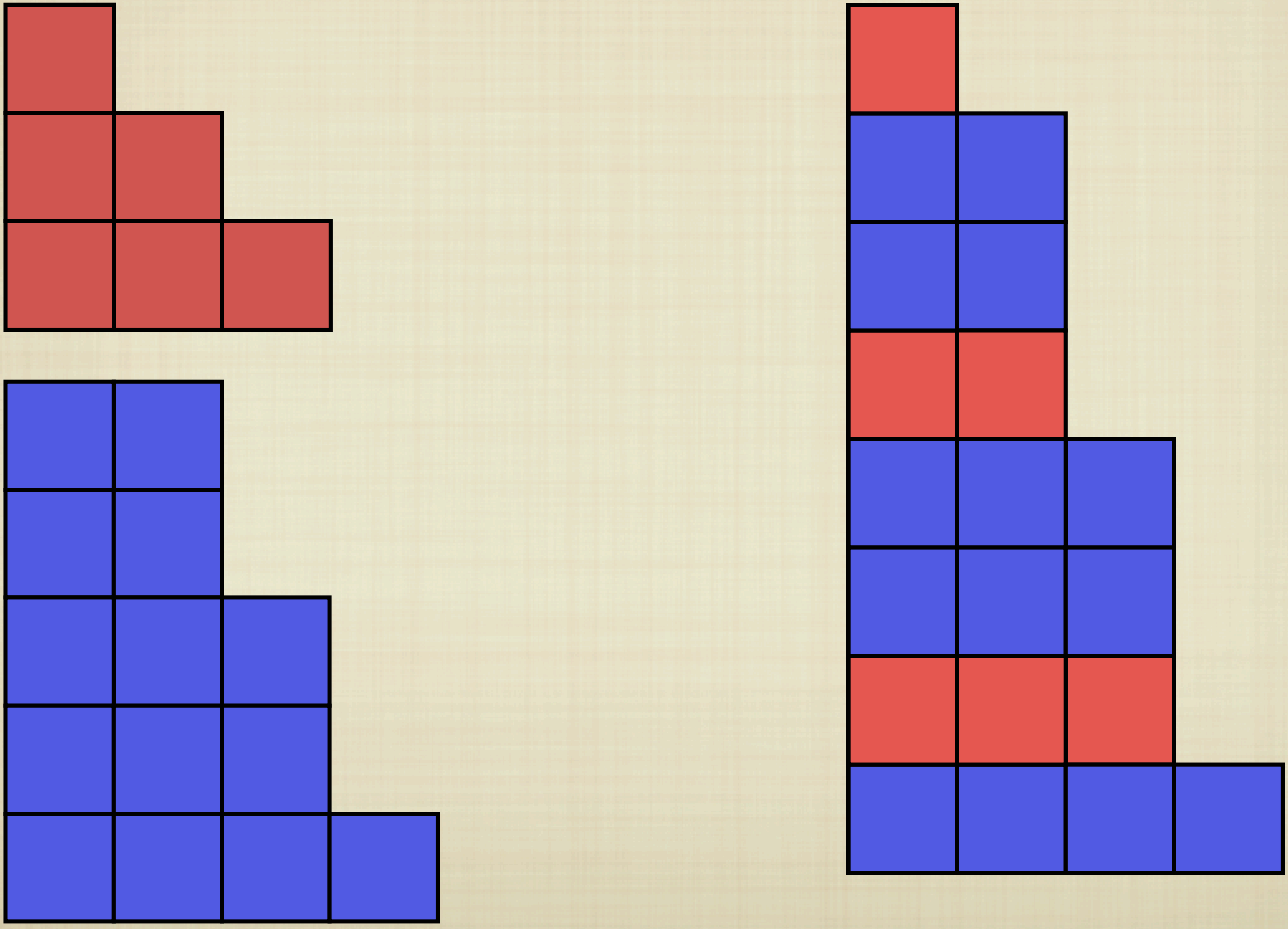
Define a commutative product



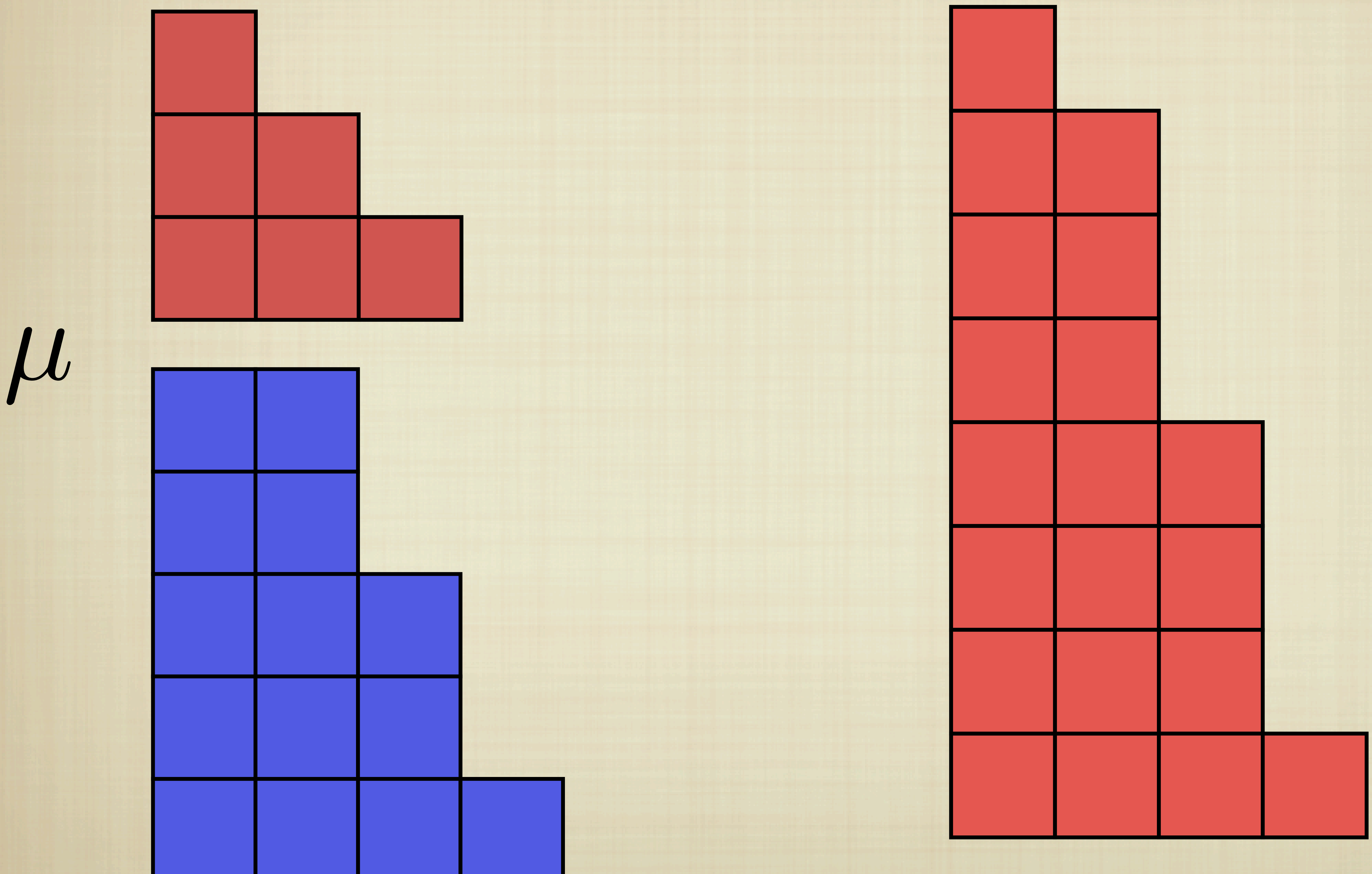
Define a commutative product



Define a commutative product

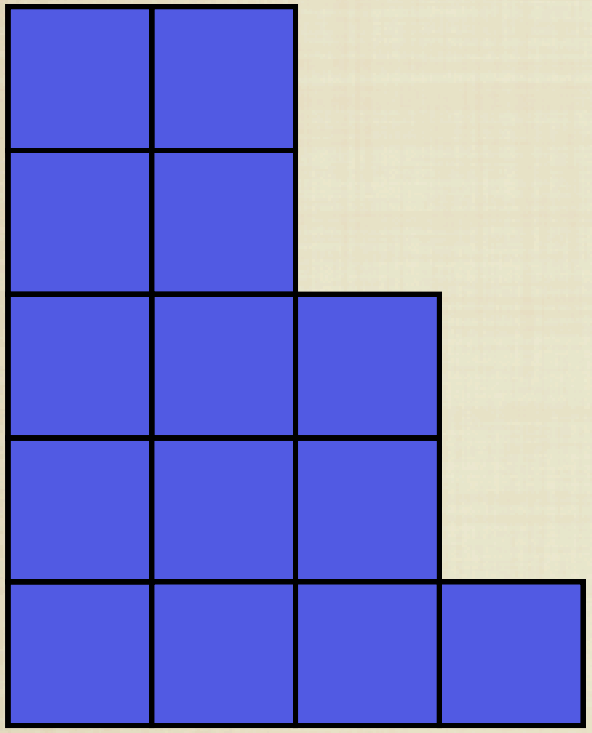


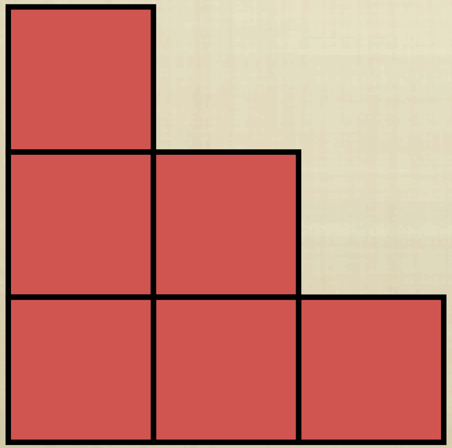
Define a commutative product



PROPERTIES OF THIS ALGEBRA

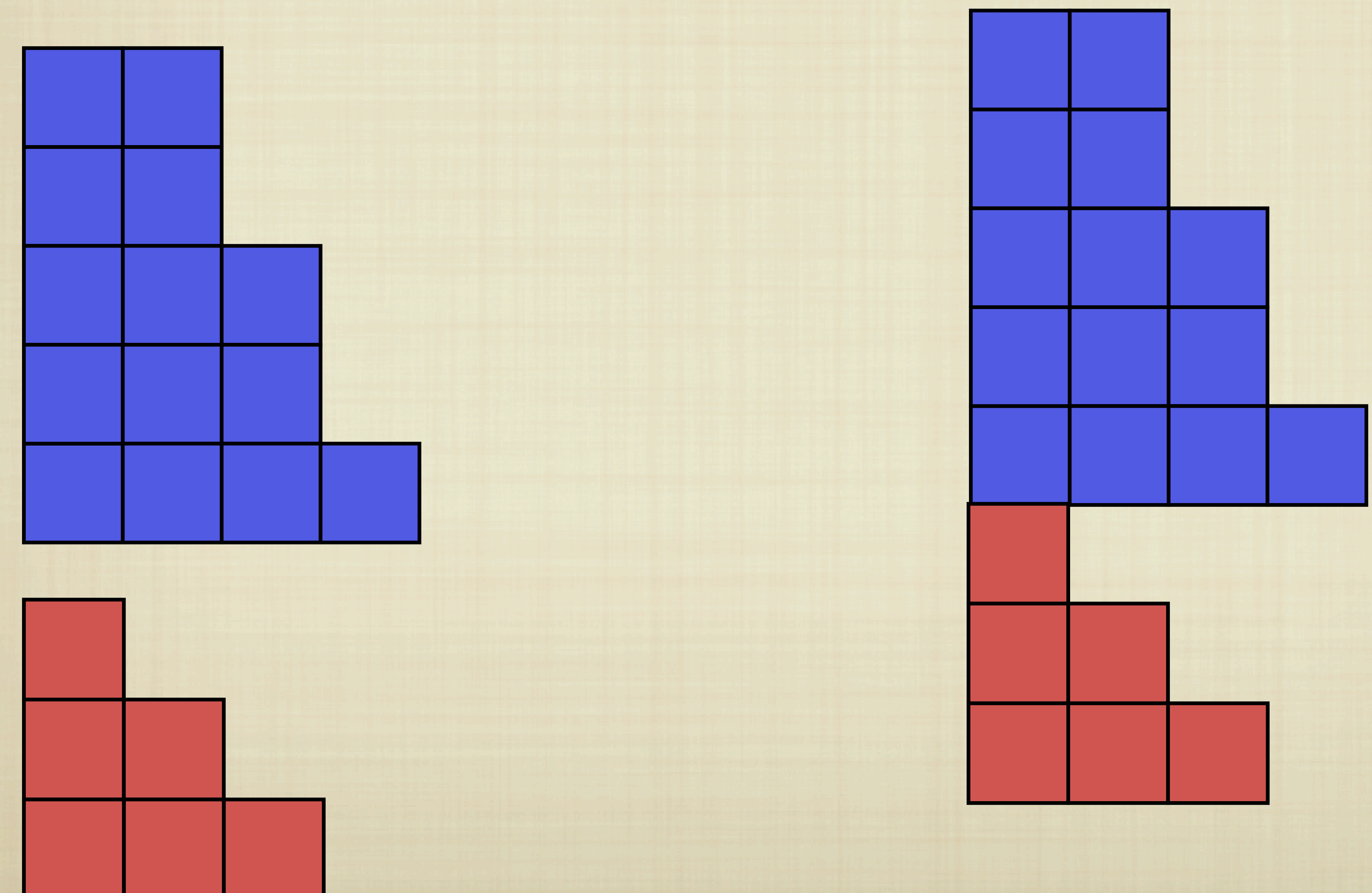
- Commutative and graded





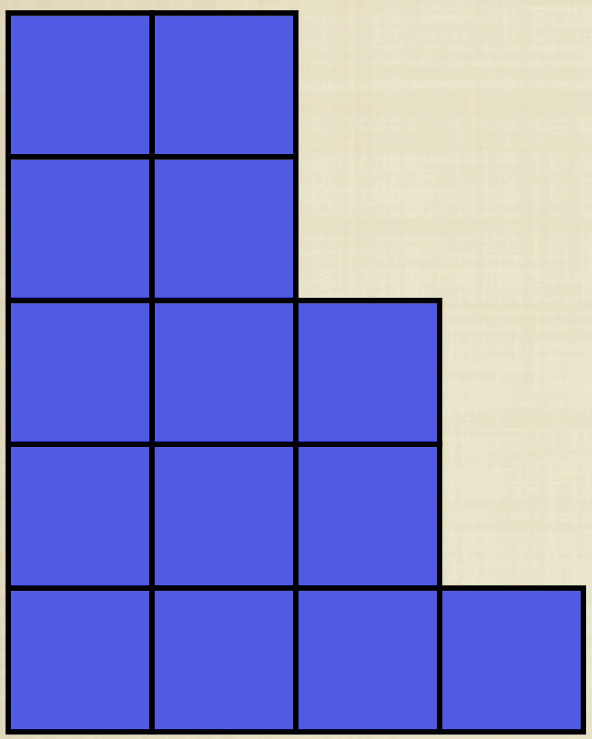
PROPERTIES OF THIS ALGEBRA

- **Commutative and graded**

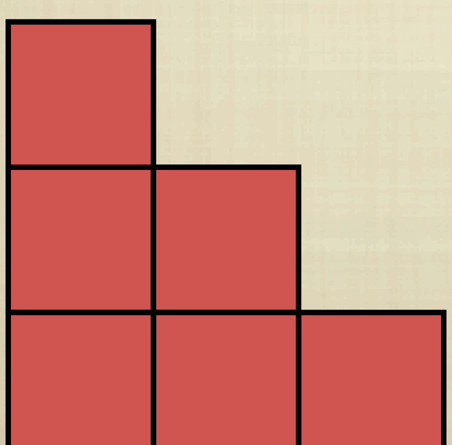


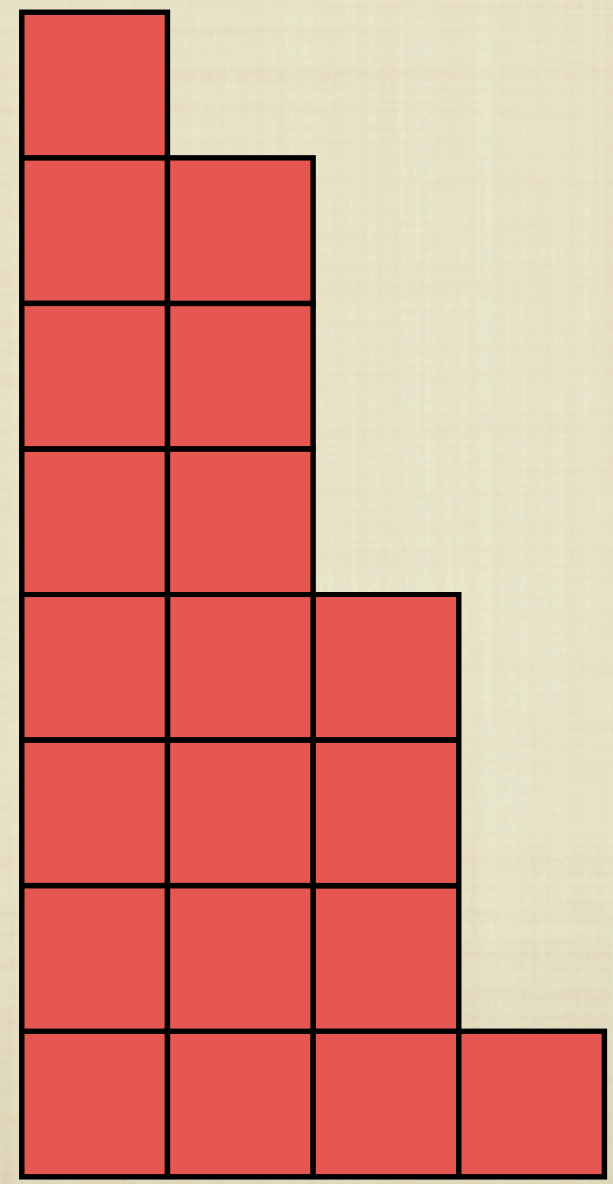
PROPERTIES OF THIS ALGEBRA

- **Commutative and graded**



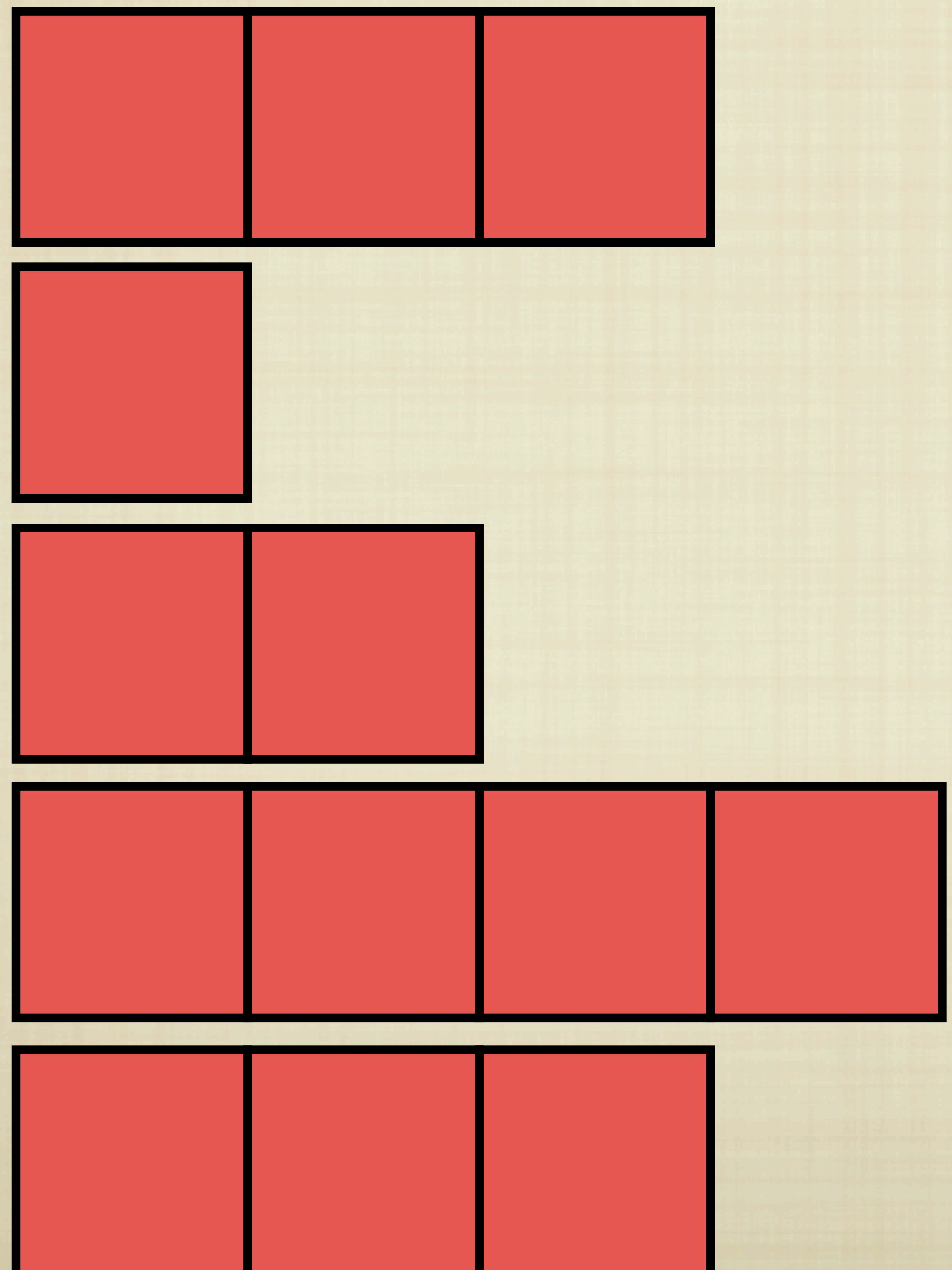
μ





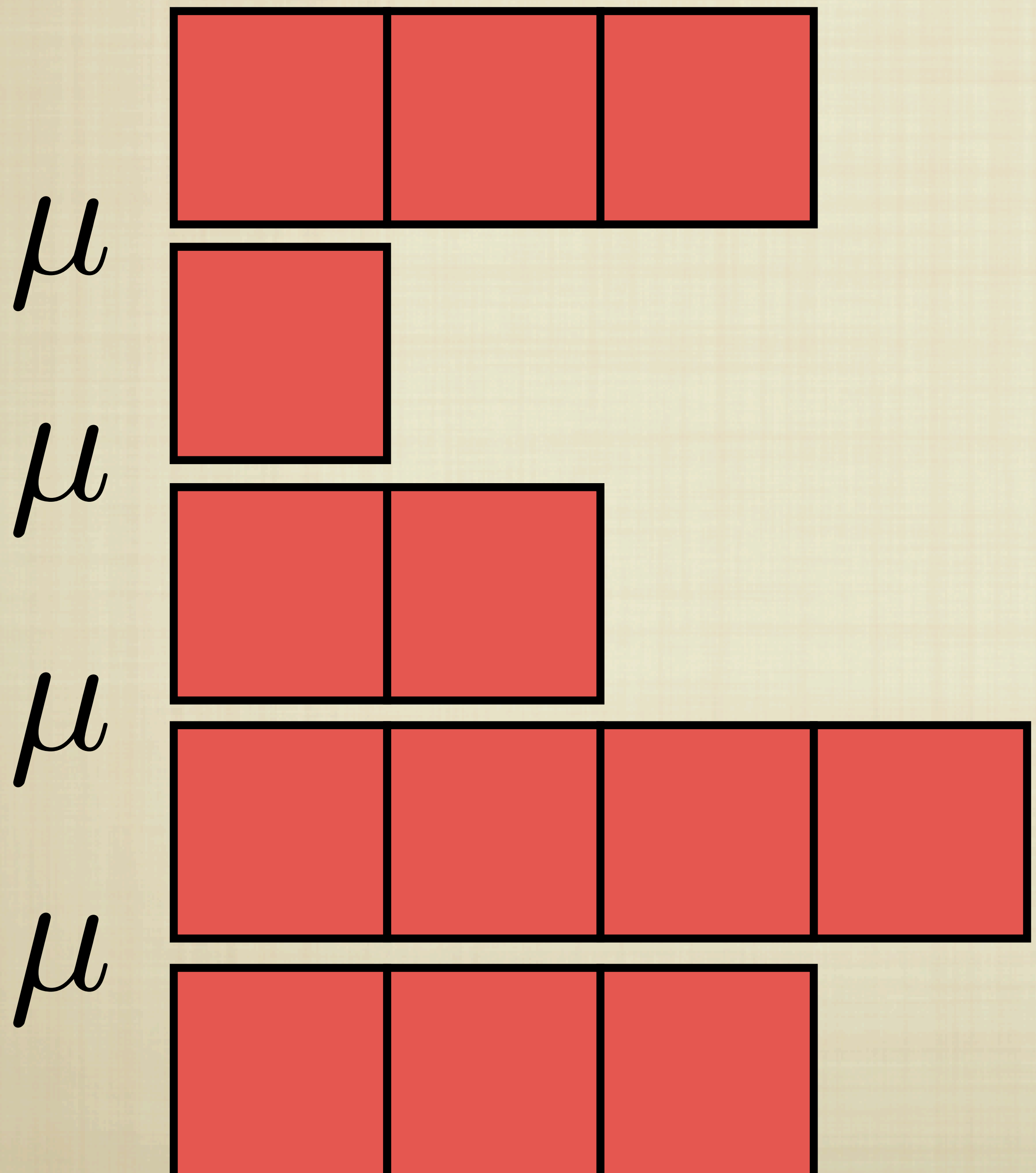
Properties of this algebra

- Freely (commutative) generated by building blocks of rows



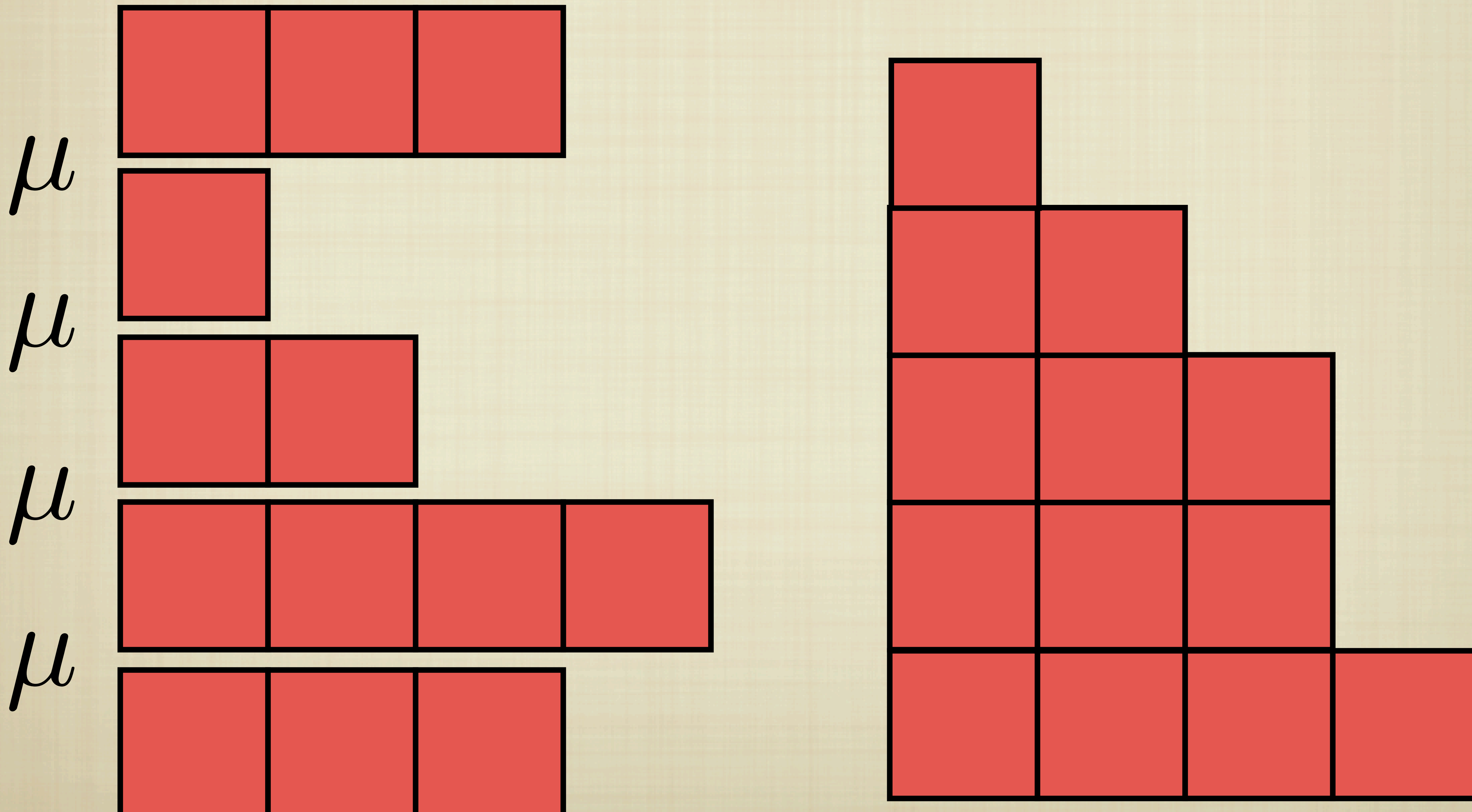
Properties of this algebra

- Freely (commutative) generated by building blocks of rows



Properties of this algebra

- Freely (commutative) generated by building blocks of rows



PROPERTIES OF THIS ALGEBRA

PROPERTIES OF THIS ALGEBRA

- Bilinear

PROPERTIES OF THIS ALGEBRA

- **Bilinear**

$$\mu : \Lambda \bigotimes \Lambda \longrightarrow \Lambda$$

Properties of this algebra

- Bilinear

$$\mu : \Lambda \bigotimes \Lambda \longrightarrow \Lambda$$

- Isomorphic to the free polynomial algebra with one generator at each degree

Properties of this algebra

- Bilinear

$$\mu : \Lambda \bigotimes \Lambda \longrightarrow \Lambda$$

- Isomorphic to the free polynomial algebra with one generator at each degree

$$\Lambda \simeq K[p_1, p_2, p_3, \ldots]$$

Properties of this algebra

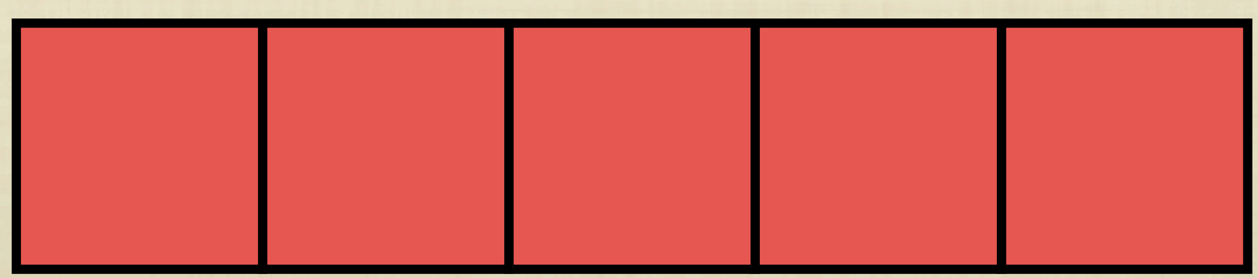
- Bilinear

$$\mu : \Lambda \bigotimes \Lambda \to \Lambda$$

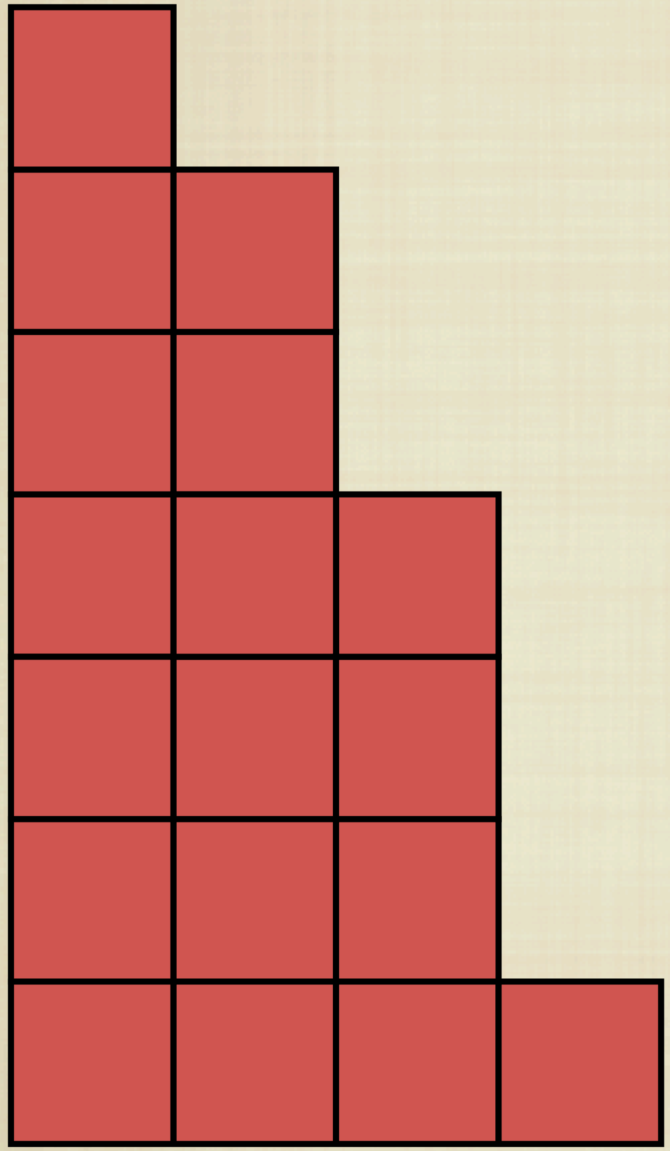
- Isomorphic to the free polynomial algebra with one generator at each degree

$$\Lambda \simeq K[p_1, p_2, p_3, \ldots]$$

$$p_5 \leftrightarrow$$

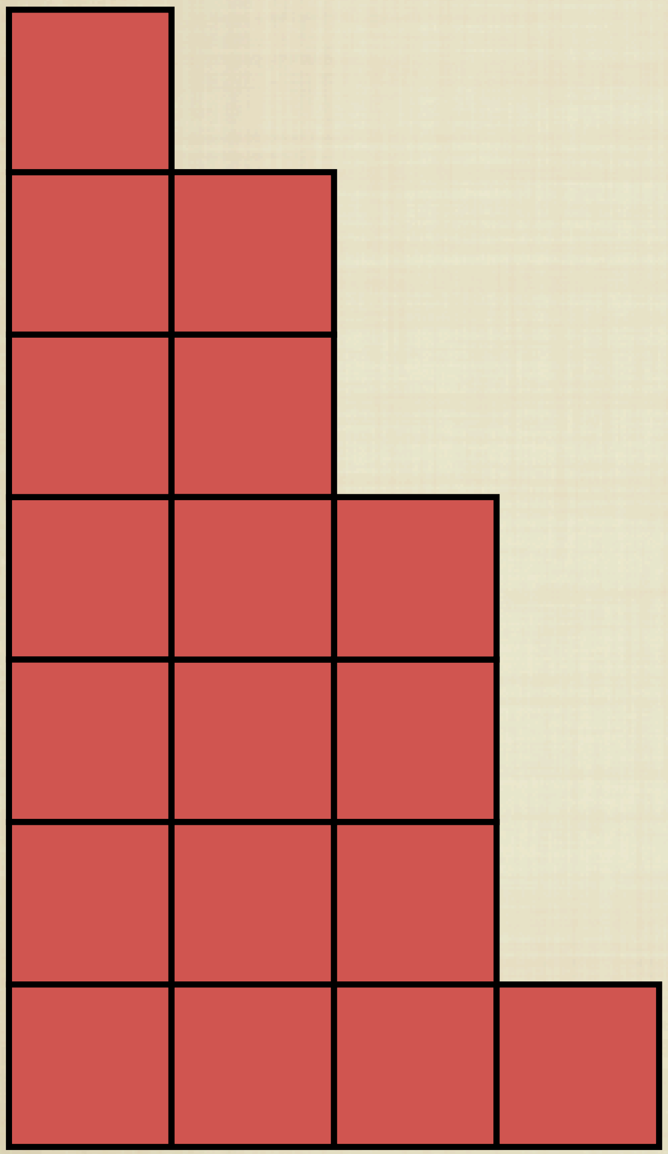


From combinatorial object to algebra



partitions

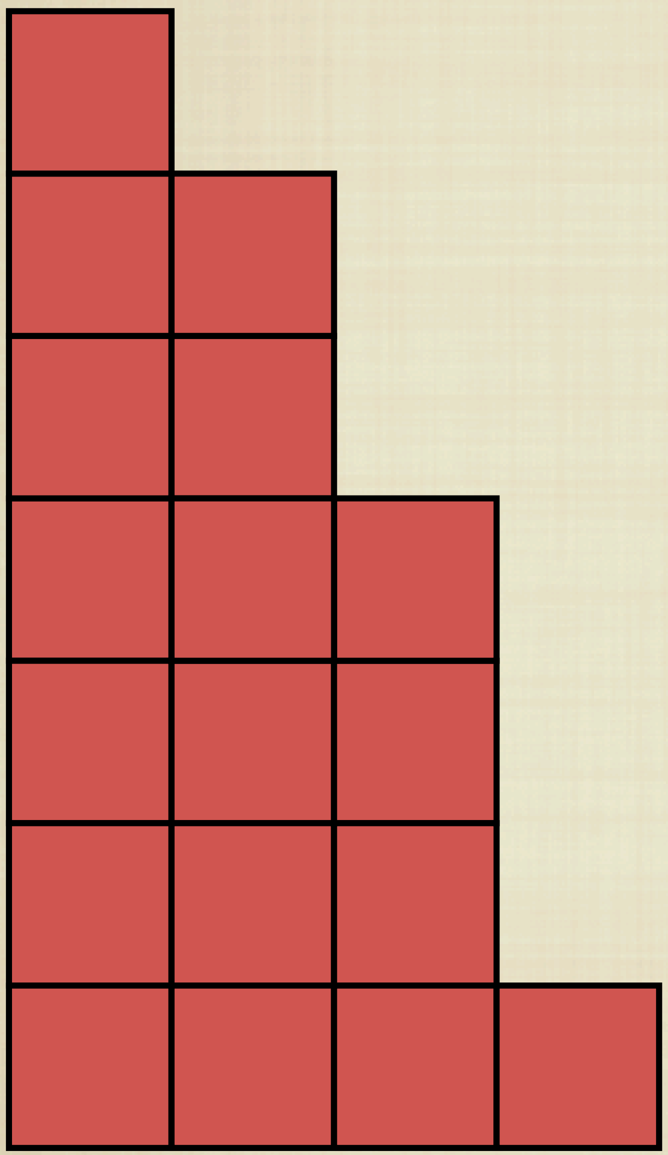
From combinatorial object to algebra



partitions

some commutative product

From combinatorial object to algebra



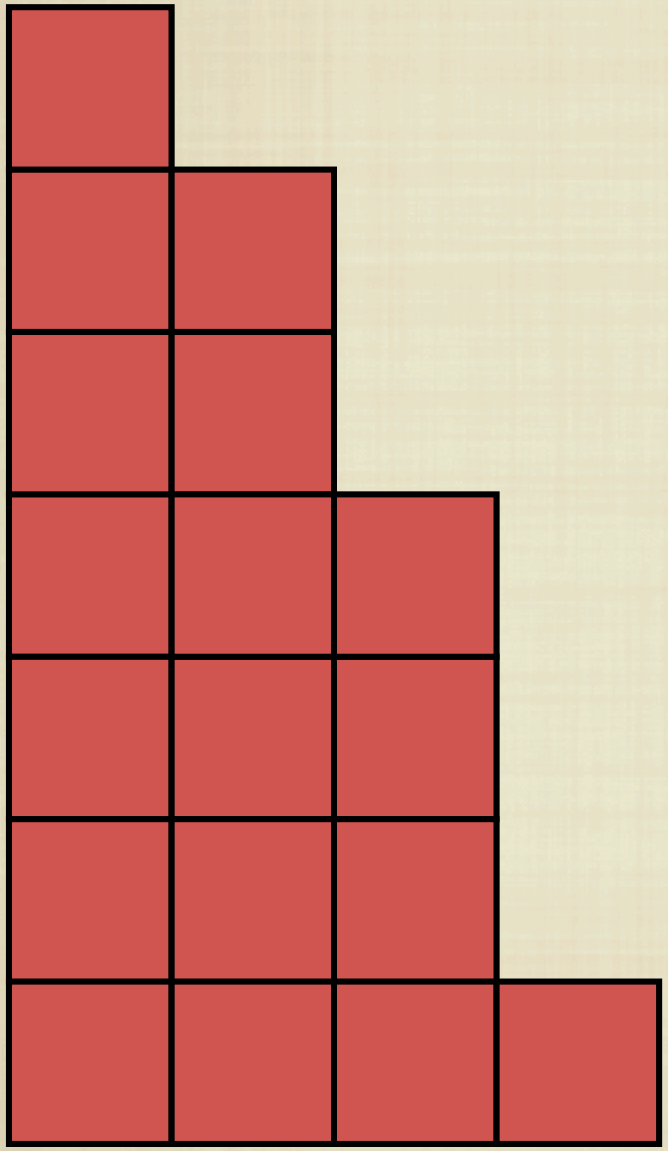
partitions

some commutative product

$K[p_1, p_2, p_3, \ldots]$

Algebra

From combinatorial object to algebra



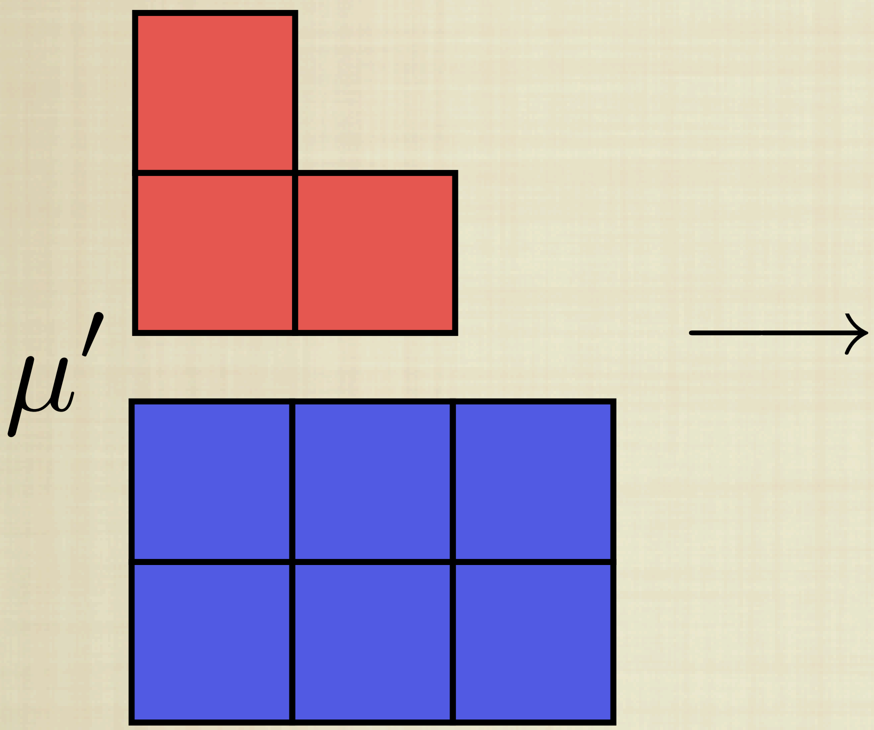
partitions

another commutative product

$$K[p_1, p_2, p_3, \ldots]$$

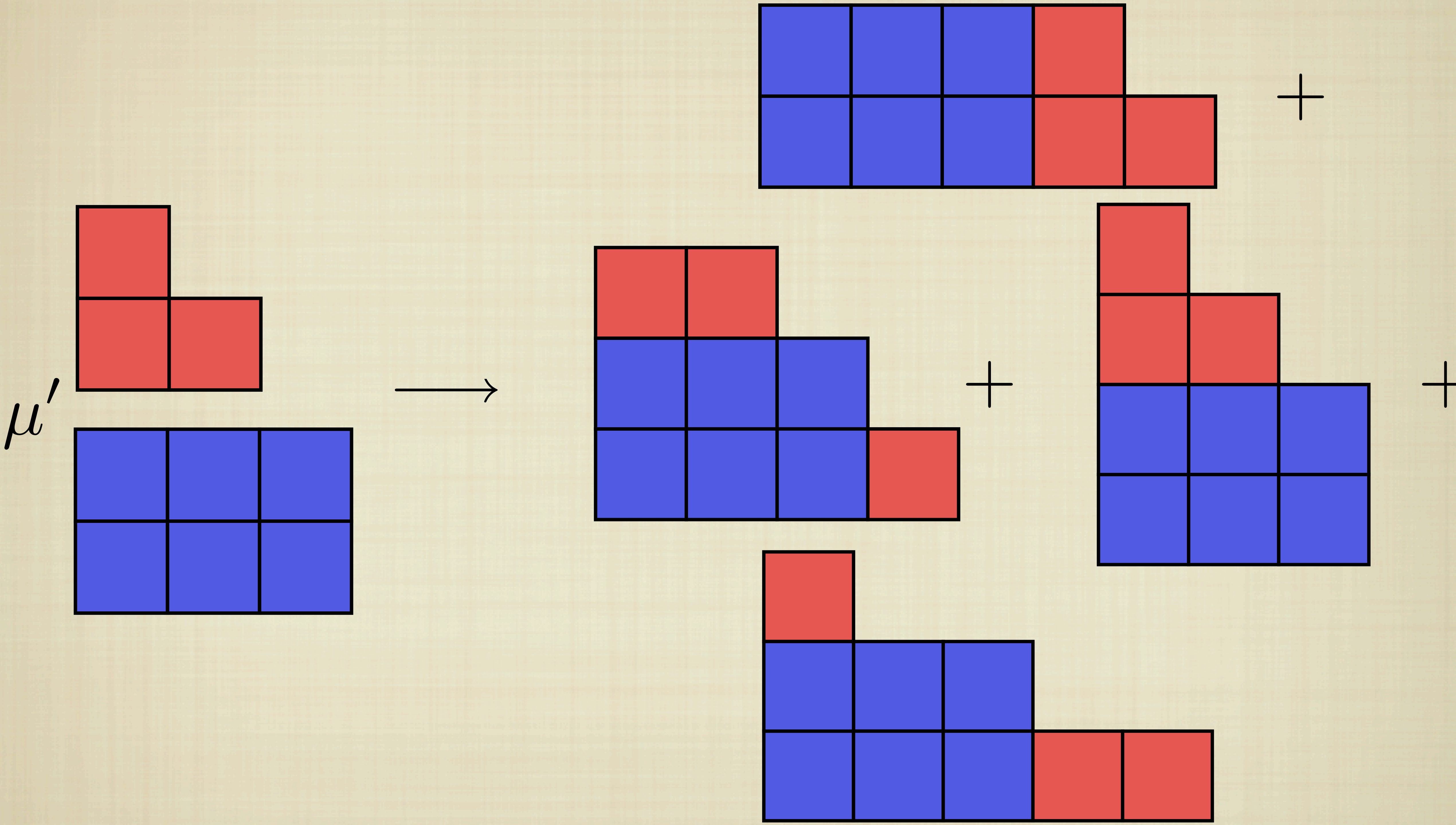
Algebra

A different commutative product



And yet the algebra which arises is isomorphic to $K[p_1, p_2, p_3, \ldots]$

A different commutative product



And yet the algebra which arises is isomorphic to $K[p_1, p_2, p_3, \ldots]$

This algebra is special

This algebra is special

- Just about any algebra with basis indexed by partitions is Isomorphic: e.g. symmetric functions, representation ring of symmetric group, ring of characters of GLn modules, cohomology rings of the grassmannians, etc.

This algebra is special

- Just about any algebra with basis indexed by partitions is Isomorphic: e.g. symmetric functions, representation ring of symmetric group, ring of characters of GLn modules, cohomology rings of the grassmannians, etc.

- Has a Hopf algebra structure product + coproduct + antipode which all interact nicely with each other

What is a Hopf algebra?

What is a Hopf algebra?

- Start with a Bialgebra

What is a Hopf algebra?

Start with a Bialgebra

Product

$$\mu : H \bigotimes H \rightarrow H$$

Coproduct

$$\Delta : H \rightarrow H \bigotimes H$$

What is a Hopf algebra?

Start with a Bialgebra

Product

$$\mu : H \bigotimes H \rightarrow H$$

Coproduct

$$\Delta : H \rightarrow H \bigotimes H$$

with unit

$$\eta : K \rightarrow H$$

and counit

$$\varepsilon : H \rightarrow K$$

What is a Hopf algebra?

Start with a Bialgebra

Product

$$\mu : H \bigotimes H \to H$$

Coproduct

$$\Delta : H \to H \bigotimes H$$

with unit

$$\eta : K \to H$$

and counit

$$\varepsilon : H \to K$$

and an antipode map

$$S : H \to H$$

What is a Hopf algebra?

Start with a Bialgebra

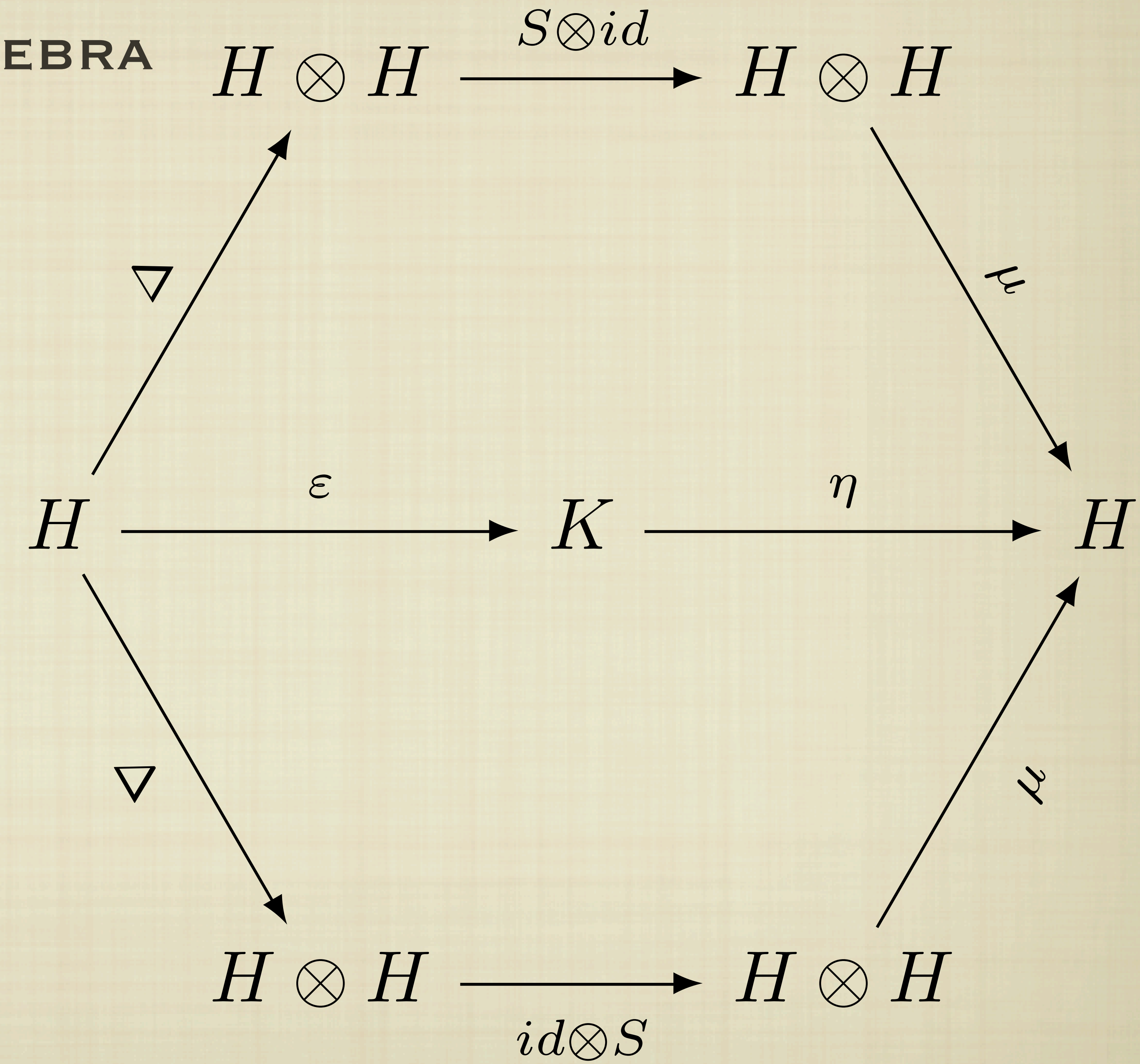
$$\mu : H \otimes H \longrightarrow H$$

$$\Delta : H \longrightarrow H \otimes H$$

$$\eta : K \longrightarrow H$$

$$\varepsilon : H \longrightarrow K$$

$$S : H \longrightarrow H$$



This diagram commutes

What is so good about a Hopf algebra?

- The graded kind associated with combinatorial objects have lots of structure
- There seems to be just "one" graded combinatorial hopf algebra for each type of combinatorial object
- Many of the combinatorial operations are reflected in the algebraic structure

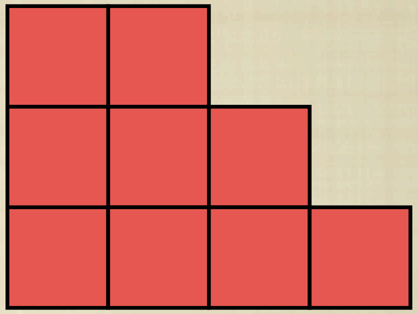
What is so good about a Hopf algebra?

- There is "usually" an internal product structure and on some bases of some algebras this is hard (but important) to explain

- Aguiar-Bergeron-Sottile says
A combinatorial Hopf algebra is a graded connected Hopf algebra with a multiplicative linear function.

CHA's in the mid-90s

THE SYMMETRIC FUNCTIONS ARE COMMUTATIVE AND GENERATED BY ONE ELEMENT AT EACH DEGREE

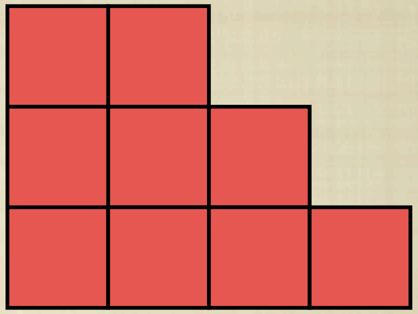


PARTITIONS

$$K[p_1, p_2, p_3, \ldots]$$

CHA's in the mid-90s

the symmetric functions are commutative and generated by one element at each degree



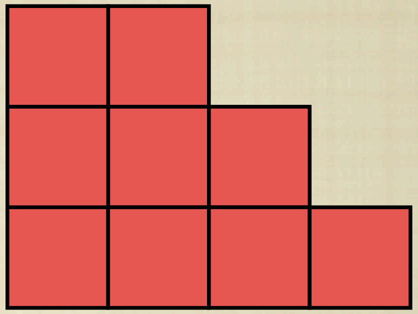
partitions

$$K[p_1, p_2, p_3, \ldots]$$

Non-commutative symmetric functions will be non-commutative and generated by one element at each degree

CHA's in the mid-90s

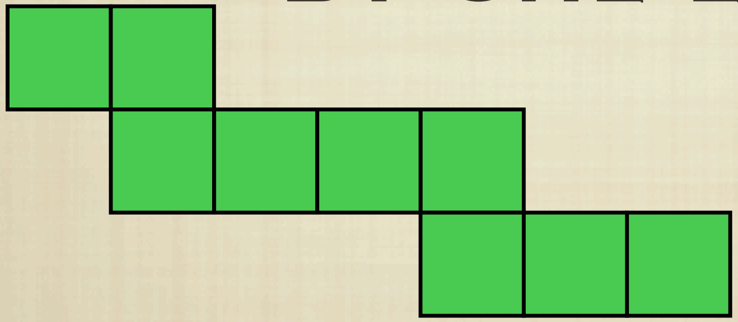
The symmetric functions are commutative and generated by one element at each degree



partitions

$$K[p_1, p_2, p_3, \ldots]$$

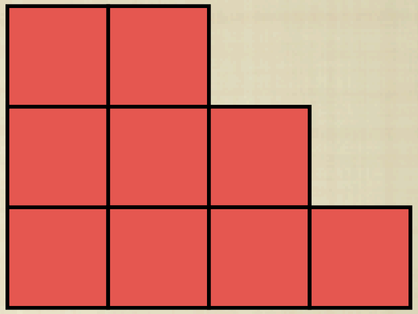
Non-commutative symmetric functions will be non-commutative and generated by one element at each degree



compositions

CHA's in the mid-90s

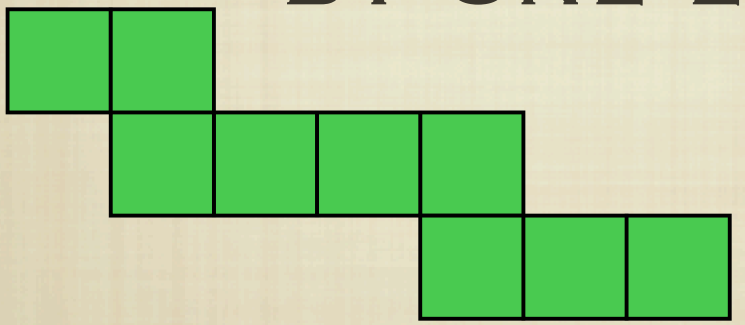
The symmetric functions are commutative and generated by one element at each degree



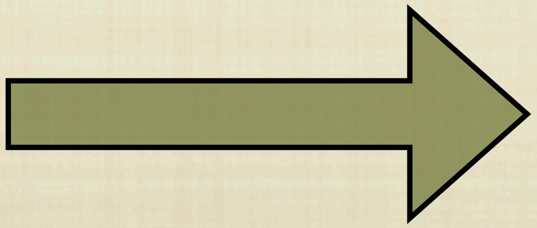
PARTITIONS

$$K[p_1, p_2, p_3, \ldots]$$

Non-commutative symmetric functions will be non-commutative and generated by one element at each degree



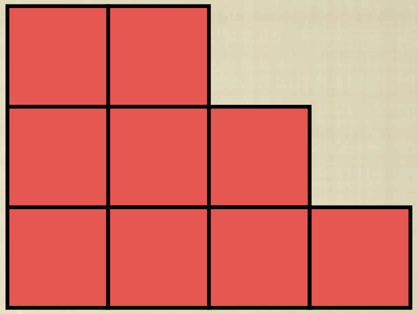
COMPOSITIONS



CONCATENATION PRODUCT

CHA's in the mid-90s

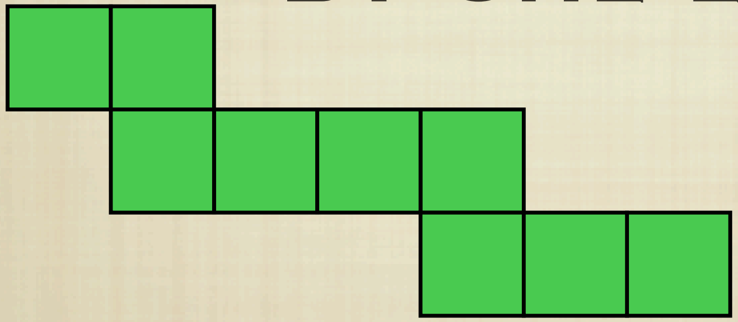
The symmetric functions are commutative and generated by one element at each degree



Partitions

$$K[p_1, p_2, p_3, \ldots]$$

Non-commutative symmetric functions will be non-commutative and generated by one element at each degree



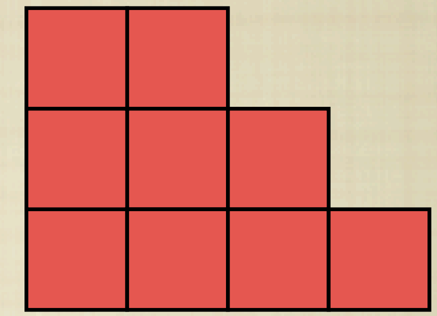
Compositions

Concatenation product

$$K\langle p_1, p_2, p_3, \ldots \rangle$$

CHA's in the mid-90s

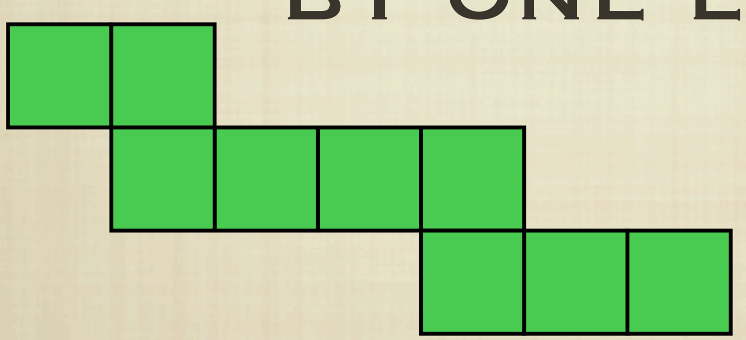
The symmetric functions are commutative and generated by one element at each degree



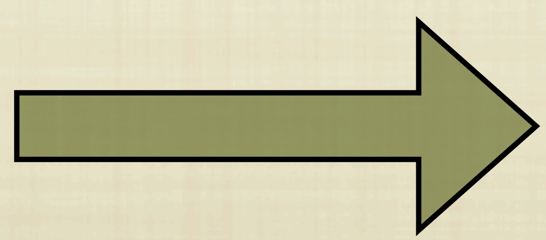
partitions

$$K[p_1, p_2, p_3, \ldots]$$

Non-commutative symmetric functions will be non-commutative and generated by one element at each degree



compositions

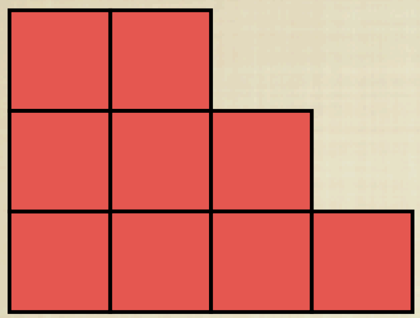


concatenation product

$$K\langle p_1, p_2, p_3, \ldots\rangle$$

I. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, and J.-Y. Thibon

CHA's in the mid-90s

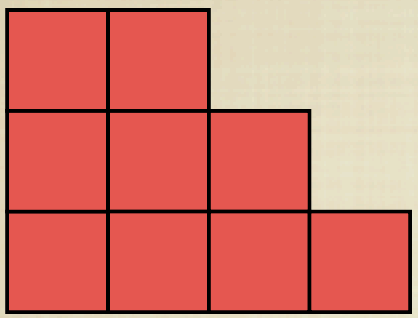


PARTITIONS

$K[p_1, p_2, p_3, \ldots]$

Sym

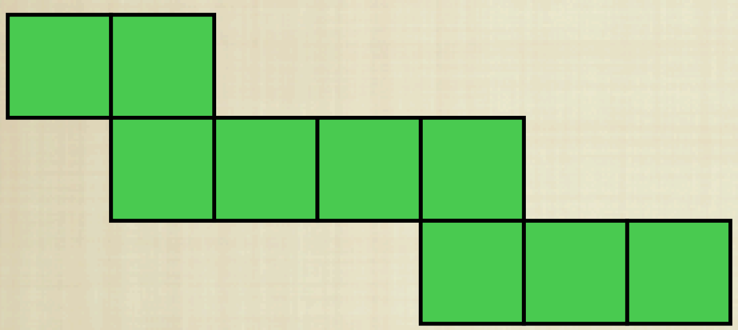
CHA's in the mid-90s



PARTITIONS

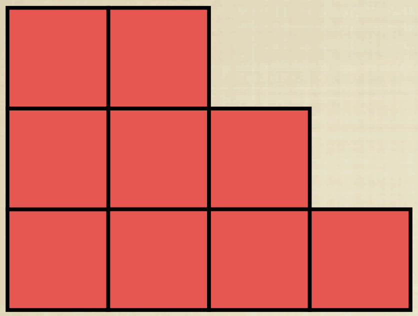
$K[p_1, p_2, p_3, \ldots]$

Sym



COMPOSITIONS

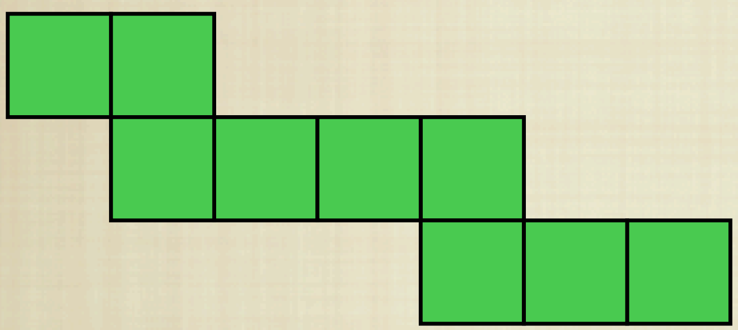
CHA's in the mid-90s



PARTITIONS

$$K[p_1, p_2, p_3, \ldots]$$

Sym

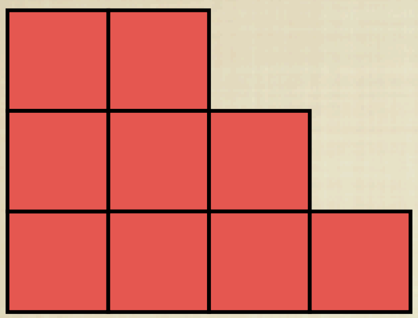


COMPOSITIONS

$$K \langle p_1, p_2, p_3, \ldots \rangle$$

GKLLRT ('95)

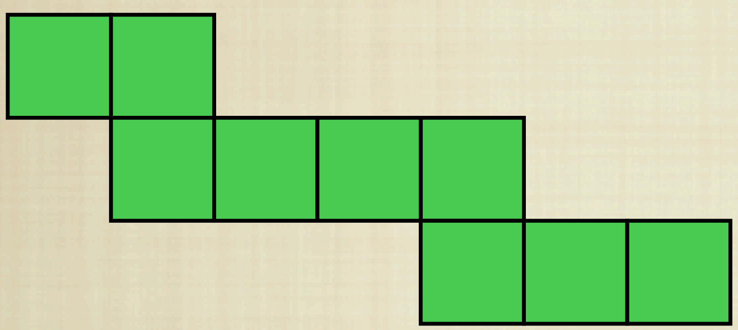
CHA's in the mid-90s



PARTITIONS

$$K[p_1, p_2, p_3, \ldots]$$

$$Sym$$



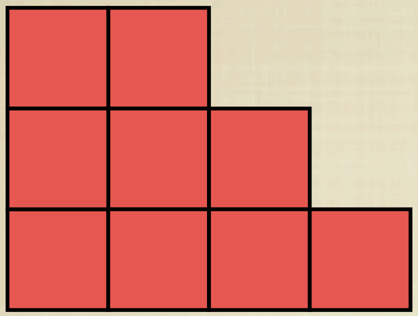
COMPOSITIONS

$$K\left\langle p_1, p_2, p_3, \ldots \right\rangle$$

$$NSym$$

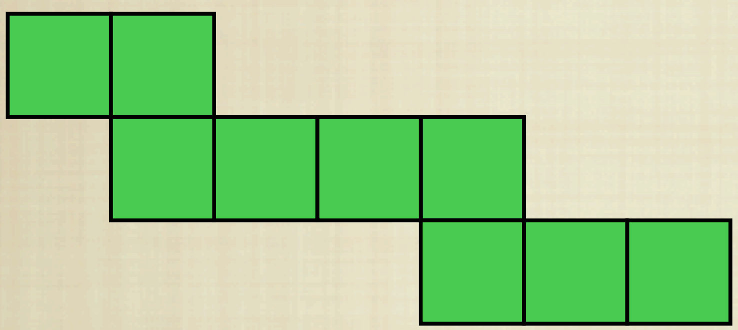
GKLLRT ('95)

CHA's in the mid-90s



PARTITIONS

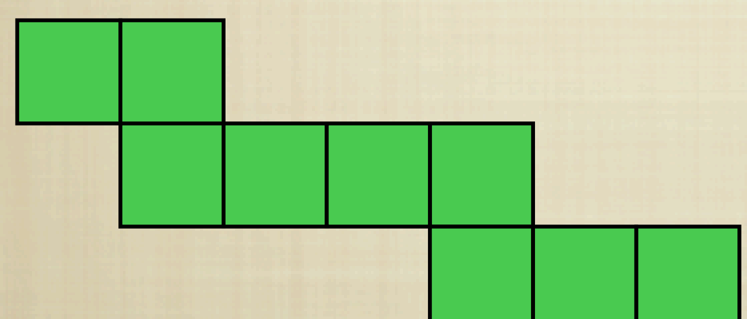
$K[p_1, p_2, p_3, \ldots]$ Sym



COMPOSITIONS

$K\langle p_1, p_2, p_3, \ldots \rangle$ $NSym$

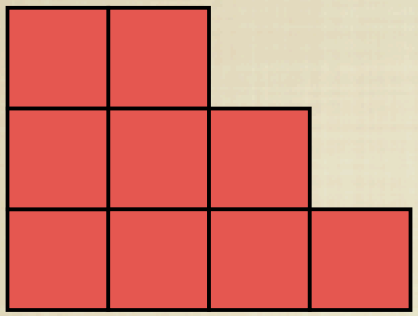
GKLLRT ('95)



COMPOSITIONS

$QSym$

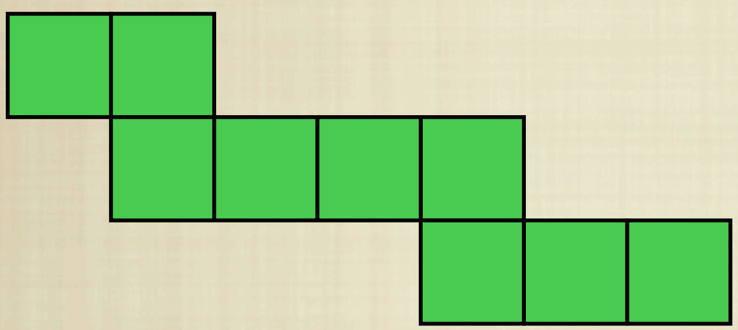
CHA's in the mid-90s



PARTITIONS

$K[p_1, p_2, p_3, \ldots]$

Sym

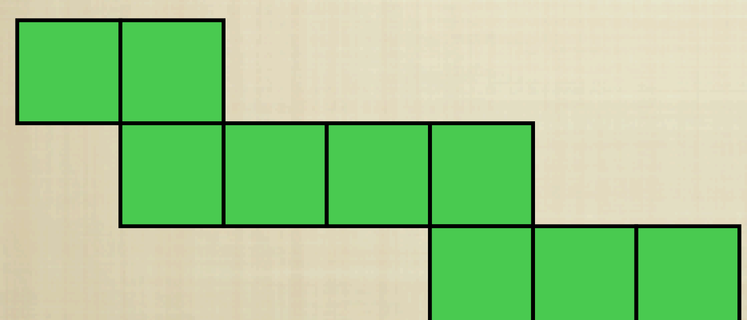


COMPOSITIONS

$K\langle p_1, p_2, p_3, \ldots \rangle$

$NSym$

GKLLRT ('95)

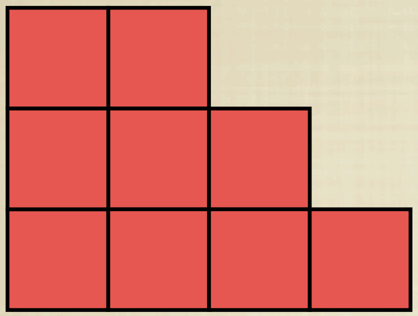


COMPOSITIONS

Commutative algebra
of Quasi-symmetric functions

$QSym$

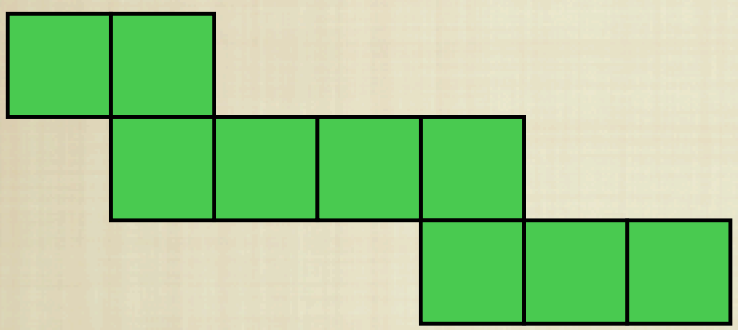
CHA's in the mid-90s



PARTITIONS

$K[p_1, p_2, p_3, \ldots]$

Sym

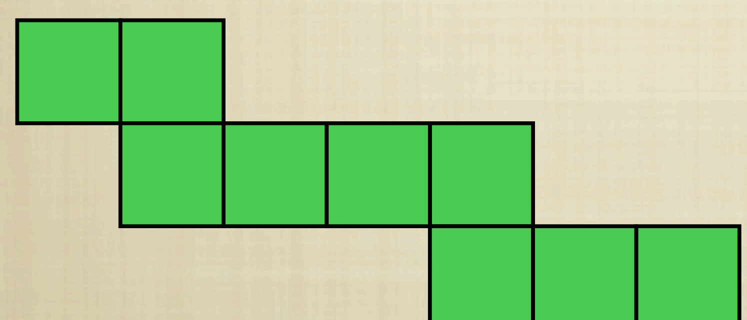


COMPOSITIONS

$K\langle p_1, p_2, p_3, \ldots \rangle$

$NSym$

GKLLRT ('95)



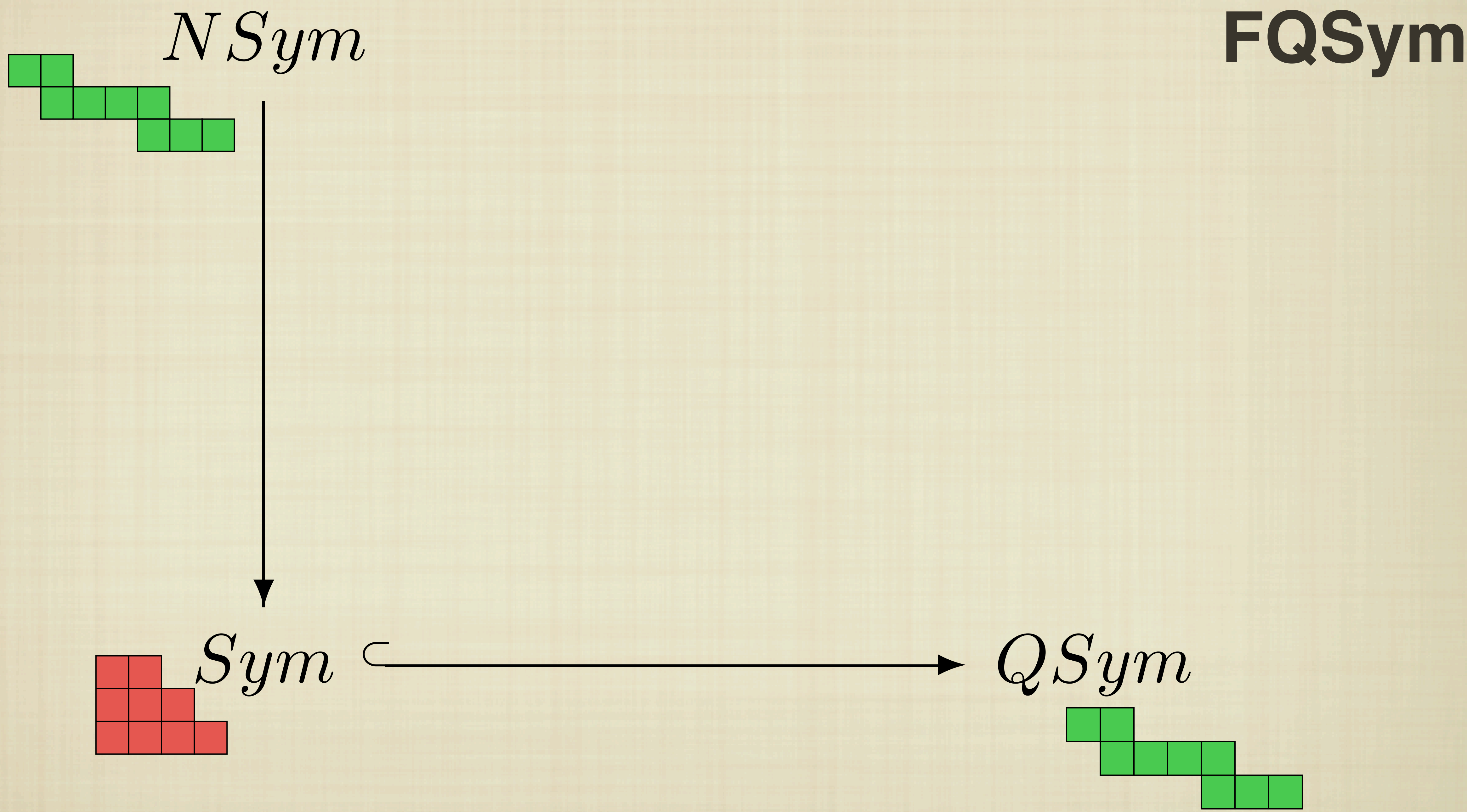
COMPOSITIONS

Commutative algebra of Quasi-symmetric functions

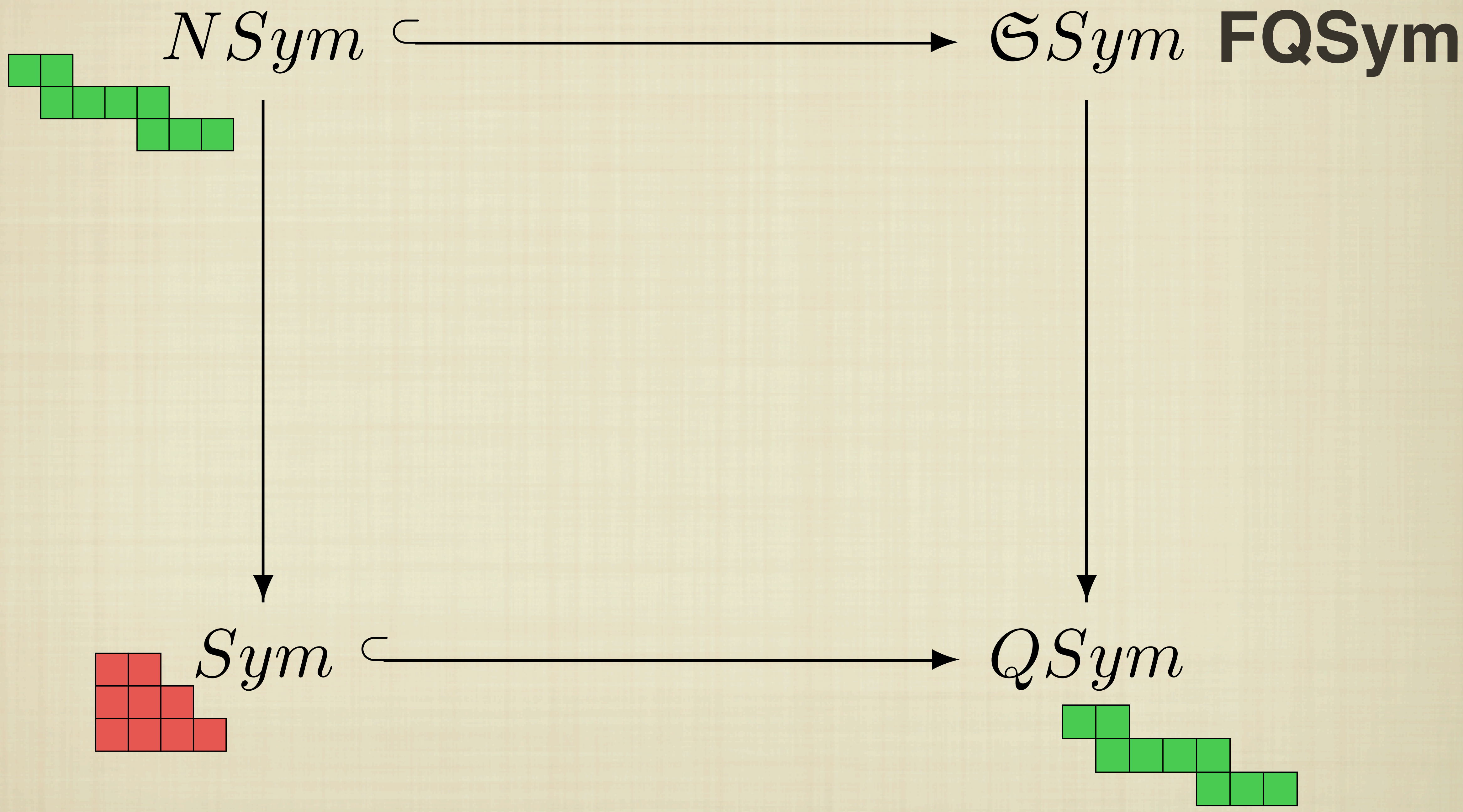
$QSym$

Gessel ('84)

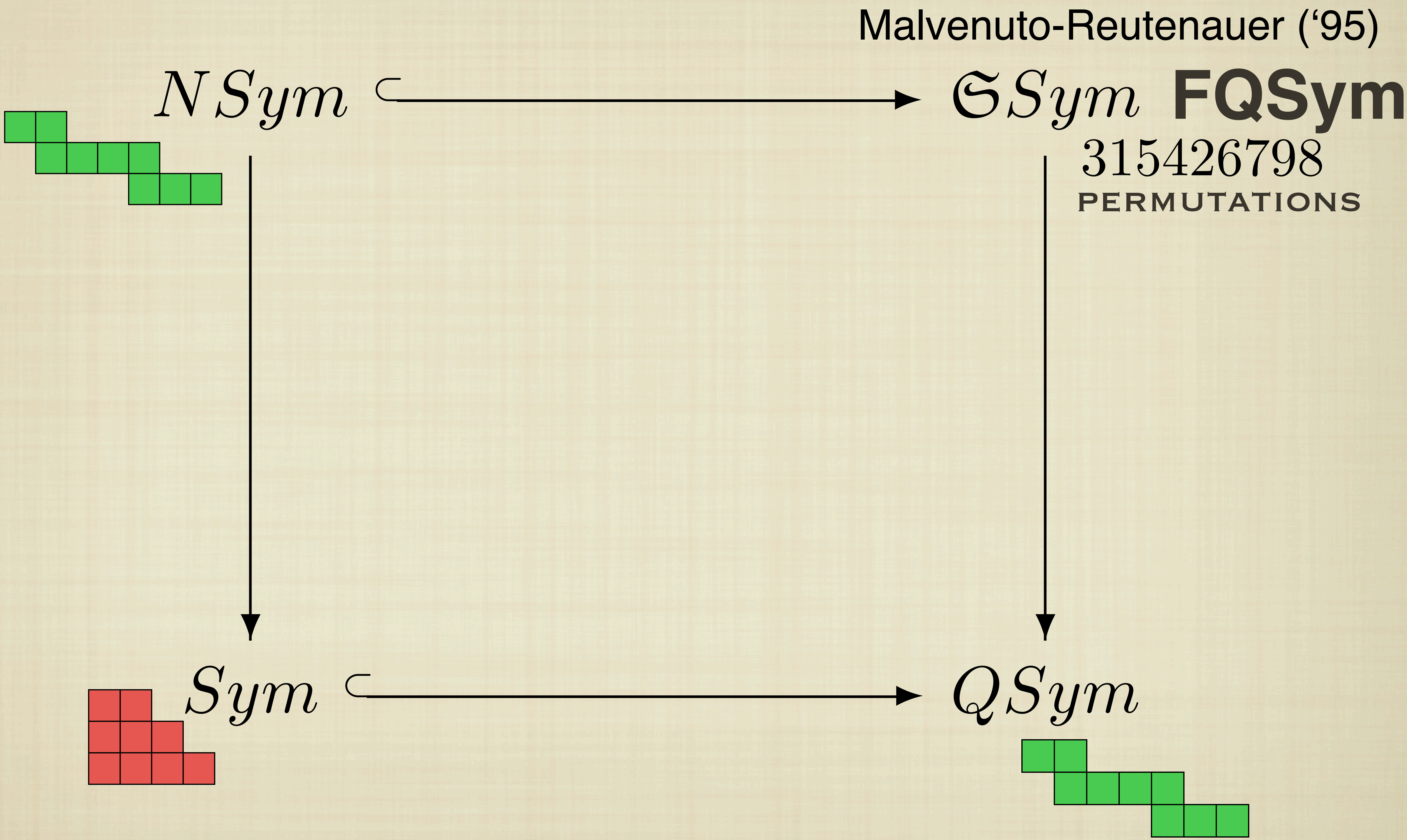
CHA's in the mid-90s



CHA's in the mid-90s



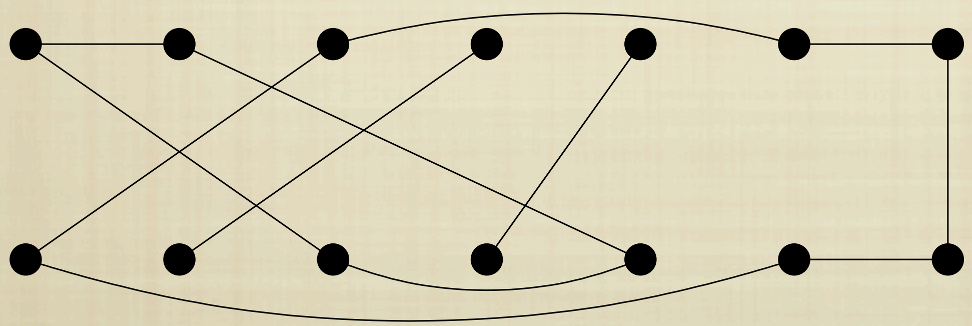
CHA's in the mid-90s



CHA's in the 90s+

CHA's in the 90s+

uniform block permutations



Aguiar-Orellana '05

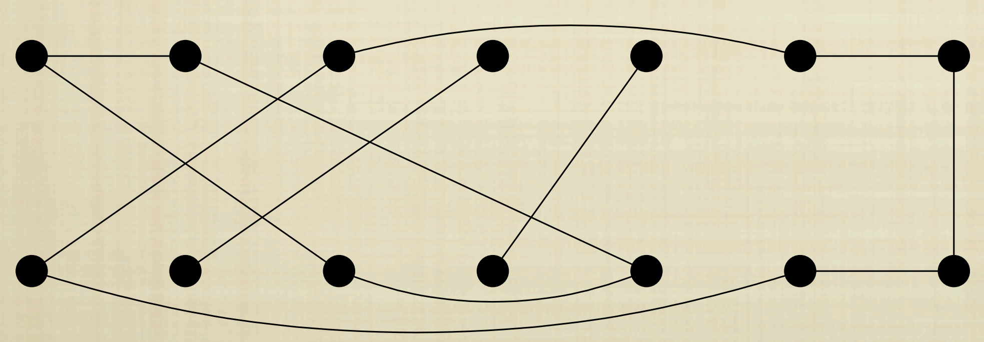
CHA's in the 90s+

Tableaux

7				
6	8	11		
4	5	9	12	14
1	2	3	10	13

Poirier-Reutenauer '95

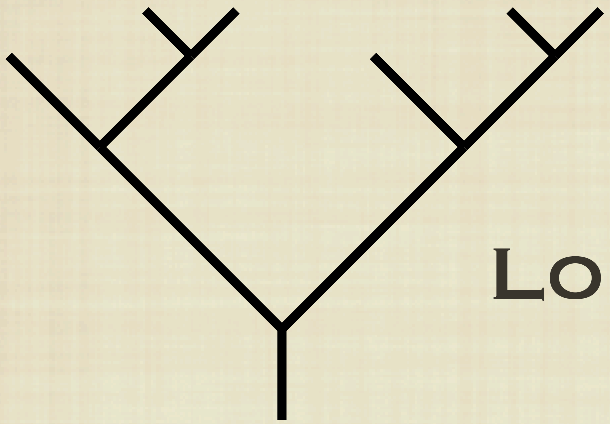
uniform block permutations



Aguiar-Orellana '05

CHA's in the 90s+

BINARY TREES



Connes-Kreimer '98
Grossman-Larson '89

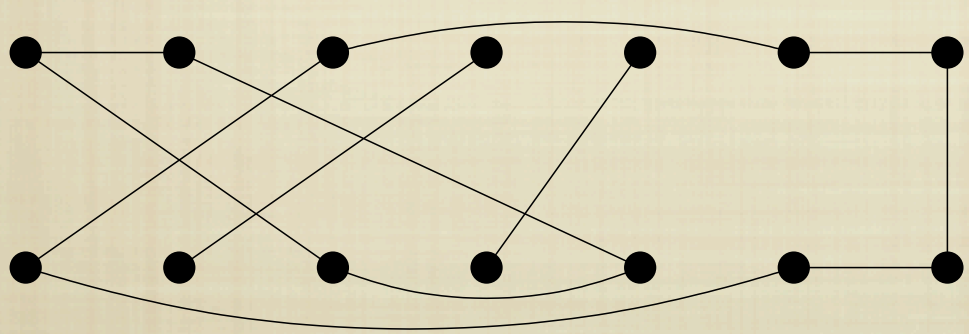
Loday-Ronco '98

TABLEAUX

7				
6	8	11		
4	5	9	12	14
1	2	3	10	13

Poirier-Reutenauer '95

UNIFORM BLOCK PERMUTATIONS

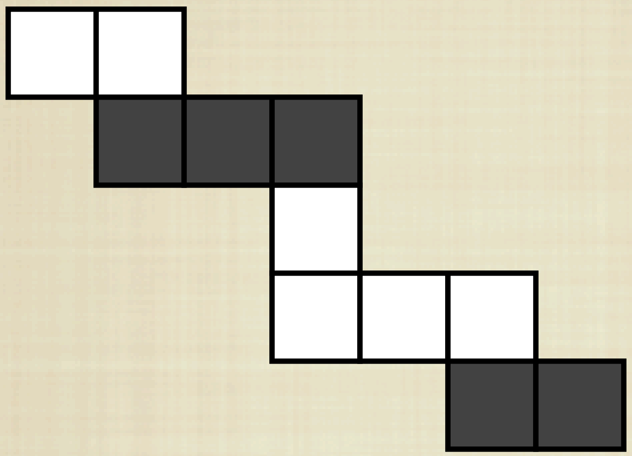


Aguiar-Orellana '05

CHA's in the 90s+

CHA's in the 90s+

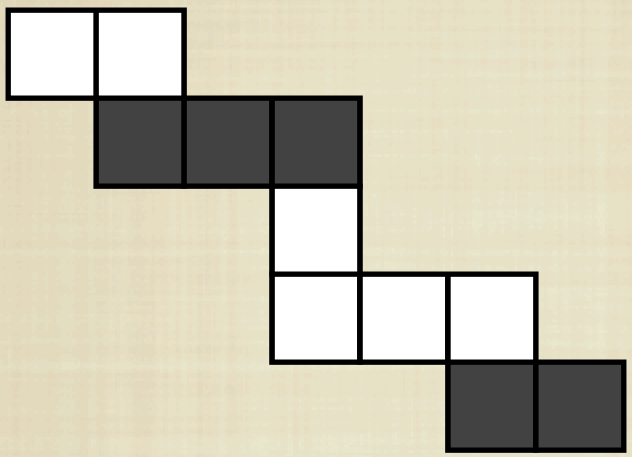
Signed Compositions



Mantaci-Reutenauer '95

CHA's in the 90s+

Signed Compositions



Mantaci-Reutenauer '95

Packed Words
Set Compositions

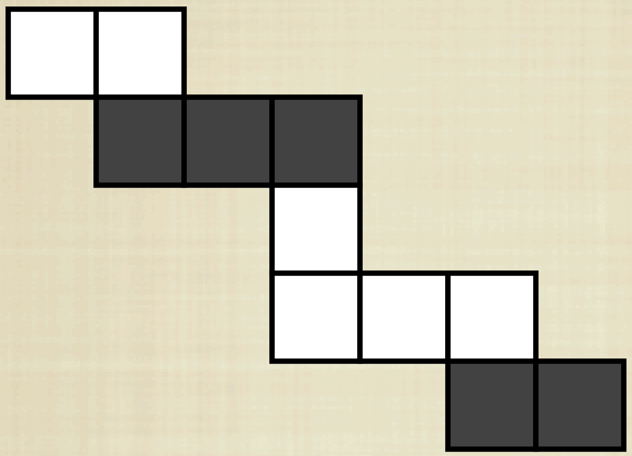
$$(\{5,6\},\{2,8\},\{1,3,4\},\{7\})$$

$$1133212331$$

Hivert '99

CHA's in the 90s+

Signed Compositions



Mantaci-Reutenauer '95

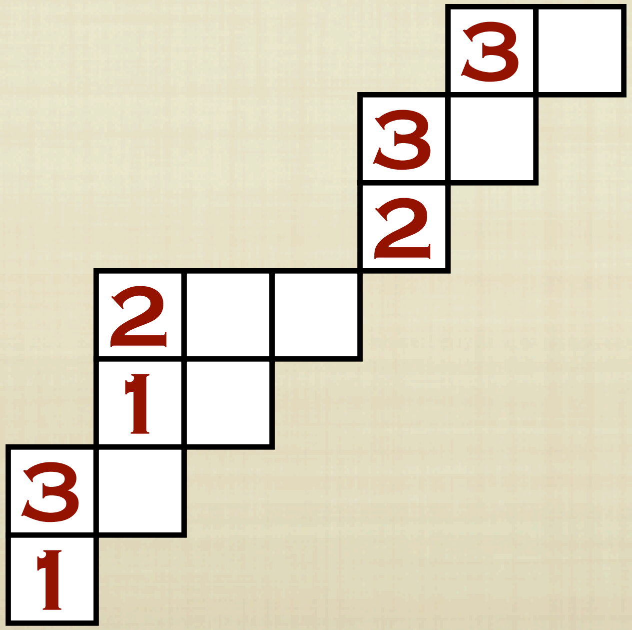
Packed Words
Set Compositions

$$(\{5,6\},\{2,8\},\{1,3,4\},\{7\})$$

$$1133212331$$

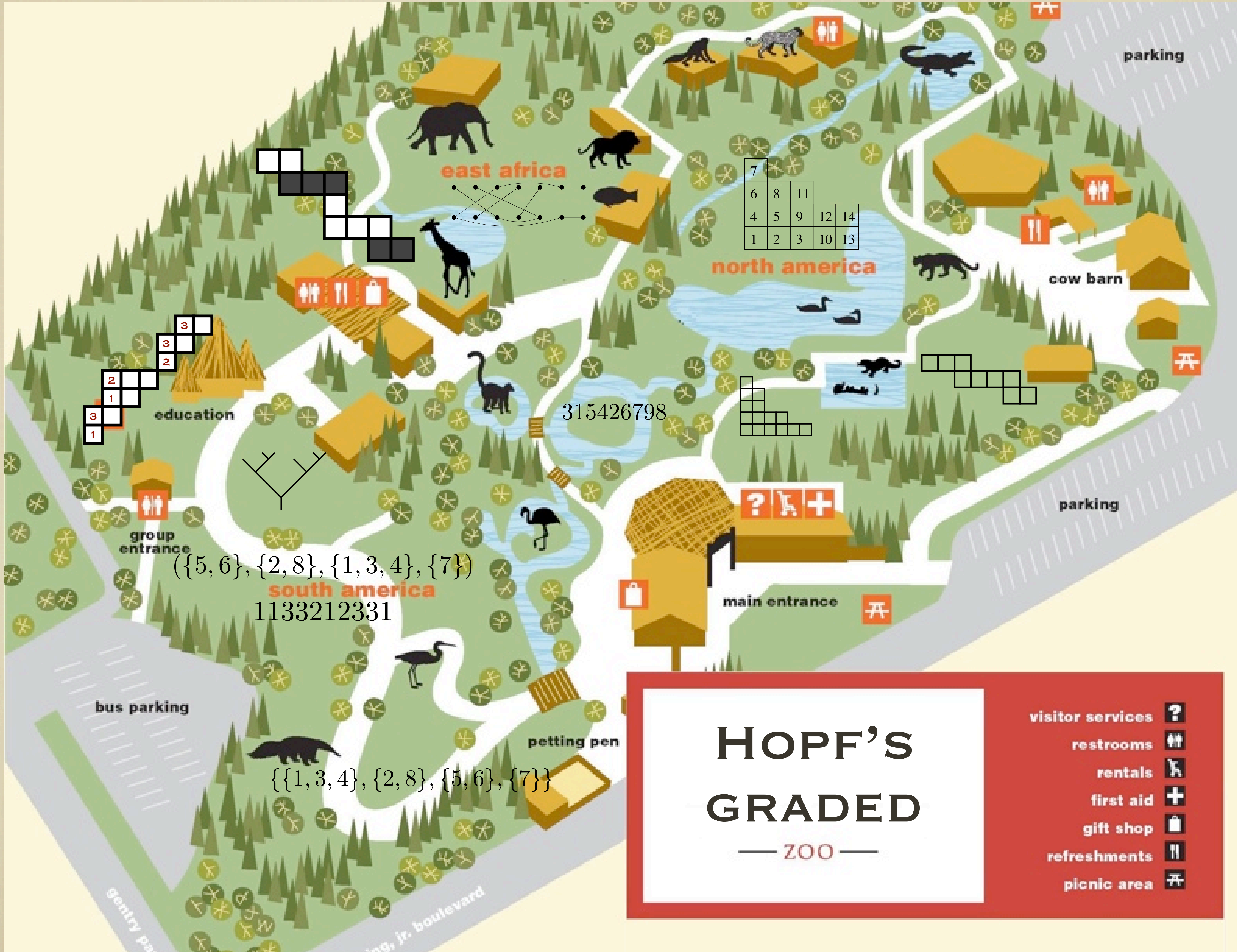
Hivert '99

Parking Functions



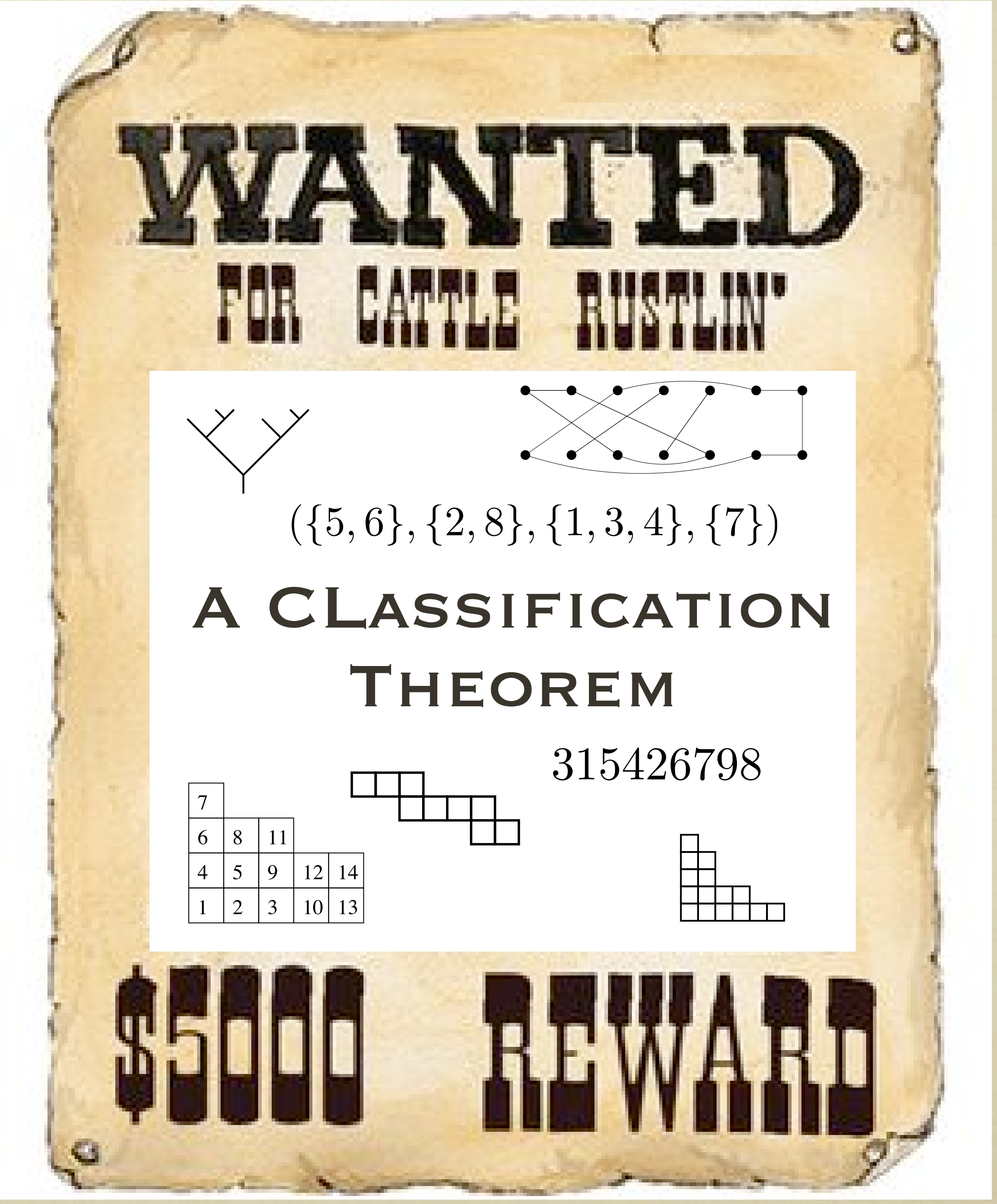
Novelli-Thibon '04

A zoo of Hopf Algberas



A goal of research on graded Hopf algebras?

- One Goal of determining the Hopf algebras associated to combinatorial objects is to try and arrive at a classification theorem for Graded combinatorial Hopf algebras

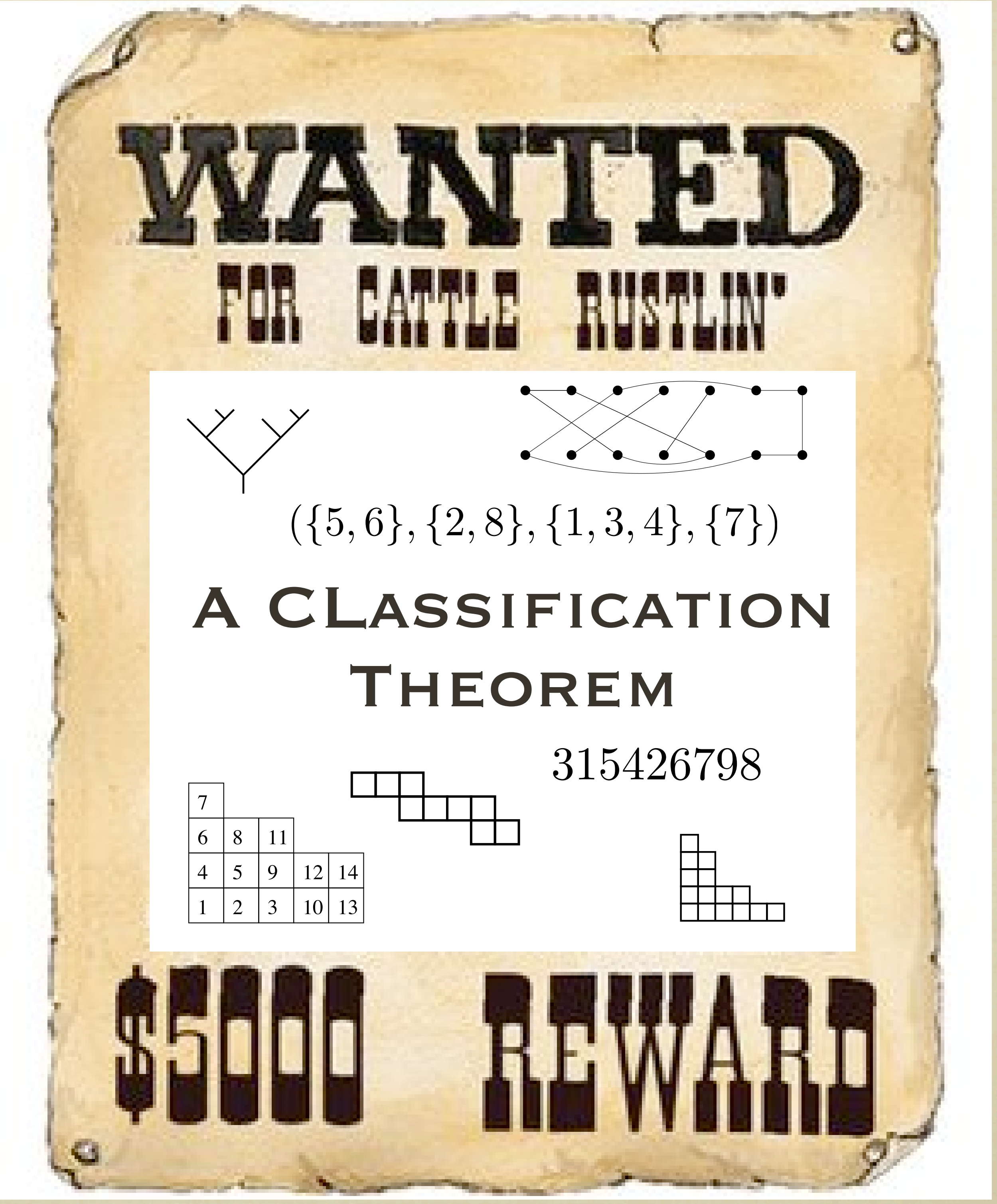


A goal of research on graded Hopf algebras?

- One Goal of determining the Hopf algebras associated to combinatorial objects is to try and arrive at a classification theorem for Graded combinatorial Hopf algebras

Aguiar

Species $\longleftrightarrow$ CHAs



A goal of research on graded Hopf algebras?

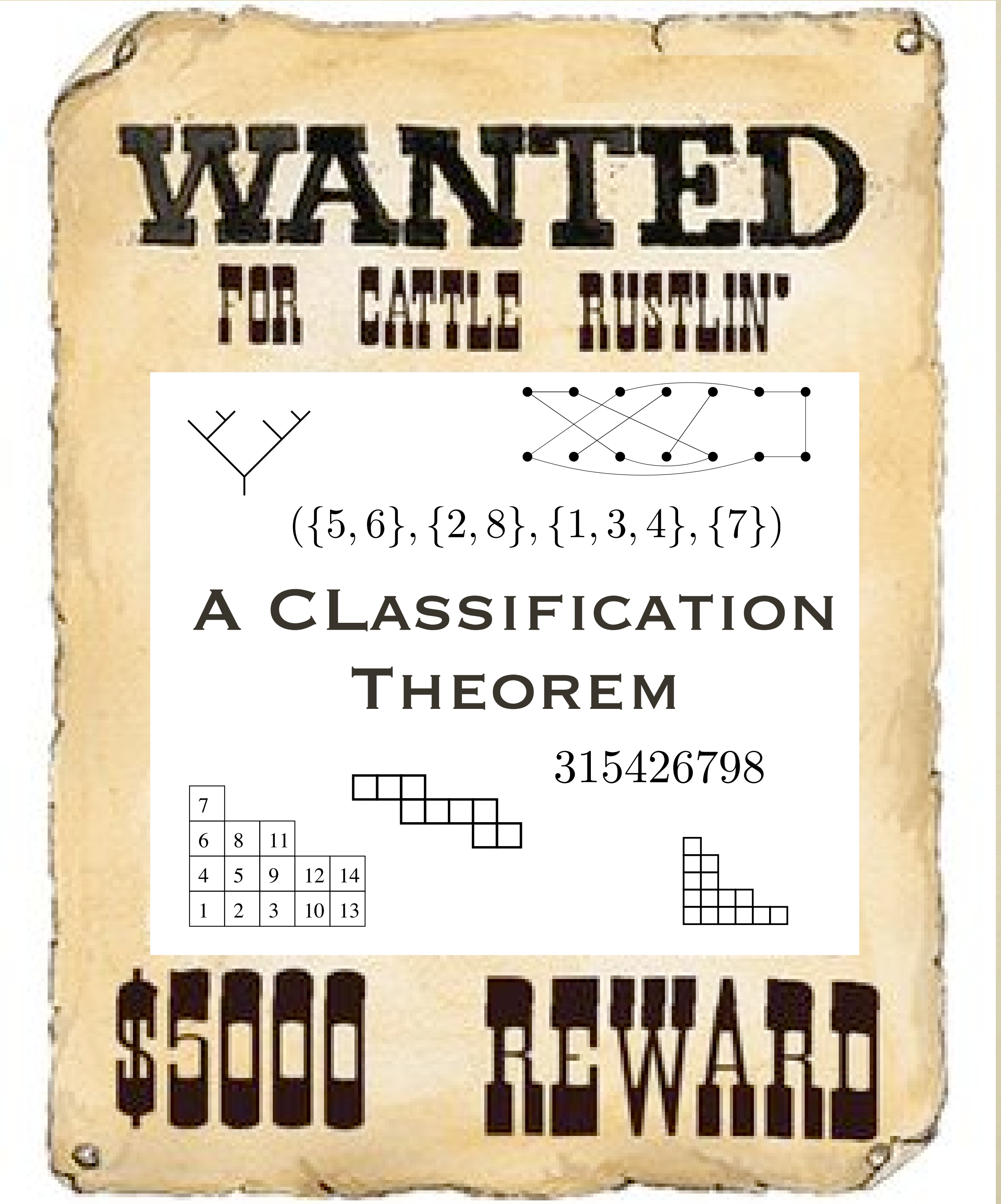
- One Goal of determining the Hopf algebras associated to combinatorial objects is to try and arrive at a classification theorem for Graded combinatorial Hopf algebras

Aguiar

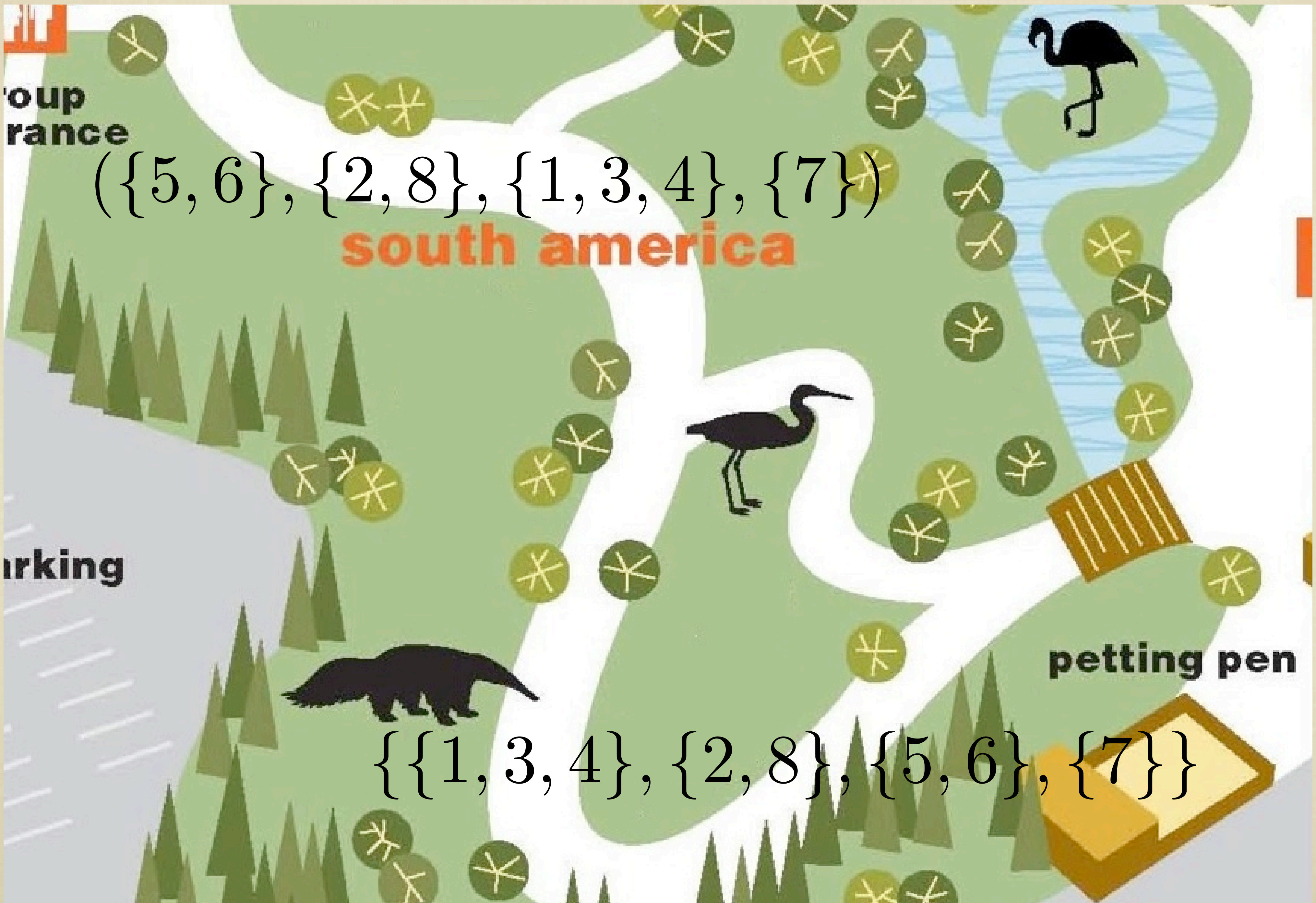
Species $\longleftrightarrow$ CHAs

(N)Bergeron-Lam-Li

Rep theory



Hopf algebras of set partitions/compositions



Hopf algebras of set partitions/compositions

$$(\{5,6\},\{2,8\},\{1,3,4\},\{7\})$$

$$\{\{1,3,4\},\{2,8\},\{5,6\},\{7\}\}$$

Hopf algebras of set partitions/compositions

$$(\{5,6\},\{2,8\},\{1,3,4\},\{7\})$$

Set composition definition:

$$(S_1, S_2, \ldots, S_k) : S_1 \uplus S_2 \uplus \cdots \uplus S_k = \{1, 2, \cdots, n\}$$

$$\{\{1,3,4\},\{2,8\},\{5,6\},\{7\}\}$$

Hopf algebras of set partitions/compositions

$$(\{5,6\},\{2,8\},\{1,3,4\},\{7\})$$

Set composition definition:

$$(S_1, S_2, \ldots, S_k) : S_1 \uplus S_2 \uplus \cdots \uplus S_k = \{1, 2, \cdots, n\}$$

Set partition definition:

$$\{S_1, S_2, \ldots, S_k\} : S_1 \uplus S_2 \uplus \cdots \uplus S_k = \{1, 2, \cdots, n\}$$

$$\{\{1,3,4\},\{2,8\},\{5,6\},\{7\}\}$$

Another type of non-commutative symmetric functions

Sym Invariants under the left action on the polynomial ring $\mathbb{Q}[X_n]$

$$\sigma(x_i) = x_{\sigma(i)}$$

$NSym$ Free algebra generated by one element at each degree

$NCSym$ Invariants under the left action on the non-comm poly ring $\mathbb{Q}\langle X_n\rangle$

Wolf 1936, Rosas-Sagan '03

Monomial $\longrightarrow$ set partition

For each monomial

$$x_{i_1} x_{i_2} \cdots x_{i_n}$$

Associate a set partition

$$\nabla(i_1, i_2, \ldots, i_n) = A \vdash [n]$$
$$r, s \in A_d \text{ whenever } i_r = i_s$$

$$m_A[X_n] = \sum_{\nabla(i_1, i_2, \ldots, i_n) = A} x_{i_1} x_{i_2} \cdots x_{i_n}$$

Example:

$$m_{\{\{1,3,5\},\{2,4\}\}}[X_n] = x_1x_2x_1x_2x_1 + x_2x_1x_2x_1x_2 + x_1x_3x_1x_3x_1 +$$

$$x_3x_1x_3x_1x_3 + x_2x_3x_2x_3x_2 + x_3x_2x_3x_2x_3 + x_1x_4x_1x_4x_1 + \cdots$$

Example:

$$m_{\{\{1,3,5\},\{2,4\}\}}[X_n] = x_1 x_2 x_1 x_2 x_1 + x_2 x_1 x_2 x_1 x_2 + x_1 x_3 x_1 x_3 x_1 +$$

$$x_3 x_1 x_3 x_1 x_3 + x_2 x_3 x_2 x_3 x_2 + x_3 x_2 x_3 x_2 x_3 + x_1 x_4 x_1 x_4 x_1 + \cdots$$

Example:

$$m_{\{\{1,3,5\},\{2,4\}\}}[X_n] = x_1 x_2 x_1 x_2 x_1 + x_2 x_1 x_2 x_1 x_2 + x_1 x_3 x_1 x_3 x_1 +$$

$$x_3 x_1 x_3 x_1 x_3 + x_2 x_3 x_2 x_3 x_2 + x_3 x_2 x_3 x_2 x_3 + x_1 x_4 x_1 x_4 x_1 + \cdots$$

Example:

$$m_{\{\{1,3,5\},\{2,4\}\}}[X_n] = x_1 x_2 x_1 x_2 x_1 + x_2 x_1 x_2 x_1 x_2 + x_1 x_3 x_1 x_3 x_1 +$$

$$x_3 x_1 x_3 x_1 x_3 + x_2 x_3 x_2 x_3 x_2 + x_3 x_2 x_3 x_2 x_3 + x_1 x_4 x_1 x_4 x_1 + \cdots$$

This is a non commutative polynomial for a given n

Example:

$$m_{\{\{1,3,5\},\{2,4\}\}}[X_n] = x_1x_2x_1x_2x_1 + x_2x_1x_2x_1x_2 + x_1x_3x_1x_3x_1 +$$

$$x_3x_1x_3x_1x_3 + x_2x_3x_2x_3x_2 + x_3x_2x_3x_2x_3 + x_1x_4x_1x_4x_1 + \cdots$$

This is a non commutative polynomial for a given n

Consider the elements m_A to be the object by letting the number of variables $n \to \infty$ in $m_A[X_n]$

Combinatorial rule for product

$$m_{\{\{1,3\},\{2,4\}\}} \cdot m_{\{\{1,2,4\},\{3\}\}} =$$

Combinatorial rule for product

$$m_{\{\{1,3\},\{2,4\}\}} \cdot m_{\{\{1,2,4\},\{3\}\}} =$$

$$m_{\{\{1,3\},\{2,4\},\{5,6,8\},\{7\}\}} + m_{\{\{1,3,5,6,8\},\{2,4\},\{7\}\}} +$$

$$m_{\{\{1,3\},\{2,4,5,6,8\},\{7\}\}} + m_{\{\{1,3,7\},\{2,4\},\{5,6,8\}\}} +$$

$$m_{\{\{1,3\},\{2,4,7\},\{5,6,8\}\}} + m_{\{\{1,3,7\},\{2,4,5,6,8\}\}} +$$

$$m_{\{\{1,3,5,6,8\},\{2,4,7\}\}}$$

Combinatorial rule for product

$$m_{\{\{1,3\},\{2,4\}\}} \cdot m_{\{\{1,2,4\},\{3\}\}} =$$

$$m_{\{\{1,3\},\{2,4\},\{5,6,8\},\{7\}\}} + m_{\{\{1,3,5,6,8\},\{2,4\},\{7\}\}} +$$

$$m_{\{\{1,3\},\{2,4,5,6,8\},\{7\}\}} + m_{\{\{1,3,7\},\{2,4\},\{5,6,8\}\}} +$$

$$m_{\{\{1,3\},\{2,4,7\},\{5,6,8\}\}} + m_{\{\{1,3,7\},\{2,4,5,6,8\}\}} +$$

$$m_{\{\{1,3,5,6,8\},\{2,4,7\}\}}$$

Coproduct

Inspiration:

$$m_A[X_n, Y_m]$$

$$\Delta(m_{\{\{1\},\{2,4,5,6\},\{3,7\}\}}) =$$

Coproduct

Inspiration:

$$m_A[X_n, Y_m]$$

$$\Delta(m_{\{\{1\},\{2,4,5,6\},\{3,7\}\}}) =$$

$$m_{\{\{1\},\{2,4,5,6\},\{3,7\}\}} \otimes 1 + m_{\{\{1,3,4,5\},\{2,6\}\}} \otimes m_{\{\{1\}\}} +$$

$$m_{\{\{1\},\{2,3\}\}} \otimes m_{\{\{1,2,3,4\}\}} + m_{\{\{1\},\{2,3,4,5\}\}} \otimes m_{\{\{1,2\}\}} +$$

$$m_{\{\{1\}\}} \otimes m_{\{\{1,3,4,5\},\{2,6\}\}} + m_{\{\{1,2,3,4\}\}} \otimes m_{\{\{1\},\{2,3\}\}} +$$

$$m_{\{\{1,2\}\}} \otimes m_{\{\{1\},\{2,3,4,5\}\}} + 1 \otimes m_{\{\{1\},\{2,4,5,6\},\{3,7\}\}}$$

Coproduct

Inspiration:

$$m_A[X_n, Y_m]$$

$$\Delta(m_{\{\{1\},\{2,4,5,6\},\{3,7\}\}}) =$$

$$m_{\{\{1\},\{2,4,5,6\},\{3,7\}\}} \otimes 1 + m_{\{\{1,3,4,5\},\{2,6\}\}} \otimes m_{\{\{1\}\}} +$$

$$m_{\{\{1\},\{2,3\}\}} \otimes m_{\{\{1,2,3,4\}\}} + m_{\{\{1\},\{2,3,4,5\}\}} \otimes m_{\{\{1,2\}\}} +$$

$$m_{\{\{1\}\}} \otimes m_{\{\{1,3,4,5\},\{2,6\}\}} + m_{\{\{1,2,3,4\}\}} \otimes m_{\{\{1\},\{2,3\}\}} +$$

$$m_{\{\{1,2\}\}} \otimes m_{\{\{1\},\{2,3,4,5\}\}} + 1 \otimes m_{\{\{1\},\{2,4,5,6\},\{3,7\}\}}$$

Definition of $NCSym$

$$NCSym = \bigoplus_{n \geq 0} \mathcal{L}\{m_A : A \vdash [n]\}$$

Non-commutative
Co-commutative
Hopf algebra of set partitions

Analogy between Sym and NCSym

$$S(V^*) = \text{symmetric tensor algebra}$$
$$\simeq \mathbb{Q}[X_n]$$

$$T(V^*) = \text{tensor algebra}$$
$$\simeq \mathbb{Q}\left\langle X_n \right\rangle$$

Sym is to $S(V^*)$ as $NCSym$ is to $T(V^*)$

Properties of NCSym

Properties of NCSym

- Non-commutative and co-commutative

Properties of NCSym

- Non-commutative and co-commutative
- Has bases analogous to power, elementary, homogeneous, monomial symmetric functions in the algebra of Sym (Rosas-Sagan '03, Bergeron-Brlek)

Properties of NCSym

- Non-commutative and co-commutative
- Has bases analogous to power, elementary, homogeneous, monomial symmetric functions in the algebra of Sym (Rosas-Sagan '03, Bergeron-Brlek)
- The dimension of the part of degree n are the bell numbers

  1, 1, 2, 5, 15, 52, 203, 877, 4140, ...

Some questions to ask

Some questions to ask

- what is a fundamental basis of this space (analogue of Schur basis)?

Some questions to ask

- what is a fundamental basis of this space (analogue of Schur basis)?

- What is the connection with representation theory?

Some questions to ask

- what is a fundamental basis of this space (analogue of Schur basis)?

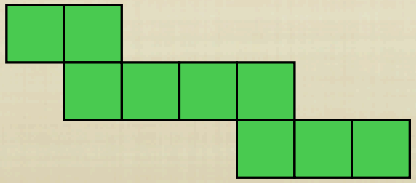
- What is the connection with representation theory?

- What is the structure of this algebra?

Some questions to ask

- what is a fundamental basis of this space (analogue of Schur basis)?

- What is the connection with representation theory?
- What is the structure of this algebra?
- What is the relationship with the 'other' non-commutative symmetric functions? (NSym)



compositions

$$\{\{1,3,4\},\{2,8\},\{5,6\},\{7\}\}$$

set partitions

The connection between NSym and NCSym

NCSym has graded dimensions

1, 1, 2, 5, 15, 52, 203, 877, 4140, ...

NSym has graded dimensions

1, 1, 2, 4, 8, 16, 32, 64, 128, ...

The connection between NSym and NCSym

NCSym has graded dimensions

1, 1, 2, 5, 15, 52, 203, 877, 4140, ...

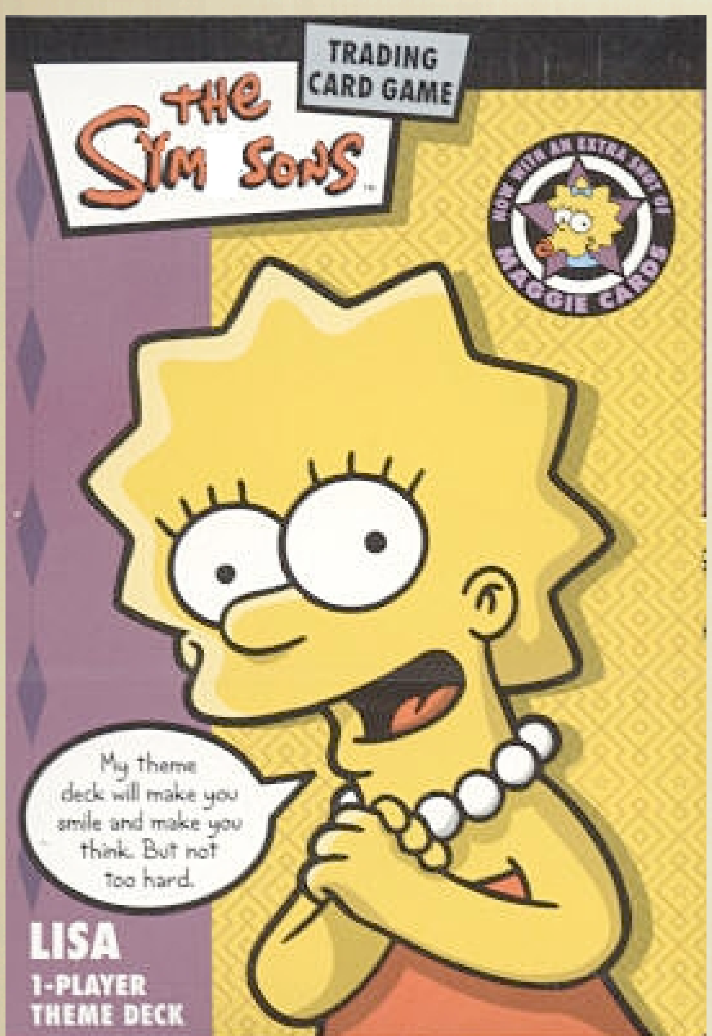
NSym has graded dimensions

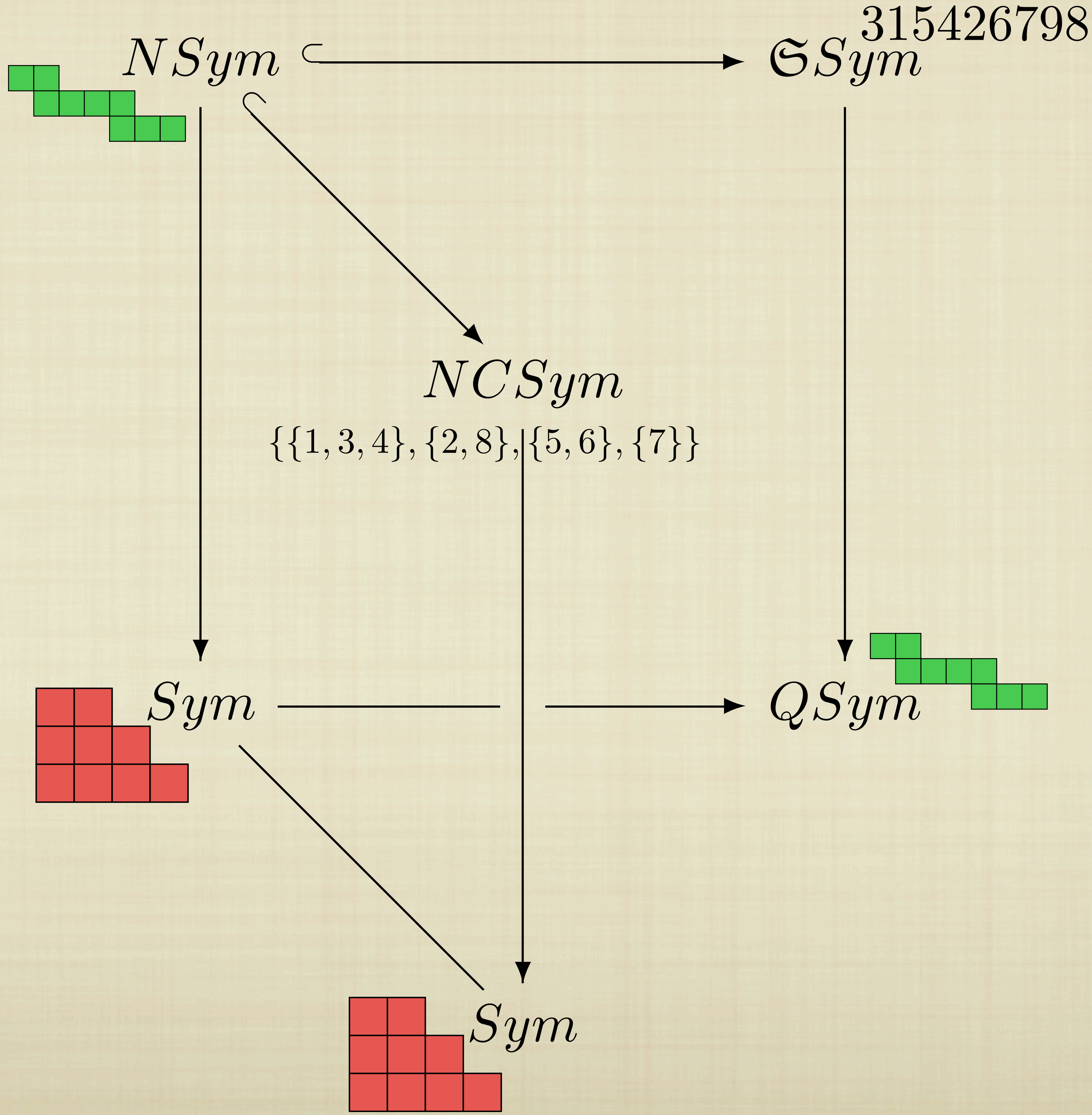
1, 1, 2, 4, 8, 16, 32, 64, 128, ...

There exists a Hopf morphism

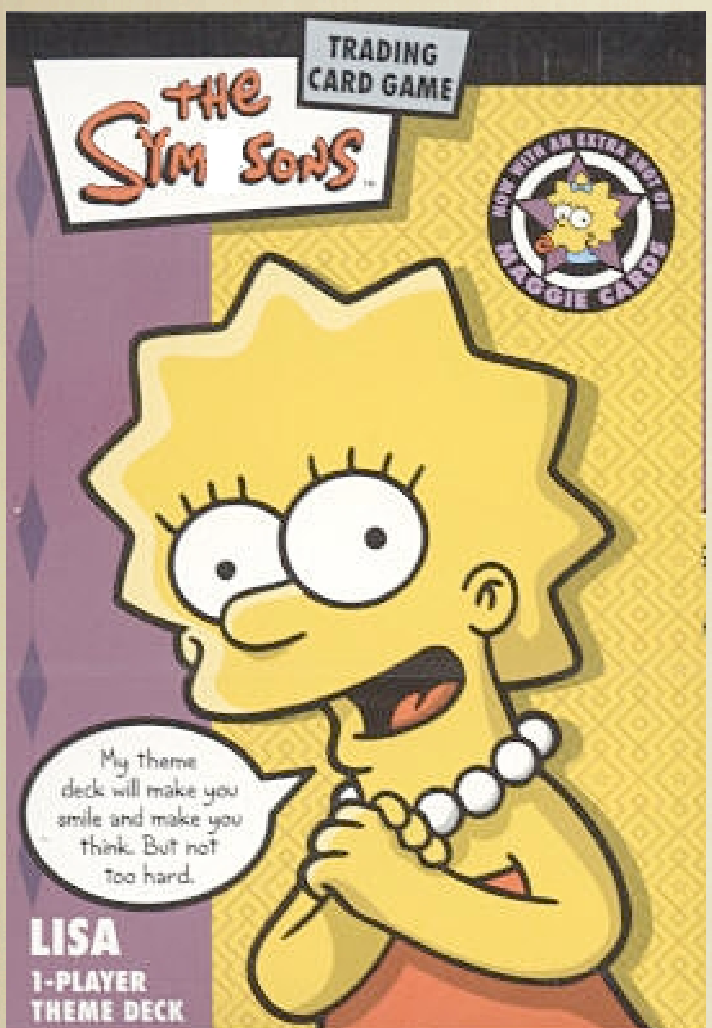
$$NSym \hookrightarrow NCSym$$

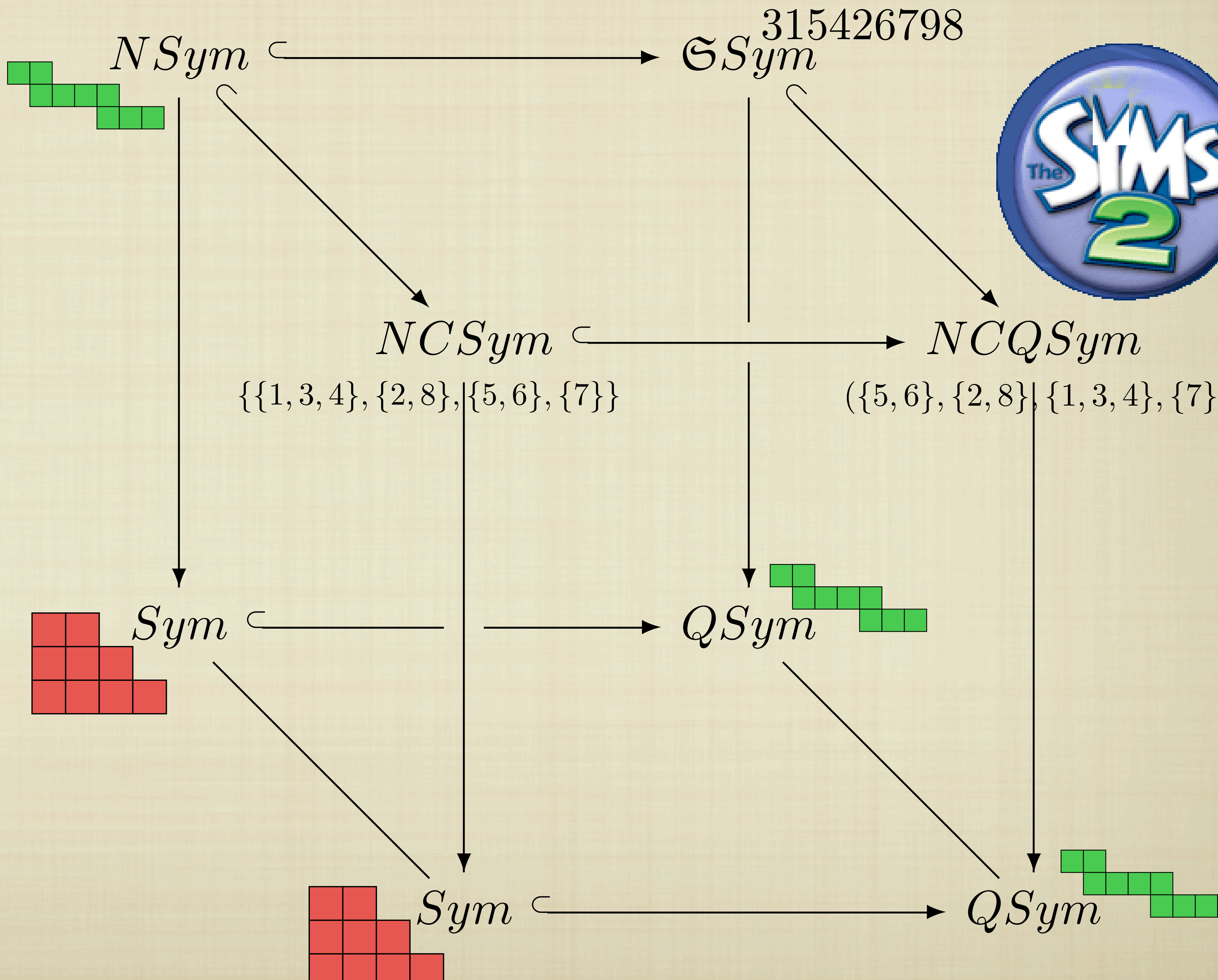
Families of morphisms

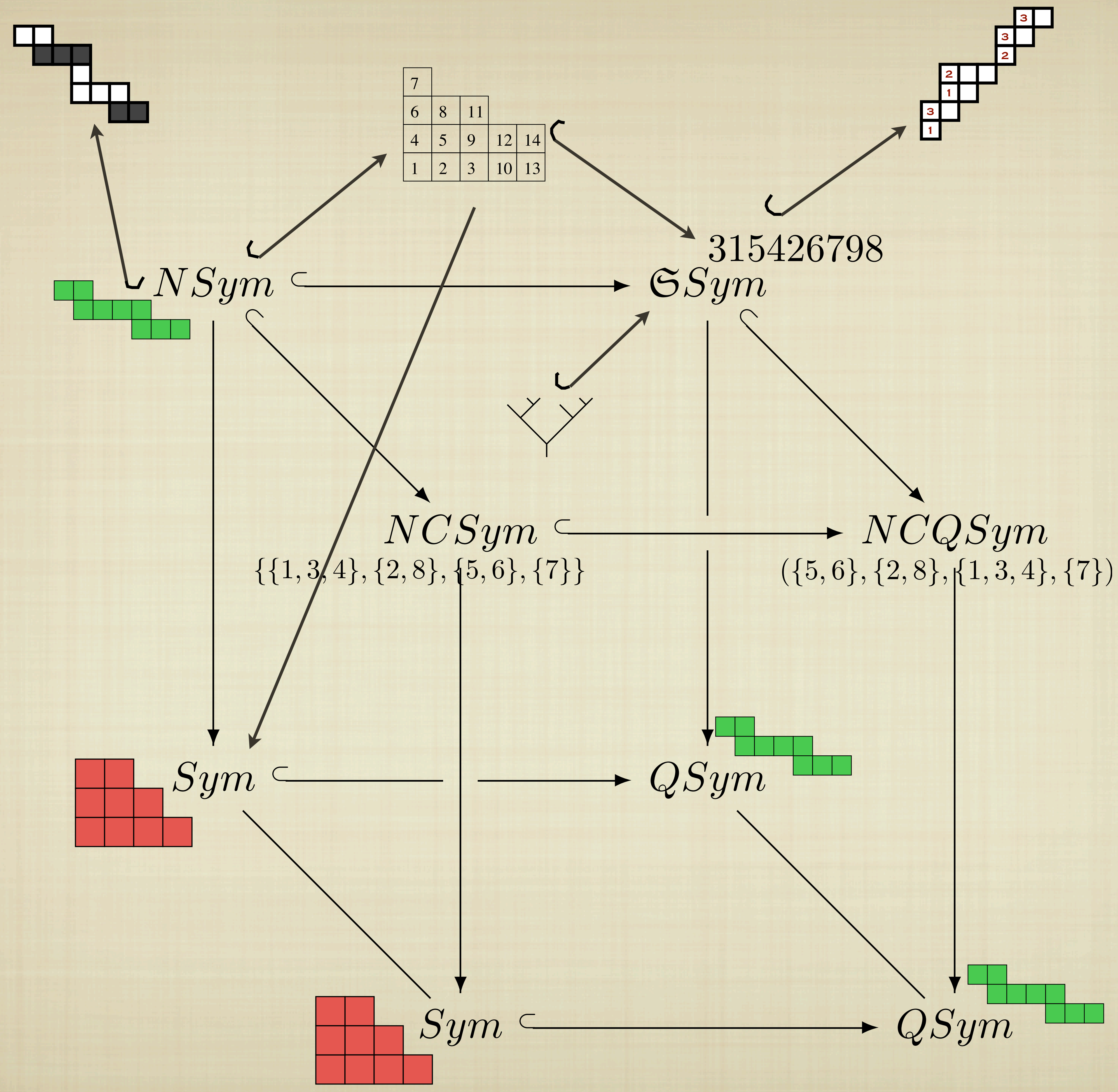




Families of morphisms







7
6 8 11
4 5 9 12 14
1 2 3 10 13
315426798
$NSym$
$\mathfrak{S}Sym$
$NCSym$
$NCQSym$
$\{\{1,3,4\},\{2,8\},\{5,6\},\{7\}\}$
$(\{5,6\},\{2,8\},\{1,3,4\},\{7\})$
Sym
$QSym$
Sym
$QSym$

One last open question

One last open question

- What is the Hopf algebra of lions?

One last open question

- What is the Hopf algebra of lions?

One last open question

- What is the Hopf algebra of lions?

Rowr!