q-Analogs of Hopf Algebra Structures

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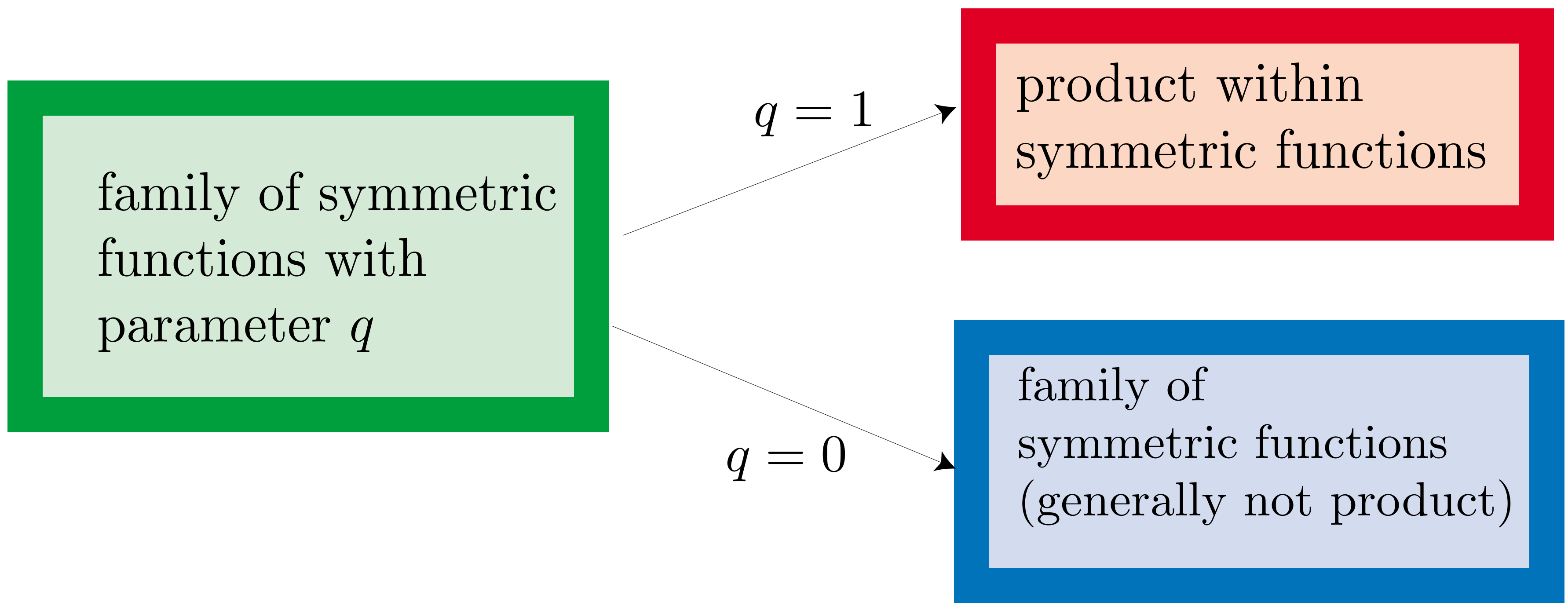
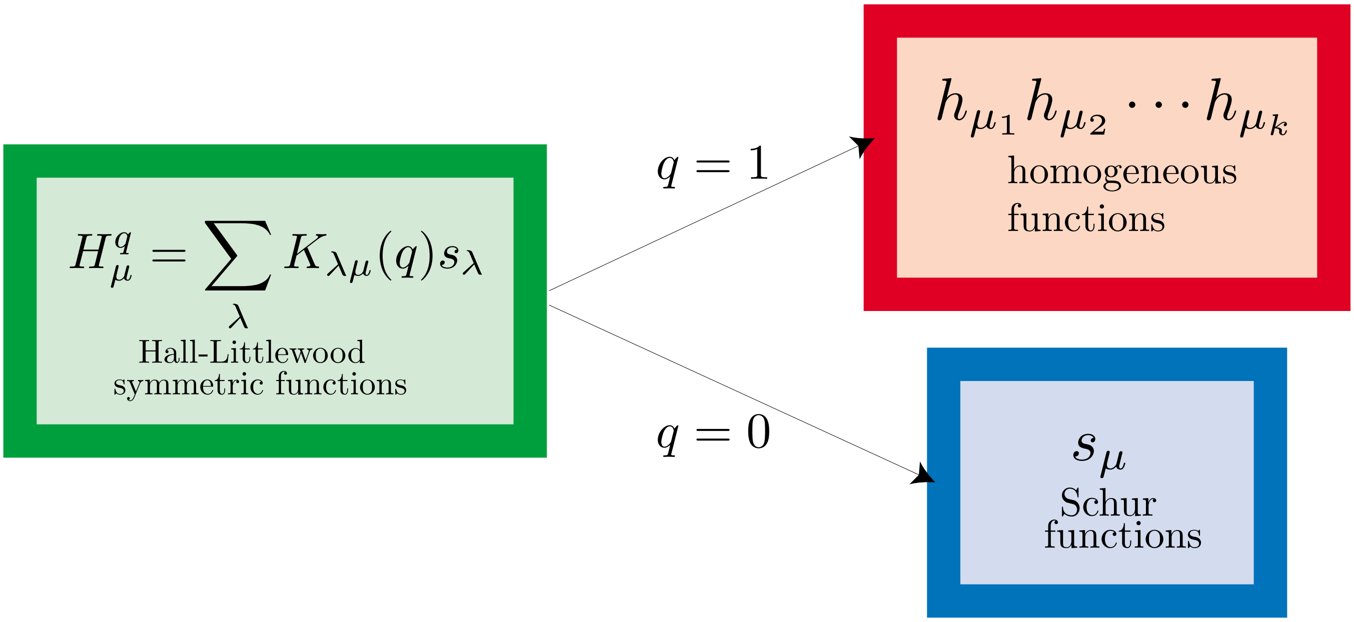
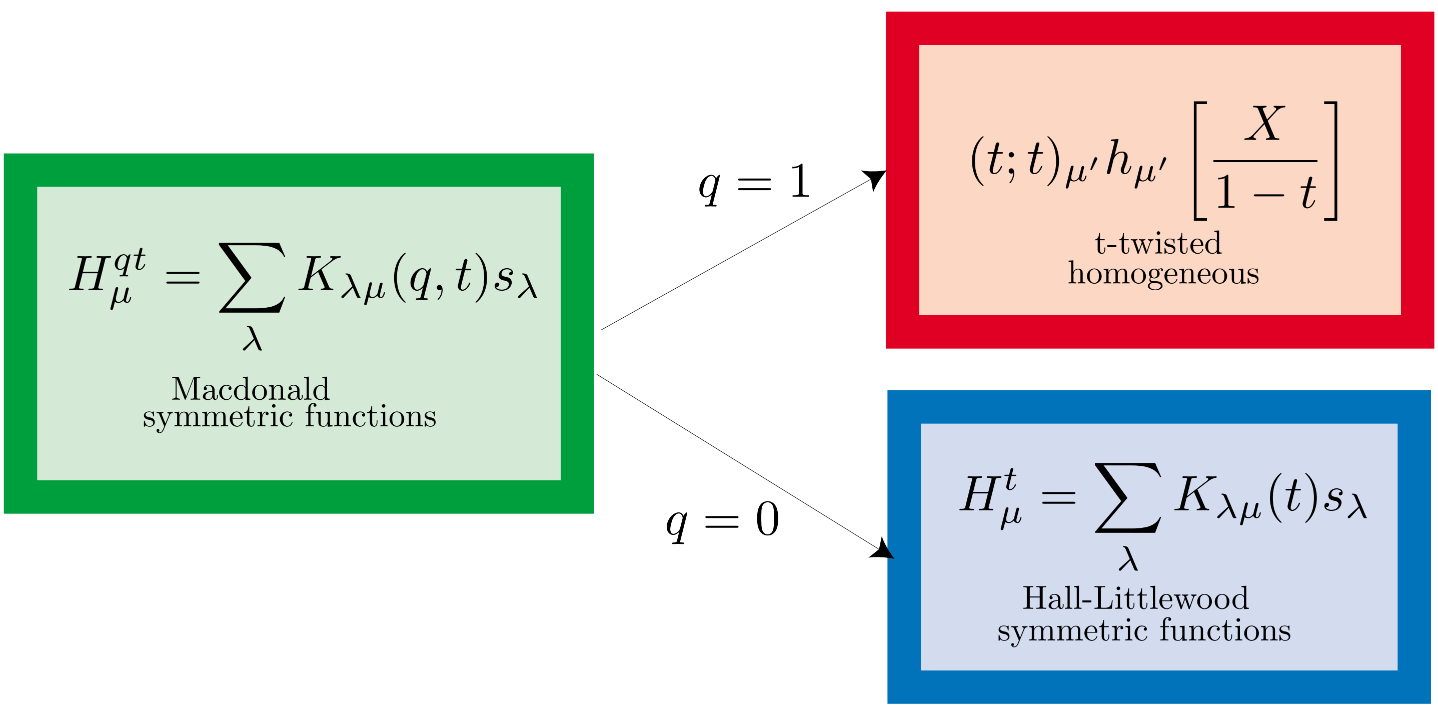
q-Analogs in Symmetric Functions



Examples:


$H_\mu^q = \sum_\lambda K_{\lambda\mu}(q) s_\lambda$
Hall-Littlewood
symmetric functions
$q = 1$
$h_{\mu_1} h_{\mu_2} \cdots h_{\mu_k}$
homogeneous
functions
$q = 0$
s_μ
Schur
functions


$H_\mu^{qt} = \sum_\lambda K_{\lambda\mu}(q,t) s_\lambda$
Macdonald
symmetric functions
$q = 1$
$(t;t)_{\mu'} h_{\mu'} \left[\frac{X}{1-t} \right]$
t-twisted
homogeneous
$q = 0$
$H_\mu^t = \sum_\lambda K_{\lambda\mu}(t) s_\lambda$
Hall-Littlewood
symmetric functions

Why look at q-SF's like this?

1. Derive algebraic relations and recurrences

2. Create general means for defining q-analog

   Identify other natural q-analog families.

3. Identify when this is occuring.

   Can we classify q-analog 'types?'

4. Derive properties on abstract level.

   When will coefficients be positive?

q-Analogs in graded Hopf Algebras

generating function for coefficients with a parameter q

$q = 1$

product within Hopf algebra

$q = 0$

expression that is generally not a product

Examples:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$$

$q = 1$

$$(1+x)^n$$

$q = 0$

$$1 + x + \cdots + x^n$$

Hopf Algebra Notation/Definitions

Algebra

A - module over a ring R
μ - multiplication $\qquad \mu : A \otimes A \to A$
η - unit $\qquad \eta : R \to A$

satisfying:
$\mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id)$ associativity
$\mu \circ (\eta \otimes id) = id = \mu \circ (id \otimes \eta)$

Let $\tau : A \otimes A \to A \otimes A$ by $\tau(f \otimes g) = g \otimes f$
A is commutative if $\mu \circ \tau = \mu$.

Co-algebra

C - module over a ring R
Δ - co-multiplication $\qquad \Delta : C \to C \otimes C$
ε - co-unit $\qquad \varepsilon : C \to R$

satisfying:
$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ associativity
$(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta$

C is co-commutative if $\tau \circ \Delta = \Delta$

Hopf Algebra

A bi-algebra is an algebra and a co-algebra at the same time such that Δ is a homomorphism with respect to μ.

H is a Hopf algebra if it is a bi-algebra with a S - antipode $\quad S : H \to H$

satisfying
$\mu \circ id \otimes S \circ \Delta = \mu \circ S \otimes id \circ \Delta = \eta \circ \varepsilon$

If H is commutative or co-commutative then S is an involution.

product	$\mu : H \otimes H \to H$
unit	$\eta : R \longrightarrow H$
co-product	$\Delta : H \longrightarrow H \otimes H$
co-unit	$\varepsilon : H \longrightarrow R$
antipode	$S : H \longrightarrow H$

The q-Analog for a graded Hopf algebra

Convolution $U, V \in \ Hom(H,H)$

$$U * V = \mu \circ U \otimes V \circ \Delta$$

this operation is associative and satisfies
$V * (\eta\varepsilon) = (\eta\varepsilon) * V = V$

Bar operation

$V \in \ Hom(H,H)$

$$\overline{V} = id * (VS)$$

if H is co-commutative then
$\overline{\overline{V}} = V$

q-Analog

H is graded, then there exists an operator $R^q \in Hom(H,H)$
For $f \in H$ and f of degree d $\quad R^q(f) = q^d f \quad$ Let $F^q = \overline{R^q}$

$$\widetilde{V}^q = \overline{\overline{V}\ \overline{R^q}} = \overline{\overline{V}\ F^q}$$

Example 1: Binomial coefficients

Hopf algebra: polynomials ring in x $\quad \mathbb{Q}[x]$

product: $\mu(x^i \otimes x^j) = x^{i+j}$

co-product: $\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \otimes x^k$

antipode: $S(x^n) = (-1)^n x^n$

grading: $R^q(x^n) = q^n x^n$

Linear operators

$B(1) = 1 + x$

$B(x^n) = x^{n+1}$ for $n > 0$

$$\widetilde{B}^q(\widetilde{B}^q(\cdots \widetilde{B}^q(1)\cdots)) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$$

$q = 0$

$$B(B(\cdots B(1)\cdots)) = 1 + x + \cdots + x^n$$

$q = 1$

$$B(1)B(1)\cdots B(1) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Example 2: Partitions with distinct parts

Hopf algebra: polynomials ring in x $\quad \mathbb{Q}[x]$

product: $\mu(x^i \otimes x^j) = x^{i+j}$

co-product: $\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \otimes x^k$

antipode: $S(x^n) = (-1)^n x^n$

grading: $R^q(x^n) = q^n x^n$

Linear operators

$B_k(1) = 1 + x^k$

$B_k(x^n) = x^n$ for $n > 0$

$$\widetilde{B}_k^q(\cdots \widetilde{B}_2^q(\widetilde{B}_1^q(1))\cdots) = \sum_{m\geq 0}\sum_{\lambda} q^{n(\lambda)} x^m$$

$n(\lambda) = \sum_{i\geq 1} \lambda_i (i-1)$

the sum is over all partitions λ of m with distinct parts of size $\leq k$.

$q = 0$:

$$B_k(\cdots B_2(B_1(1))\cdots) = 1 + x + \cdots + x^k$$

$q = 1$:

$$B_1(1)B_2(1)\cdots B_k(1) = \prod_{i=1}^{k}(1 + x^i) = \sum_{m\geq 0}\sum_{\lambda} x^m$$

Example 3: Falling factorial

Hopf algebra: polynomials ring in x $\mathbb{Q}[x]$

product: $\mu(x^i \otimes x^j) = \binom{i+j}{j} x^{i+j}$

co-product: $\Delta(x^n) = \sum_{i=0}^{n} x^{n-i} \otimes x^i$

antipode: $S(x^n) = (-1)^n x^n$

grading: $R^q(x^n) = q^n x^n$

Linear operators

$B(1) = 1 + x$

$B(x^n) = x^{n+1}$ for $n > 0$

$$\widetilde{B}^q(\widetilde{B}^q(\cdots \widetilde{B}^q(1)\cdots)) = \sum_{i=0}^{n} [n]_q [n-1]_q \cdots [n-i+1]_q x^i$$

$q = 0$ $q = 1$

$$B(B(\cdots B(1)\cdots)) = 1 + x + \cdots + x^n$$

$$B(1) \bullet B(1) \bullet \cdots \bullet B(1) = \sum_{i=0}^{n} n(n-1)\cdots(n-i+1)x^i$$

Example 4: q-Koskta coefficients

Hopf algebra: Symmetric functions $\quad \mathbb{Q}[p_1, p_2, p_3, \ldots]$

product: $\mu(f \otimes g) = fg \quad \mu(s_\lambda \otimes s_\mu) = \sum_\nu c^\nu_{\lambda\mu} s_\nu$

co-product: $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$

$\Delta(s_\nu) = \sum_{\lambda,\mu} c^\nu_{\lambda\mu} s_\lambda \otimes s_\mu$

antipode: $S(p_k) = (-1)^k p_k \quad S(s_\lambda) = (-1)^{|\lambda|} s_{\lambda'}$

grading: $R^q(p_k) = q^k p_k \quad R^q(s_\lambda) = q^{|\lambda|} s_\lambda$

Linear operators

$$B_m(s_\lambda) = s_{(m,\lambda)}$$

Theorem: Jing

$$\widetilde{B}^q_{\mu_1}(\widetilde{B}^q_{\mu_2}(\cdots \widetilde{B}^q_{\mu_k}(1)\cdots)) = \sum_\lambda K_{\lambda\mu}(q) s_\lambda$$

$q = 0$:

$$B_{\mu_1}(B_{\mu_2}(\cdots B_{\mu_k}(1)\cdots)) = s_\mu$$

$q = 1$:

$$B_{\mu_1}(1) B_{\mu_2}(1) \cdots B_{\mu_k}(1) = h_{\mu_1} h_{\mu_2} \cdots h_{\mu_k} = \sum_\lambda K_{\lambda\mu} s_\lambda$$

Application: Generalized Kostka polynomials
(q-Littlewood-Richardson coefficients)

Theorem: Shimozono-Z

Instead we take as our linear operators

$$B_\mu = B_{\mu_1} B_{\mu_2} \cdots B_{\mu_k}$$
$$B_\mu(s_\lambda) = s_{(\mu\cdot\lambda)}$$

then for a sequence of partitions
$\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)}$

the q-analog is a family of symmetric functions whose coefficients at $q = 1$ are the Littlewood-Richardson coefficients

$$\widetilde{B}^q_{\mu^{(1)}}(\widetilde{B}^q_{\mu^{(2)}}(\cdots \widetilde{B}^q_{\mu^{(k)}}(1)\cdots)) = \sum_\lambda K_{\lambda;(\mu^{(1)},\mu^{(2)},\ldots,\mu^{(k)})}(q) s_\lambda$$

Shimozono-Weyman

$q = 0$

$$B_{\mu^{(1)}}(B_{\mu^{(2)}}(\cdots B_{\mu^{(k)}}(1)\cdots))$$
$$= s_{(\mu^{(1)}\cdot\mu^{(2)}\cdots\mu^{(k)})}$$

$q = 1$

$$B_{\mu^{(1)}}(1) B_{\mu^{(2)}}(1) \cdots B_{\mu^{(k)}}(1)$$
$$= s_{\mu^{(1)}} s_{\mu^{(1)}} \cdots s_{\mu^{(1)}}$$

Example 5: q,t-Kostka polynomials

Hopf algebra: Symmetric functions $\mathbb{Q}[p_1, p_2, p_3, \ldots]$

product: $\mu(f \otimes g) = fg \quad \mu(s_\lambda \otimes s_\mu) = \sum_\nu c^\nu_{\lambda\mu} s_\nu$

co-product: $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$
$\Delta(s_\nu) = \sum_{\lambda,\mu} c^\nu_{\lambda\mu} s_\lambda \otimes s_\mu$

antipode: $S(p_k) = (-1)^k p_k \quad S(s_\lambda) = (-1)^{|\lambda|} s_{\lambda'}$

grading: $R^q(p_k) = q^k p_k \quad R^q(s_\lambda) = q^{|\lambda|} s_\lambda$

Linear operators

Add a column to Hall-Littlewood SFs

$B^t_m(H^t_\mu) = H^t_{\mu+1^m}$

Theorem: Noumi-Kirillov

$$\widetilde{B^t_{\mu_1}}^q(\widetilde{B^t_{\mu_2}}^q(\cdots \widetilde{B^t_{\mu_k}}^q(1)\cdots)) = \sum_\lambda K_{\lambda\mu'}(q,t) s_\lambda$$

$q = 0$

$q = 1$

$$B^t_{\mu_1}(B^t_{\mu_2}(\cdots B^t_{\mu_k}(1)\cdots)) = \sum_\lambda K_{\lambda\mu'}(t) s_\lambda$$

$$B^t_{\mu_1}(1) B^t_{\mu_2}(1) \cdots B^t_{\mu_k}(1) = h_\mu\left[\frac{X}{1-t}\right](t;t)_\mu$$

Applications to q, t-Kostka polynomials

1. Polynomiality of coefficients

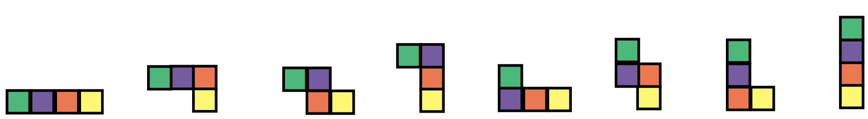
Macdonald's definition only immediately implied that coefficients were rational functions in q and t

t-Kostka are polynomials $\Longrightarrow$ q, t-Kostka are polynomials

2. Recurrences

Recursively calculate coefficients from smaller partitions

Action of B_m^t on the Schur functions known
"ribbon rule" for t-Kostka



Derive generalized ribbon rule for q, t-Kostka

Theorem: a "Macdonald-Morris" recurrence

positivity?

Non-commutative symmetric functions

$$\mathbb{Q}\langle \boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3, \ldots \rangle$$

linear basis $\boldsymbol{h}_\alpha := \boldsymbol{h}_{\alpha_1} \boldsymbol{h}_{\alpha_2} \cdots \boldsymbol{h}_{\alpha_k}$

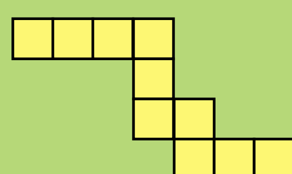
where α is a composition with non-zero parts

$$D(\alpha) = \text{descent set of } \alpha$$
$$= \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots \alpha_{\ell(\alpha)-1}\}$$

$$\alpha \geq \beta \text{ if } D(\alpha) \subseteq D(\beta)$$

Example

$\alpha = (4, 1, 2, 3)$ $\qquad D(\alpha) = \{4, 5, 7\}$



$\beta = (4, 3, 3)$ $\qquad D(\beta) = \{4, 7\}$ $\qquad \alpha \leq \beta$

Analog of Schur functions

$$\boldsymbol{s}_\alpha = \sum_{\beta \geq \alpha} (-1)^{\ell(\alpha) + \ell(\beta)} \boldsymbol{h}_\beta$$

Example 6: NC-Hall-Littlewood functions

Hopf algebra: NC-symmetric functions $\mathbb{Q}\left\langle \boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3, \ldots \right\rangle$

product: $\mu(f \otimes g) = fg$

co-product: $\Delta(\boldsymbol{h}_k) = \sum_{i=0}^{k} \boldsymbol{h}_i \otimes \boldsymbol{h}_{k-i}$

antipode: $S(\boldsymbol{h}_k) = \sum_{\alpha \models k} (-1)^{\ell(\alpha)} \boldsymbol{h}_\alpha$

grading: $R^q(\boldsymbol{h}_k) = q^k \boldsymbol{h}_k$

Linear operators

$$B_m(\boldsymbol{s}_\alpha) = \boldsymbol{s}_{(\alpha,m)}$$

Theorem: Bergeron-Z

$$\widetilde{B}^q_{\alpha_k}(\cdots \widetilde{B}^q_{\alpha_2}(\widetilde{B}^q_{\alpha_1}(1))\cdots) = \sum_{\beta \geq \alpha} q^{c(\alpha,\beta)} \boldsymbol{s}_\beta$$

$$c(\alpha,\beta) = \sum_{i \in D(\alpha) \cap D(\beta)} i$$

$q = 0$

$$B_{\alpha_k}(\cdots B_{\alpha_2}(B_{\alpha_1}(1))\cdots) = \boldsymbol{s}_\alpha$$

$q = 1$

$$B_{\alpha_1}(1) B_{\alpha_2}(1) \cdots B_{\alpha_k}(1) = \boldsymbol{h}_\alpha = \sum_{\beta \geq \alpha} \boldsymbol{s}_\beta$$