A q-analog of Schur's Q-functions

Mike Zabrocki

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Abstract: The Hall-Littlewood functions are a well studied basis of the space of symmetric functions Λ and are a q-analog of the fundamental basis of the symmetric functions, Schur's S-functions. The subalgebra of Λ generated by the odd power sum elements $\Gamma = \{p_1, p_3, p_5, \ldots\}$ is known as the Q-function algebra and it contains a fundamental basis know as Schur's Q-functions. These functions are associated to the representation theory of the spin group and to the projective represenation theory of the symmetric and alternating groups. By abstracting a definition for the Hall-Littlewood functions to a setting which also exists in the Q-function algebra we can define a q-analog of Schur's Q-functions. This basis for Γ appears to be previously unstudied in the literature and shares many analogous properties to the Hall-Littlewood symmetric functions.

The Hall-Littlewood symmetric functions can be seen as a generating function for the number of column strict tableaux of fixed content. The algebraic relations that they satisfy have been translated into combinatorial statements about column strict tableaux and used to define a poset structure on this set of elements. The q-analog of Schur's Q-functions are conjectured to be a generating function for the number of marked shifted tableaux of fixed content and suggest a similar poset structure. We will discuss some of the combinatorial aspects of this problem.

DATE: Monday, April 29, 2002

TIME: 4:00 - 4:50 pm

LOCATION: *Room: 3-203, Dr. Alvin Woods Building*

Wilfrid Laurier University

ALL ARE WELCOME!

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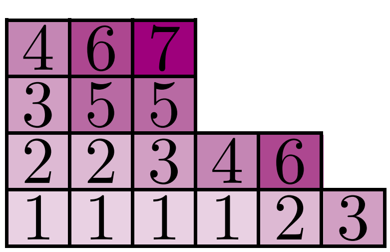
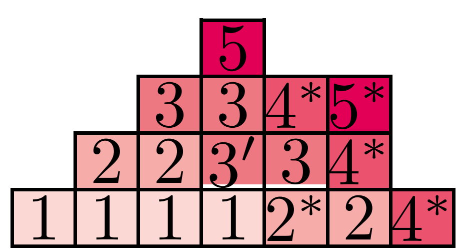
`http://garsia.math.yorku.ca/~zabrocki/posets`
joint work with Geanina Tudose (University of Minnesota)





The Symmetric Functions

$$\Lambda = \mathbb{Q}[h_1, h_2, h_3, \ldots]$$
$$deg(h_k) = k$$

The space of symmetric functions is generated algebraically by the simple homogeneous symmetric functions.

The Schur Functions

$$s_\lambda = det|h_{\lambda_i+i-j}|$$

Example:

$$s_{(2,2,1)} = \begin{vmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = {h_2}^2 h_1 - h_2 h_3 - h_3 {h_1}^2 + h_4\, h_1$$

$$h_3^2 h_7 h_2 h_1^3 h_4$$

degree $2 \cdot 3 + 7 + 2 + 3 \cdot 1 + 4 = 22$

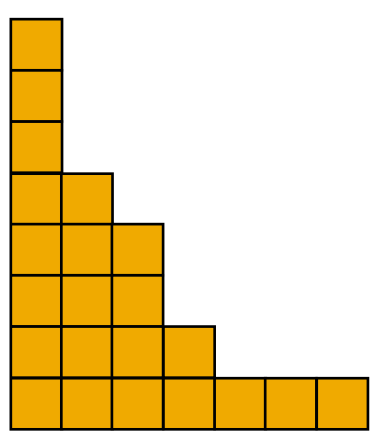
$$h_7 h_4 h_3^2 h_2 h_1^3$$

$$h_{(7,4,3,3,2,1,1,1)}$$

definition: a *partition* of n

sequence of non-negative integers $(\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$
such that $\lambda_1 + \lambda_2 + \cdots + \lambda_{\ell(\lambda)} = n$
and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0$.

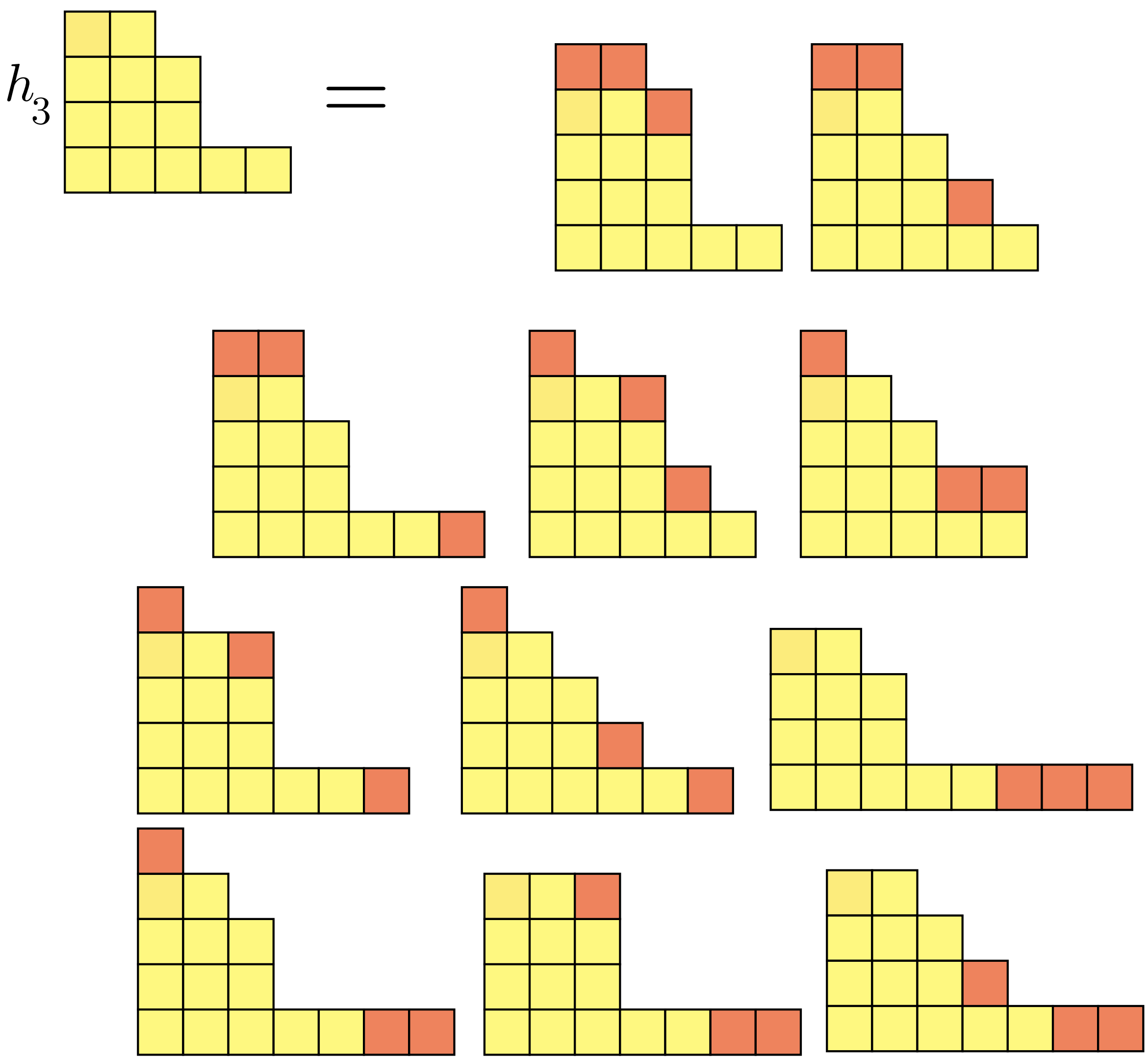
Young diagram of a partition



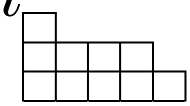
The Pieri Rule

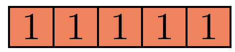
$$h_m s_\lambda = \sum_{\mu} s_\mu$$

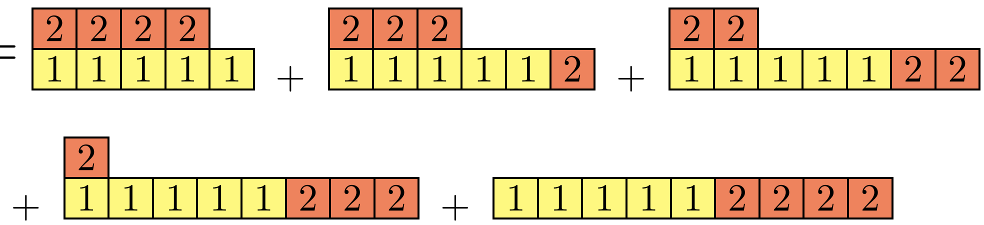
μ contains λ and the difference has at most one cell per column



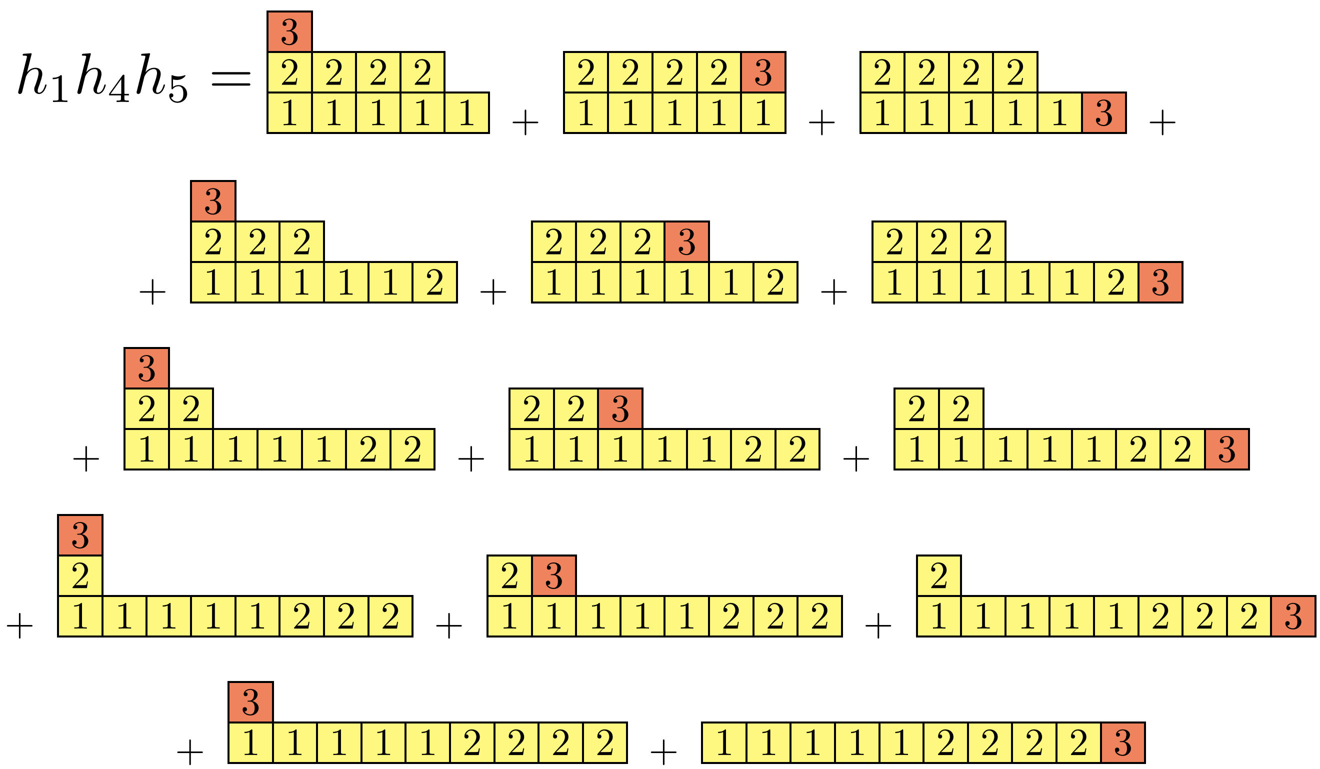
Homogeneous $\longrightarrow$ Schur

h  $= h_5 h_4 h_1$

$h_5 =$  $= s_5$

$h_4 h_5 =$ 

$$= s_9 + s_{8,1} + s_{7,2} + s_{6,3} + s_{5,4}$$

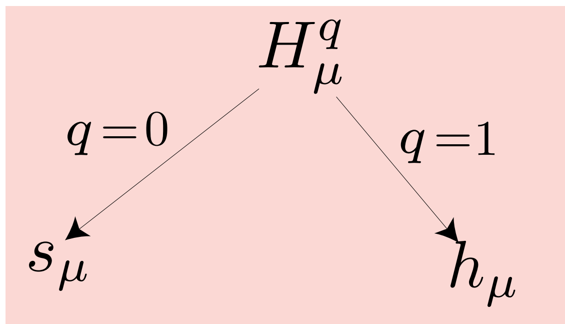
$h_1 h_4 h_5 =$ 

$$= s_{10} + 2\, s_{9,1} + 2\, s_{8,2} + s_{8,1,1} + 2\, s_{7,3} + s_{7,2,1}$$

$$+ 2\, s_{6,4} + s_{6,3,1} + s_{5,5} + s_{5,4,1}$$

Hall-Littlewood symmetric functions
Kostka-Foulkes coefficients

$$H_\mu^q = \sum_\lambda K_{\lambda\mu}(q) s_\lambda$$



$$K_{\lambda\mu}(0) = 1 \text{ if } \lambda = \mu \text{ and } 0 \text{ otherwise}$$

$$K_{\lambda\mu}(1) = h_\mu \Big|_{s_\lambda}$$

Example:

$$H^q_{(2,2,2)} = q^6 s_6 + q^4 (q+1) s_{5,1} + q^2 \left(q^2 + q + 1\right) s_{4,2}$$
$$+ q^3 s_{4,1,1} + q^3 s_{3,3} + q(q+1) s_{3,2,1} + s_{2,2,2}$$

Bernstein Schur row adding operator

$$\mathbf{S}_m(s_\mu) = s_{(m,\mu)}$$

Jing's Hall-Littlewood row adding operator

$$\mathbf{H}_m(H^q_\mu) = H^q_{(m,\mu)}$$

Hopf q-ification of linear operators

Δ coproduct	$\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$
μ multiplication	$\mu(f \otimes g) = fg$
S antipode	$S(p_k) = -p_k$
	$R^q(f) = q^{deg(f)} f$

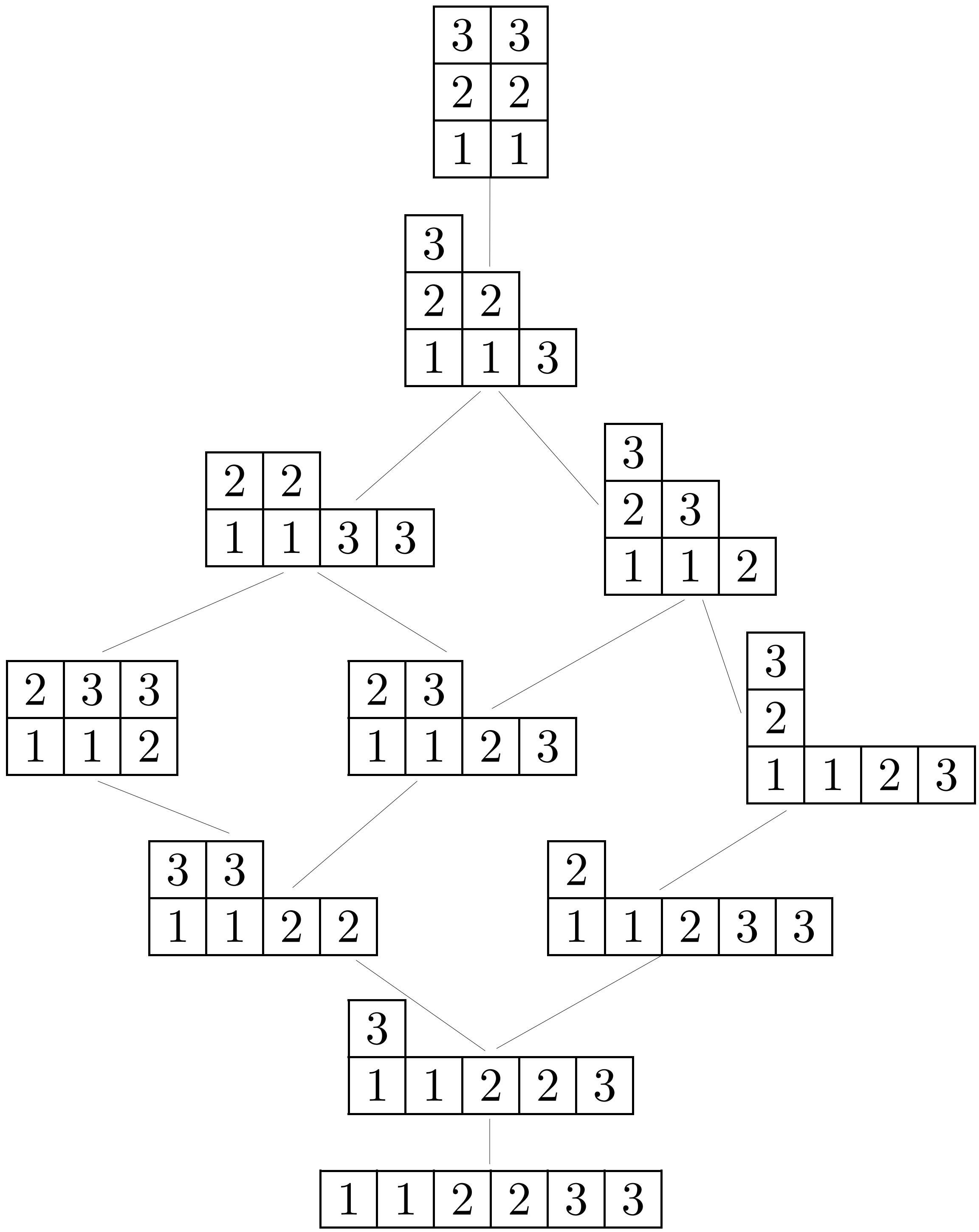
Define

$$\overline{V} := \mu \circ id \otimes (VS) \circ \Delta$$

$$\widetilde{V}^q := \overline{\overline{V}\ \overline{R^q}}$$

$$\mathbf{H}_m = \widetilde{\mathbf{S}_m}^q$$

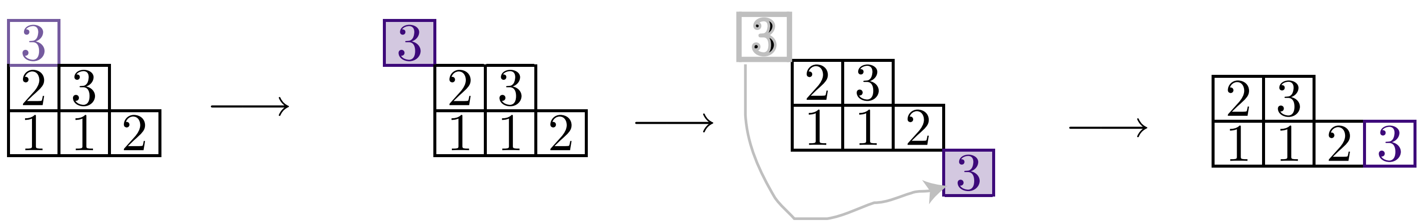
Poset structure of the column strict tableaux of content $(2,2,2)$



$$H^q_{(2,2,2)} = q^6 s_6 + q^4 (q+1)\, s_{5,1} + q^2 \left(q^2+q+1\right) s_{4,2}$$
$$+q^3 s_{4,1,1} + q^3 s_{3,3} + q\,(q+1)\, s_{3,2,1} + s_{2,2,2}$$

Covering relation - cyclage

1. Pick a corner
2. column evacuate
3. row insert



If $\mu \geq \nu$, then there is an injection from CST^{μ} to CST^{ν} which preserves the statistic

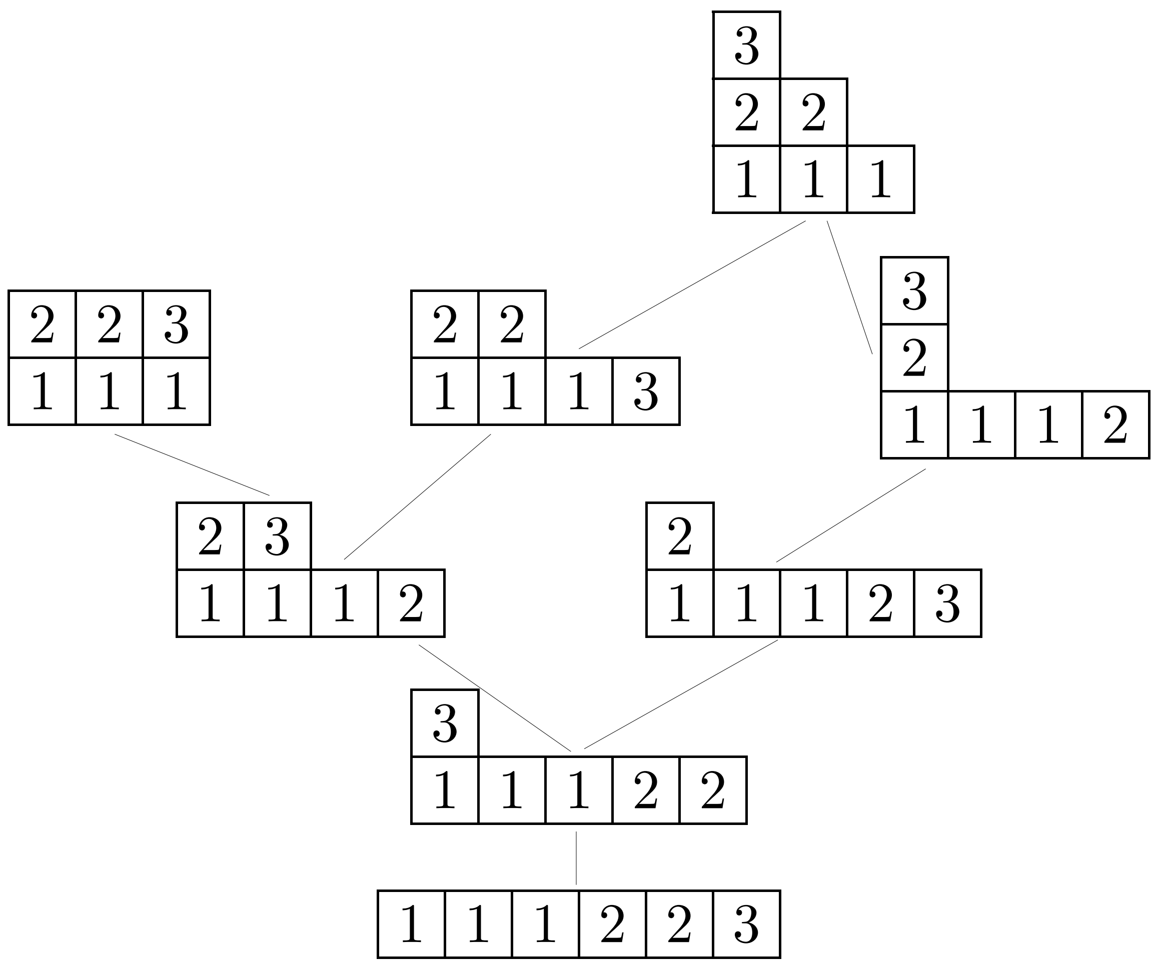
This implies

$$q^{n(\mu)} H_{\nu}^{q} - q^{n(\nu)} H_{\mu}^{q}$$

is Schur positive

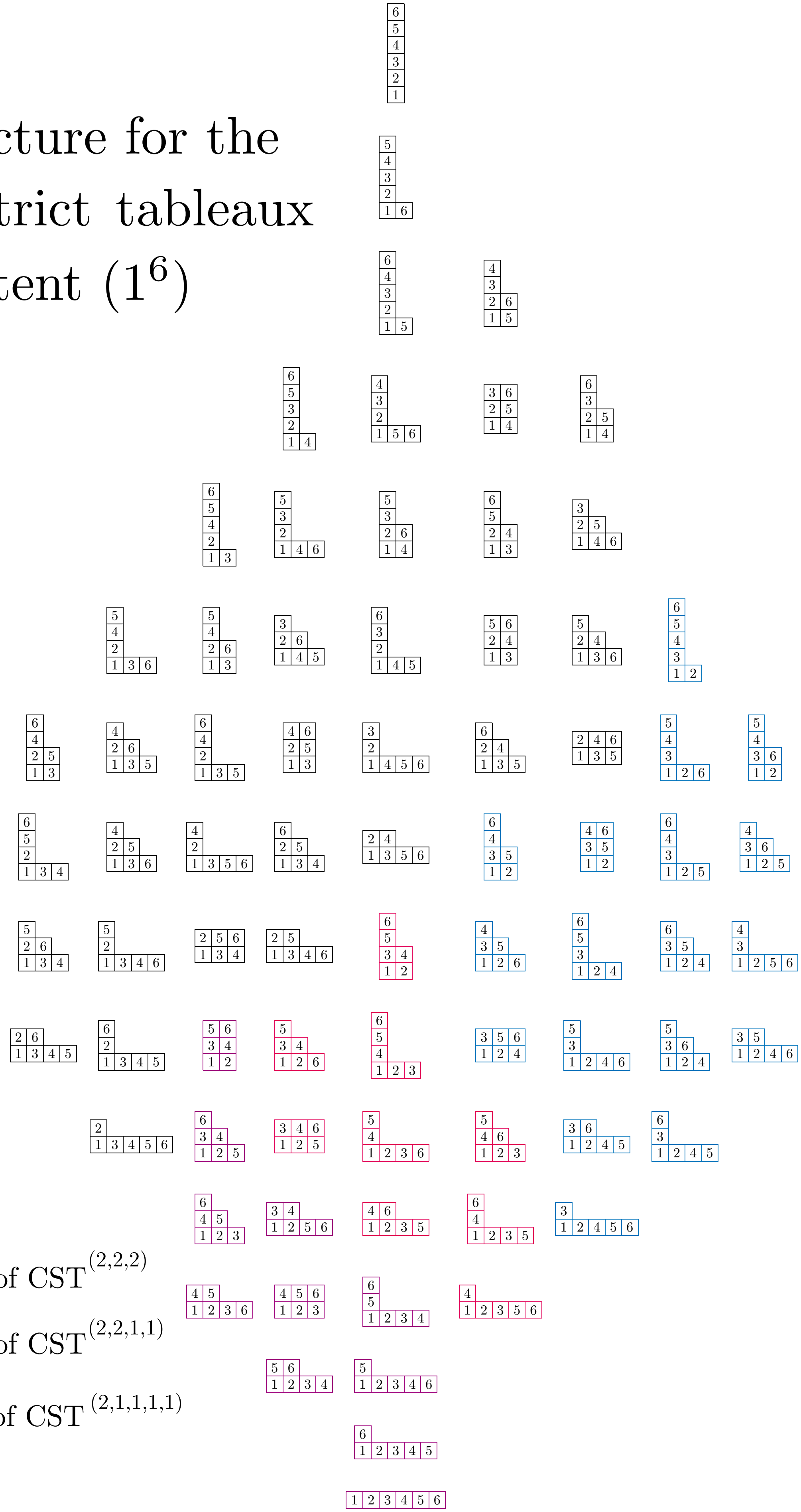
where $\quad n(\mu) := \sum_i (i-1)\mu_i$

Poset structure of the column strict tableaux of content $(3, 2, 1)$



$$H^q_{(3,2,1)} = q^4 s_6 + q^2 (q+1) s_{5,1} + q (q+1) s_{4,2} + q s_{4,1,1} + q s_{3,3} + s_{3,2,1}$$

Poset structure for the column strict tableaux of content (1^6)



Legend:

Image of CST$^{(2,2,2)}$

Image of CST$^{(2,2,1,1)}$

Image of CST$^{(2,1,1,1,1)}$

Schur's Q-functions

$$\Gamma = \mathbb{Q}[p_1, p_3, p_5, \ldots] \subset \Lambda$$

$$\theta : \Lambda \longrightarrow \Gamma$$

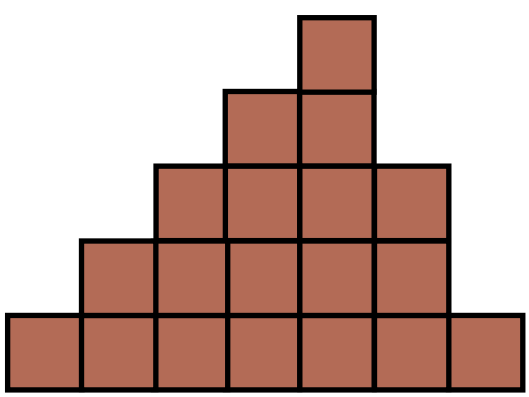
$$\theta(p_k) = (1 - (-1)^k)p_k$$

$$\theta(h_\mu) = q_\mu$$

Definition: $Q_\mu = \theta(H_\mu^q)\Big|_{q=-1}$

Basis indexed by strict partitions

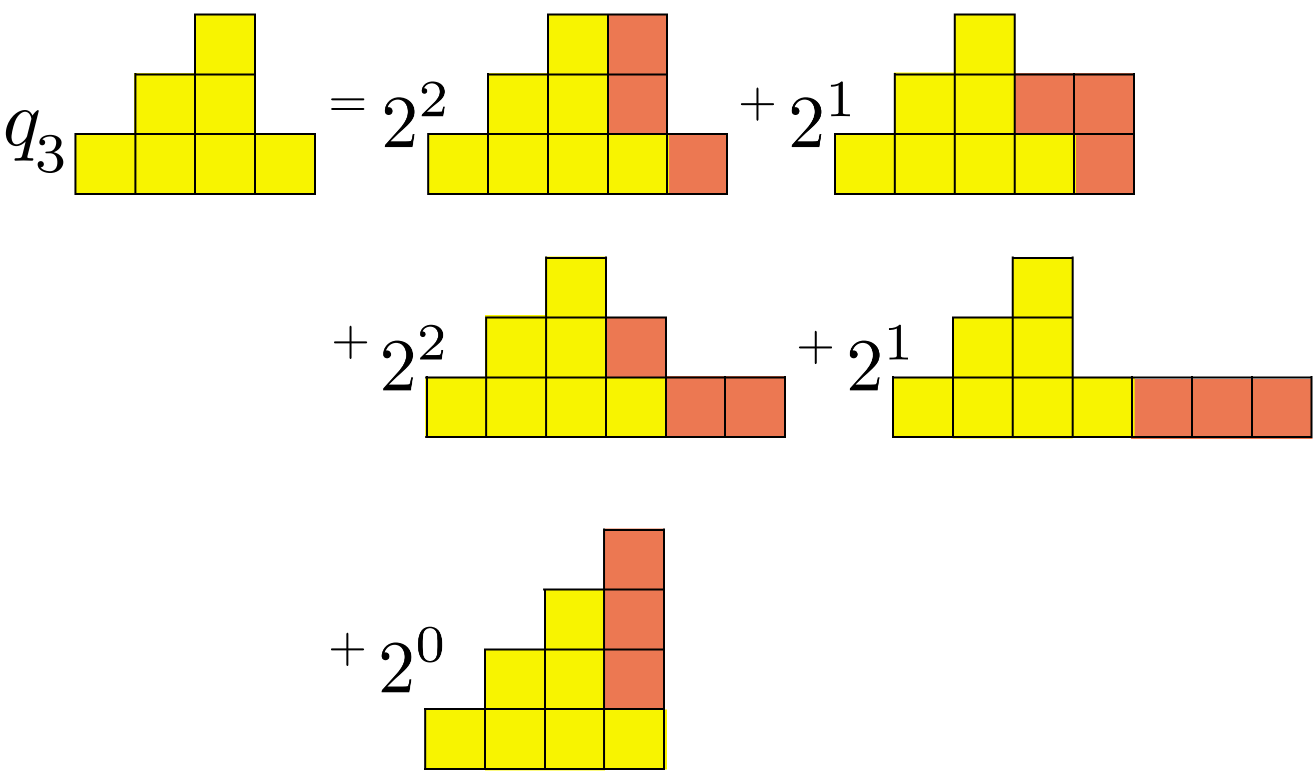
$(7, 5, 4, 2, 1)$



Pieri rule for the product $q_m Q_\lambda$

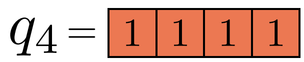
$$q_m Q_\mu = \sum_{\lambda/\mu \in \mathcal{H}_m} 2^{a(\lambda/\mu) - \ell(\lambda) + \ell(\mu)} Q_\lambda$$

$a(\lambda/\mu) = 1+$ the number of $1 < j \leq \ell(\lambda)$ such that $\lambda_j > \mu_j$ and $\mu_{j-1} > \lambda_j$

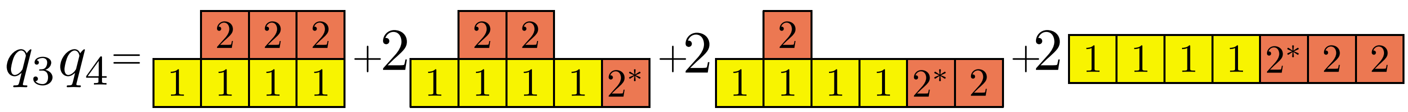


$q_\lambda \longrightarrow$ Schur's Q-functions

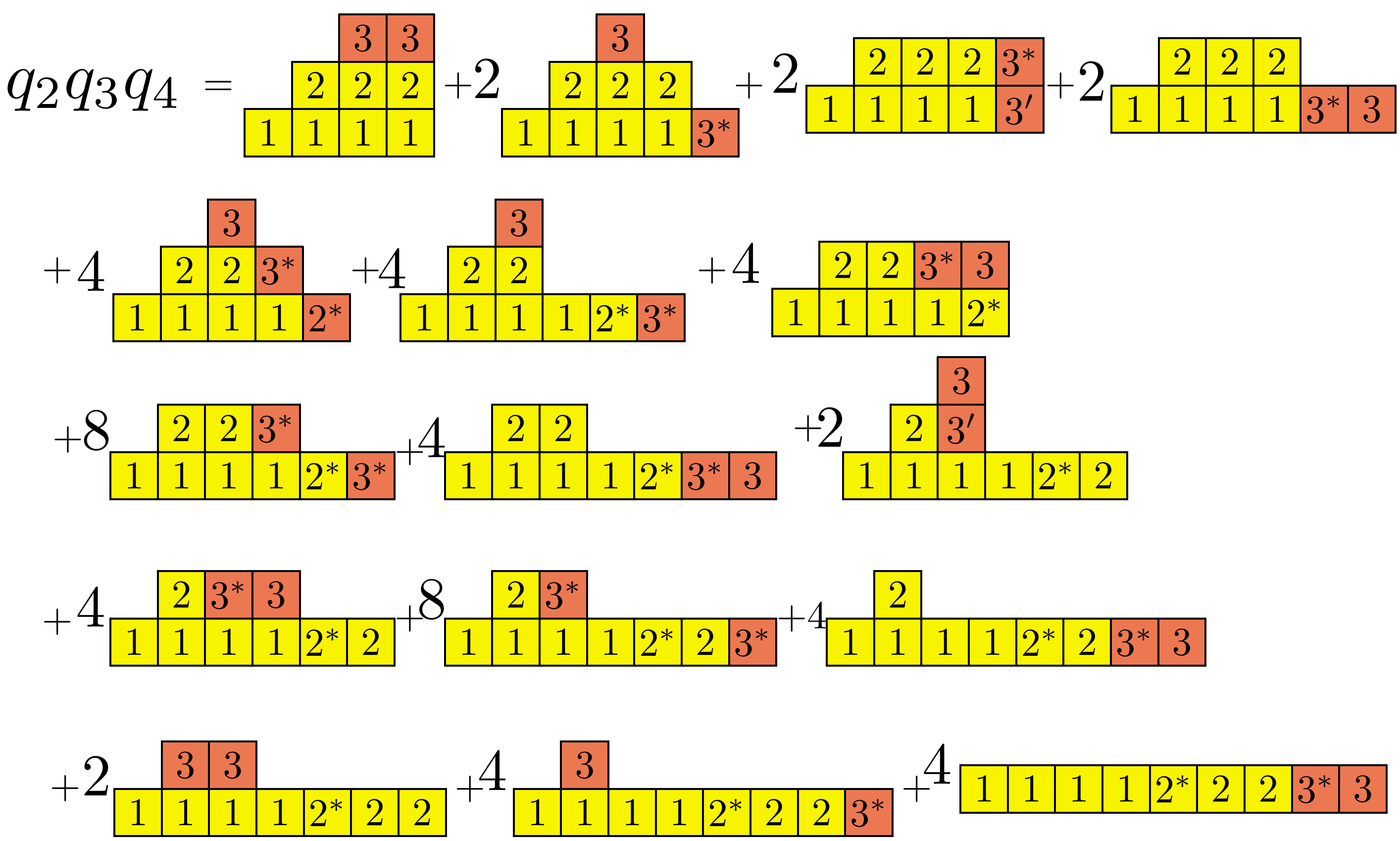
$$q_4 q_3 q_2$$



$$= Q_4$$



$$= 2\,Q_7 + 2\,Q_{6,1} + 2\,Q_{5,2} + Q_{4,3}$$



$$= 4\,Q_9 + 8\,Q_{8,1} + 14\,Q_{7,2} + 14\,Q_{6,3} + 6\,Q_{6,2,1} + 6\,Q_{5,4} + 6\,Q_{5,3,1} + Q_{4,3,2}$$

Jing

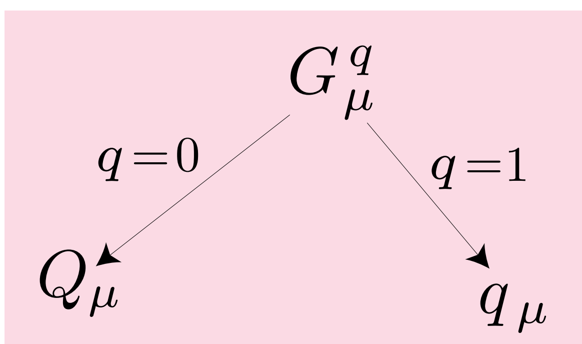
$$\mathbf{Q}_m = \theta(\mathbf{H}_m^{-1})$$

$$\mathbf{Q}_m(Q_\lambda) = Q_{(m,\lambda)} \qquad \text{for } m > \lambda_1$$

$$\mathbf{G}_m := \widetilde{\mathbf{Q}_m}^{\,q}$$

$$G_\mu^q := G_{\mu_1} G_{\mu_2} \cdots G_{\mu_k} 1$$

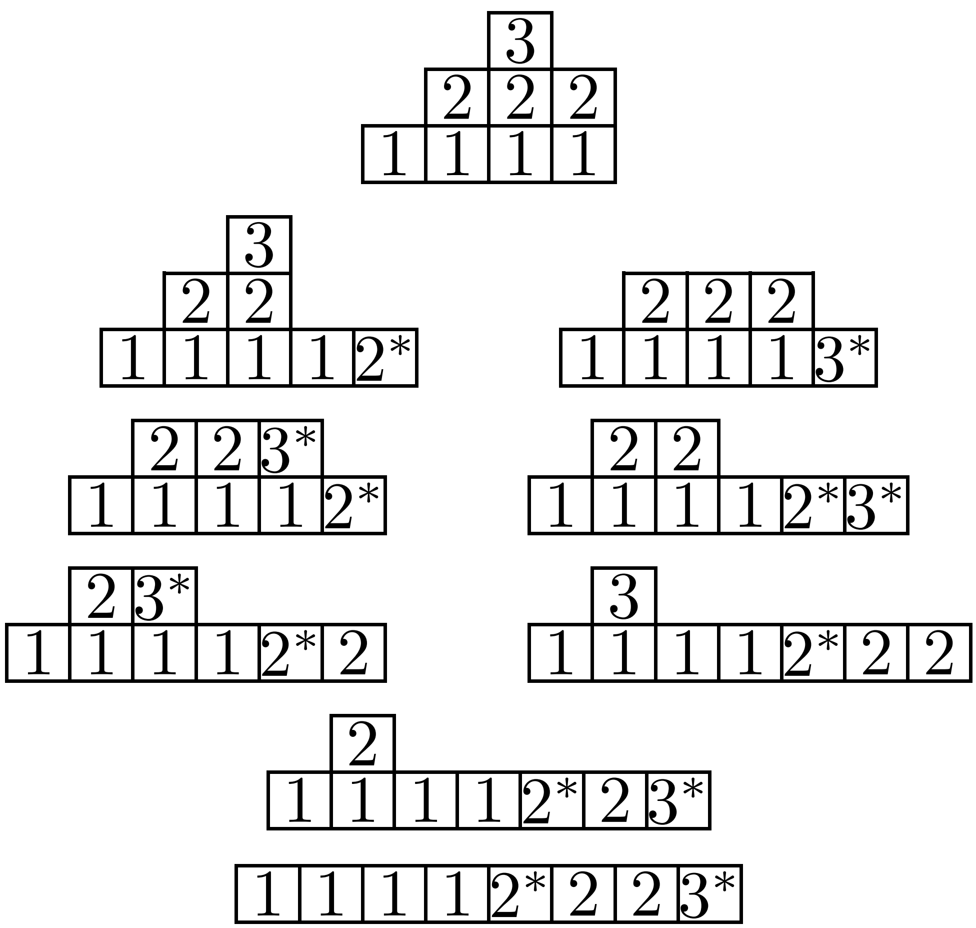
$$G_\mu^q = \sum_\lambda L_{\lambda\mu}(q) Q_\lambda$$



Example :

$$G^q_{(4,3,1)} = 4\,q^5 Q_8 + \left(2\,q^3 + 4\,q^4\right) Q_{7,1} + \left(4\,q^3 + 4\,q^2\right) Q_{6,2}$$
$$+\left(4\,q^2 + 2\,q\right) Q_{5,3} + 2\,q Q_{5,2,1} + Q_{4,3,1}$$

$$G^q_{4,3,1} = 4\,q^5 Q_8 + \left(2\,q^3 + 4\,q^4\right) Q_{7,1} + \left(4\,q^2 + 4\,q^3\right) Q_{6,2}$$
$$+ \left(2\,q + 4\,q^2\right) Q_{5,3} + 2\,q Q_{5,2,1} + Q_{4,3,1}$$



Some Results

Theorem (Tudose & Zabrocki)

$$L_{\alpha,(n,\mu)}(q) = \sum_{s=1}^{t:\alpha_t \geq n} (-1)^{s-1} q^{\alpha_s - n} \sum_{\lambda:\lambda/\alpha^{(s)} \in \mathcal{H}_{(\alpha_s - n)}} 2^{a(\lambda/\alpha^{(s)})} L_{\lambda\mu}(q)$$

where $n > \mu_1$ and $\alpha^{(s)}$ is α with part α_s removed.

Proposition

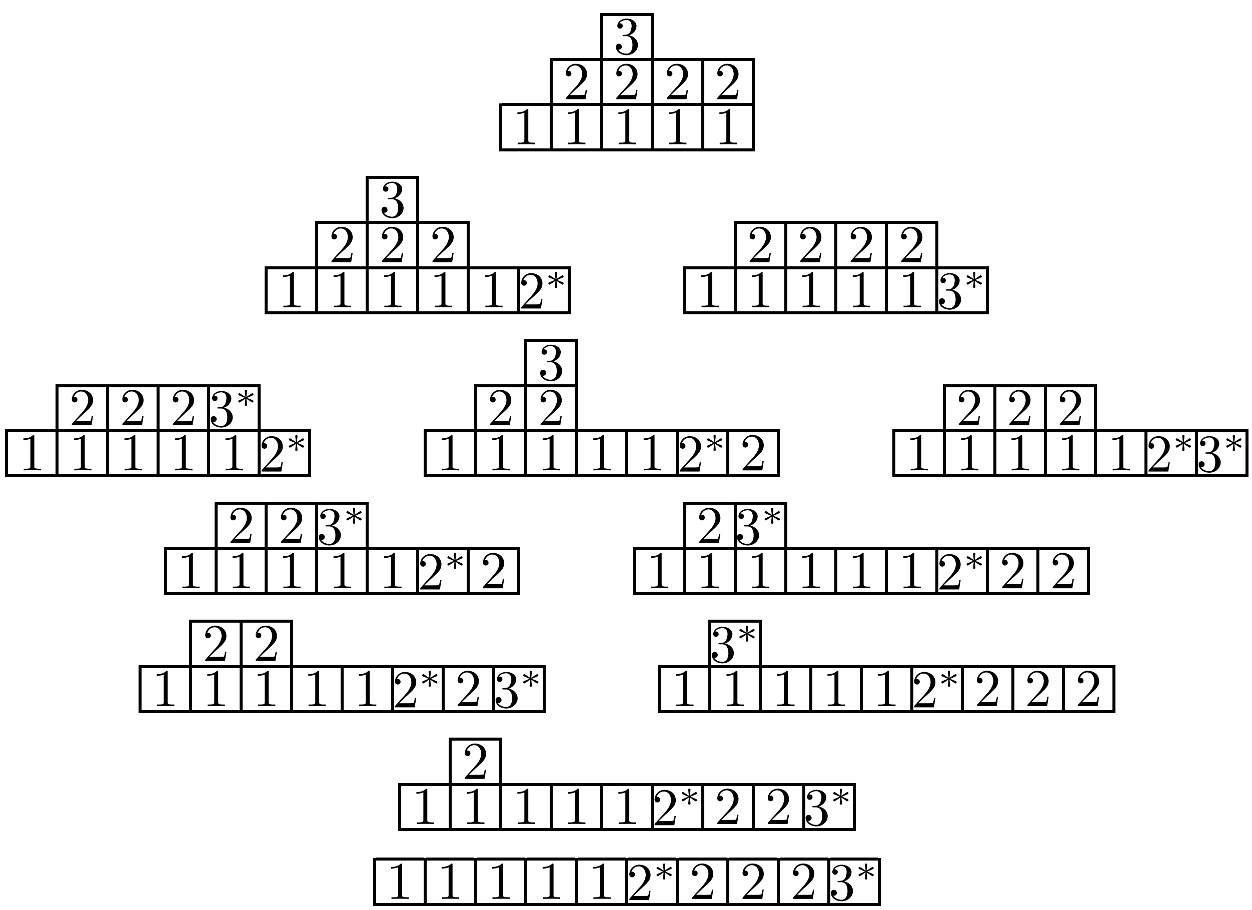
$$deg_q L_{\lambda\mu}(q) = q^{n(\mu)-n(\lambda)}$$

Proposition

$$2^{\ell(\mu)-\ell(\lambda)} \text{ divides } L_{\lambda\mu}(q)$$

Conjectured rank function for $MST^{(5,4,1)}$

$$
\begin{aligned}
G^q_{5,4,1} = {} & 4\,q^6 Q_{10} + \left(2\,q^4 + 4\,q^5\right) Q_{9,1} + \left(4\,q^4 + 4\,q^3\right) Q_{8,2} \\
& + \left(4\,q^2 + 4\,q^3\right) Q_{7,3} + 2\,q^2 Q_{7,2,1} \\
& + \left(2\,q + 4\,q^2\right) Q_{6,4} + 2\,q Q_{6,3,1} + Q_{5,4,1}
\end{aligned}
$$



Conjectures

Conjecture

$$G_\mu^q = \sum_{T \in MST^\mu} q^{d(T)} Q_{\lambda(T)}$$

where $d(T) = d(S)$ if T and S are the same tableaux except for markings.

In particular,
$L_{\lambda\mu}(q)$ is a polynomial in q with non-negative integer coefficients

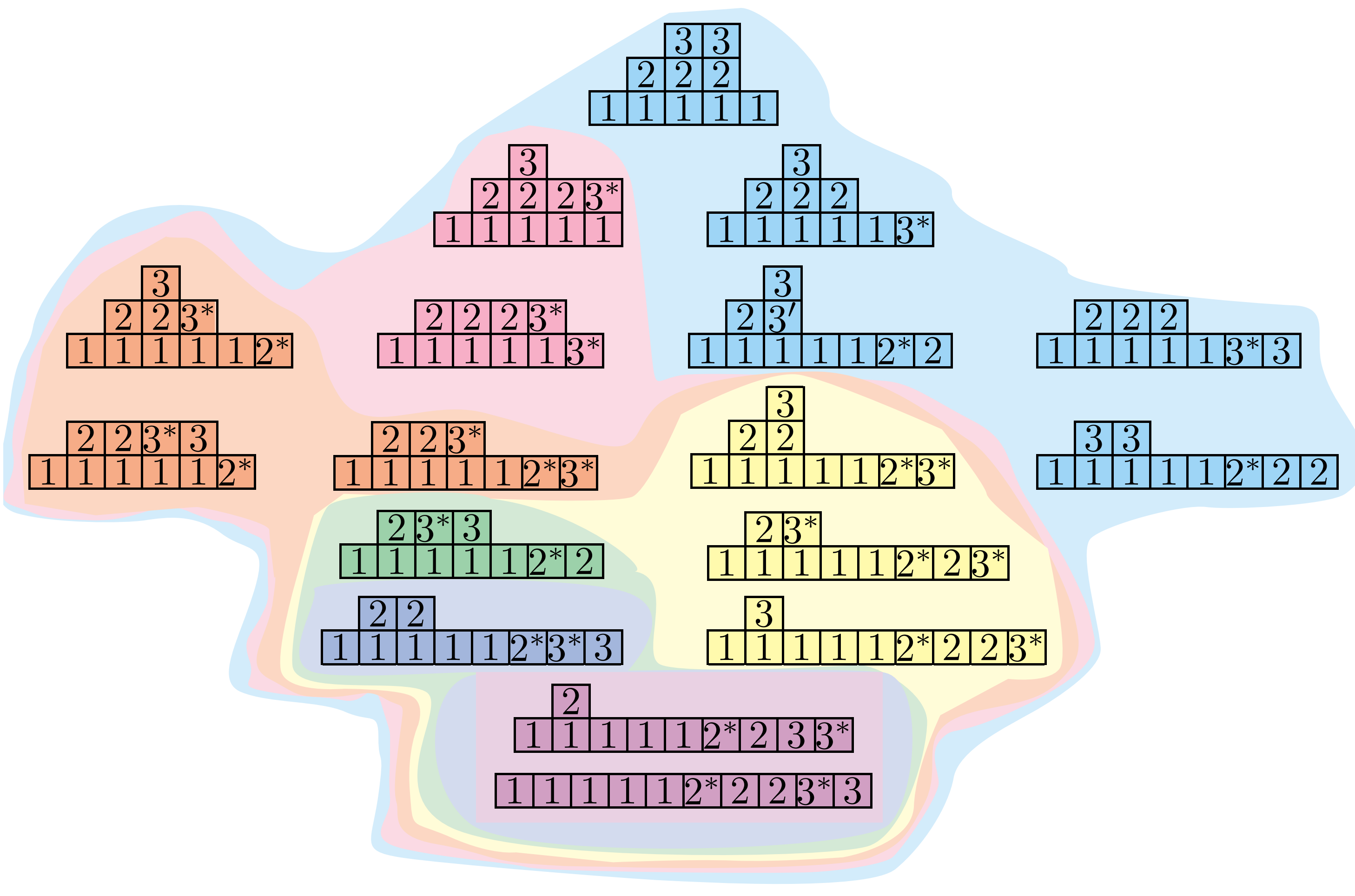
Conjecture
If $\mu > \lambda$ are strict partitions then

$$2^{\ell(\mu)} q^{n(\mu)} G_\lambda^q - 2^{\ell(\lambda)} q^{n(\lambda)} G_\mu^q$$

is Q-Schur positive

Conjectured rank function for $MST^{(5,3,2)}$

$$
\begin{aligned}
G^q_{5,3,2} = {} & 4\,q^7 Q_{10} + \left(4\,q^5 + 4\,q^6\right) Q_{9,1} + \left(4\,q^5 + 2\,q^3 + 8\,q^4\right) Q_{8,2} \\
& + \left(4\,q^4 + 2\,q^2 + 8\,q^3\right) Q_{7,3} + \left(2\,q^2 + 4\,q^3\right) Q_{7,2,1} \\
& + \left(4\,q^2 + 4\,q^3\right) Q_{6,4} + \left(2\,q + 4\,q^2\right) Q_{6,3,1} \\
& + 2\,q Q_{5,4,1} + Q_{5,3,2}
\end{aligned}
$$



Legend: