Gems of Algebra: The secret life of the symmetric group

Some things that you may not have known about permutations.

Permutation of n

$$\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

$$\sigma \text{ injective}$$

S_n = the set of permutations of n

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$$|S_n| = n!$$

Two line notation

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{pmatrix}$$

Example

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Example

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One line notation (word of a permutation)

$$\sigma(1)\sigma(2)\sigma(3)\cdots\sigma(n)$$

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Example one line notation:

$$367952184$$

Cycle notation

A cycle

$$(a_1, a_2, a_3, \ldots, a_r)$$

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$$\sigma = (a_1, a_2, \ldots, a_{r_1})(b_1, b_2, \ldots, b_{r_2}) \cdots (c_1, c_2, \ldots, c_{r_d})$$

where each of the cycles contain disjoint sets of integers

This representation is not unique, the cycles may be written in any order and any number in the cycle can be listed first.

Two line notation:

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One line notation:

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Cycle notation:

$$(137)(26)(49)(5)(8)$$

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$$(8)(94)(371)(5)(62)$$

Multiplication of permutations

Composition of two permutations as functions gives another permutation

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$$\sigma \circ \tau(i) = \sigma(\tau(i))$$

$\sigma \circ \tau$ will sometimes be written as $\sigma\tau$

Multiplication using two line notation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 9 & 7 & 6 & 3 & 1 & 5 & 8 \end{pmatrix}$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & 7 & 9 & 5 & 2 & 1 & 8 & 4 \end{pmatrix}$$

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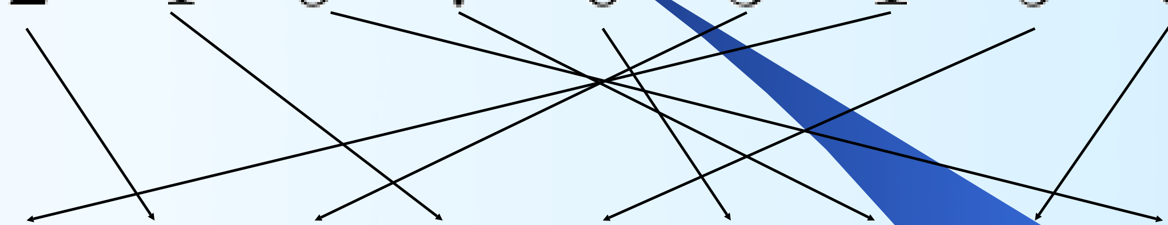

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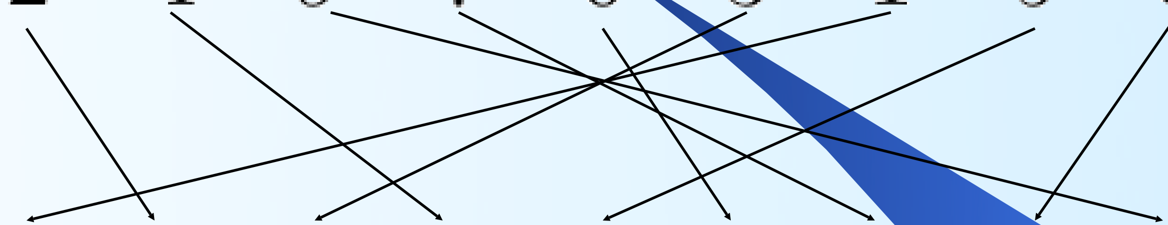

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Under the operation of multiplication S_n forms a group.

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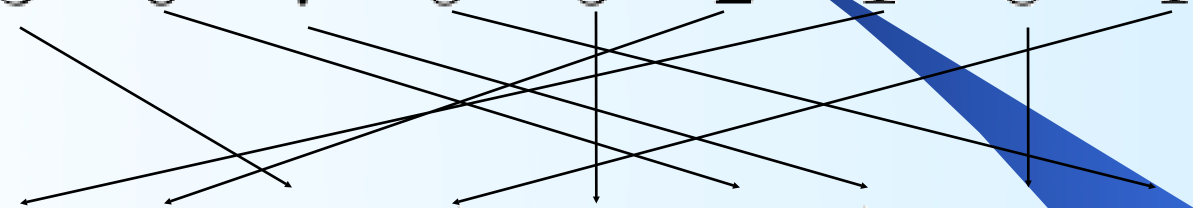

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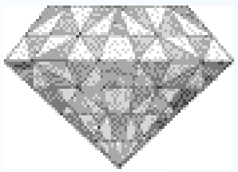
The group of permutations is called the symmetric group



The symmetric group contains every group of order n as a subgroup

$g_1, g_2, g_3, \ldots, g_n$ are the elements of a group of order n

$$g \leftrightarrow \begin{pmatrix} g_1 & g_2 & g_3 & \cdots & g_n \\ g \cdot g_1 & g \cdot g_2 & g \cdot g_3 & \cdots & g \cdot g_n \end{pmatrix}$$

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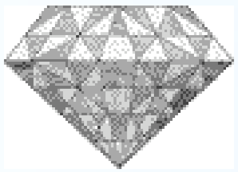

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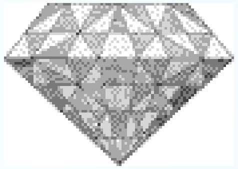

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$$g \leftrightarrow \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{pmatrix}$$

Then these corresponding elements will multiply in the symmetric group just as they do in their own group.

Generators and relations

$$s_i(i) = i + 1 \qquad s_i(i+1) = i$$

$$s_i(j) = j \text{ for } j \neq i, i+1$$

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The elements $s_1, s_2, \ldots, s_{n-1}$ generate S_n

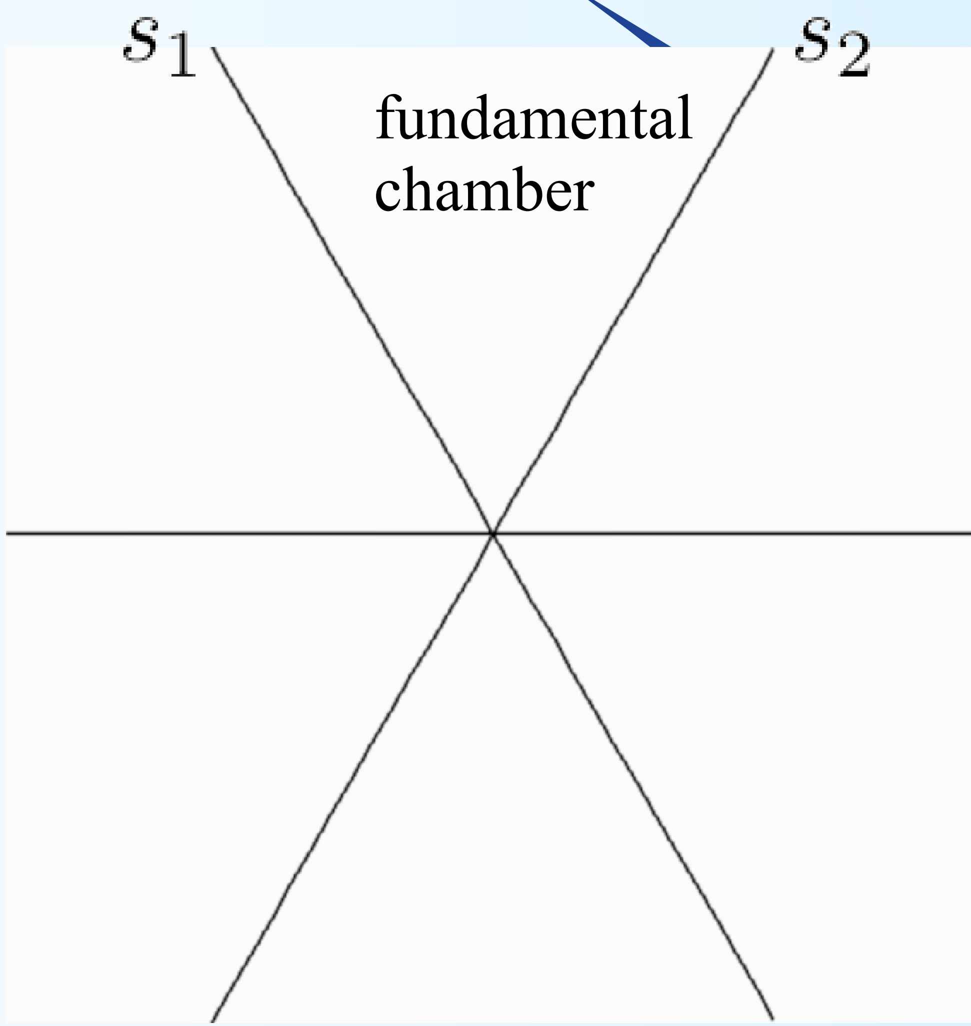
where these elements are characterized by the relations

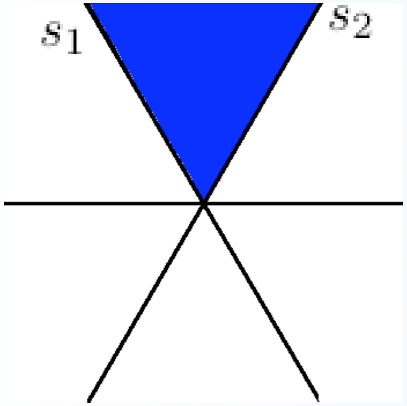
$$s_i^2 = 1$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

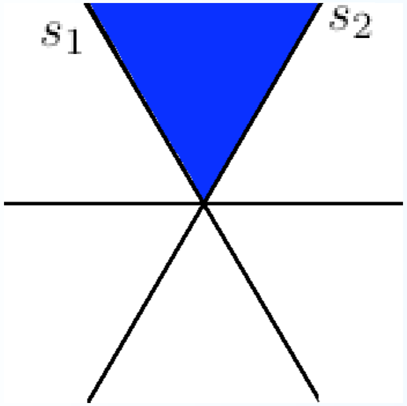
$$s_i s_j = s_j s_i \text{ for } |i - j| \geq 2$$

(Coxeter) The set of permutations can be realized as compositions of reflections across hyperplanes in $\mathbb{R}^{n-1}$ which divide the space into $n!$ chambers.





$$1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$



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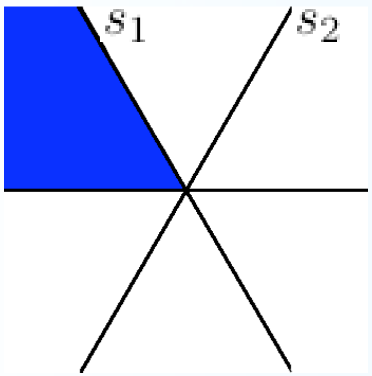
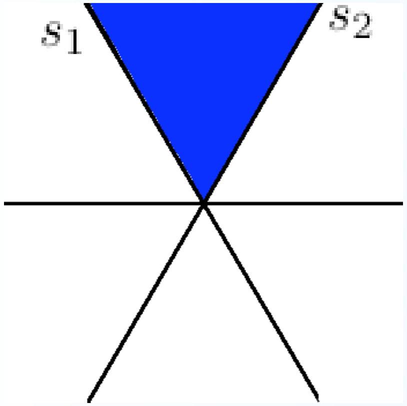


$$s_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$



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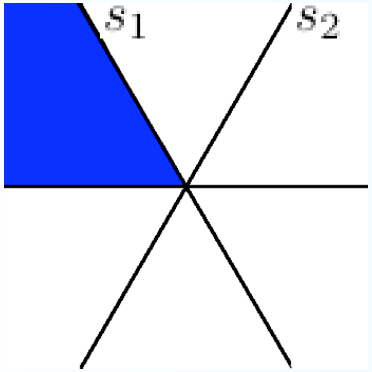

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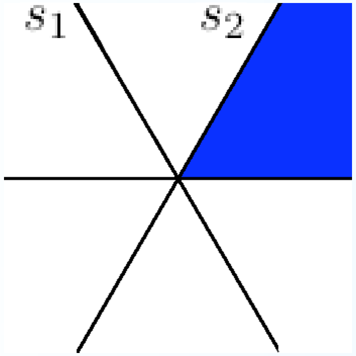
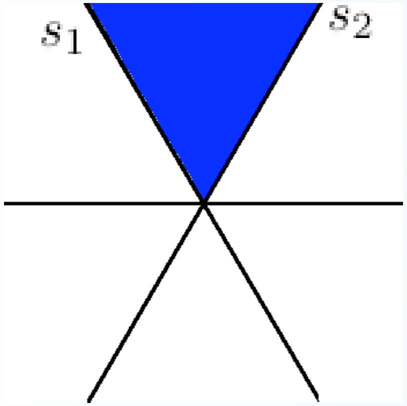


$$s_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$



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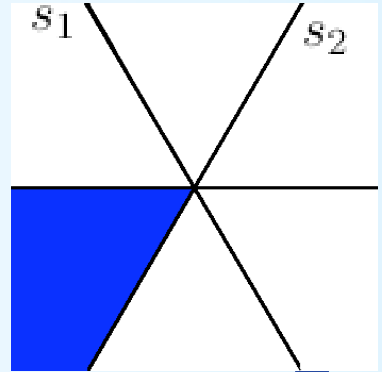

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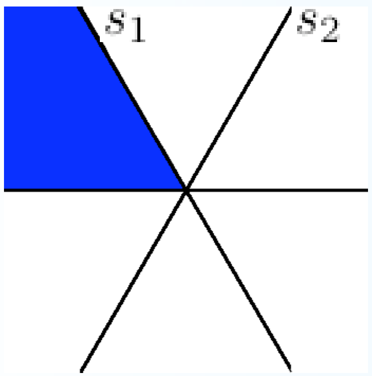

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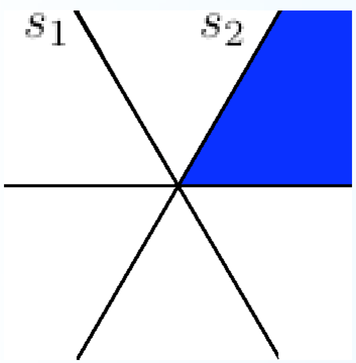

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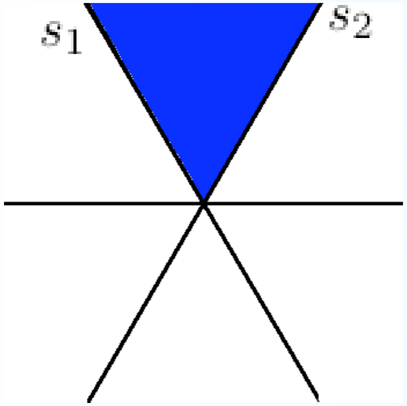

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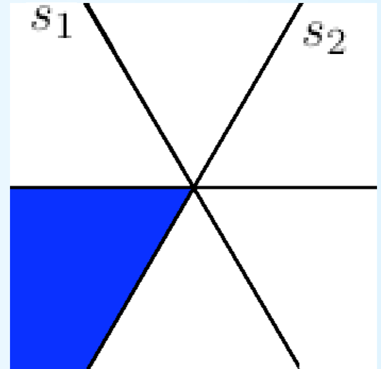

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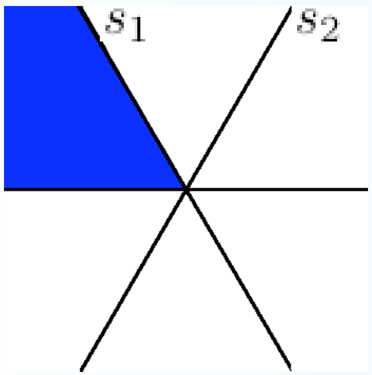

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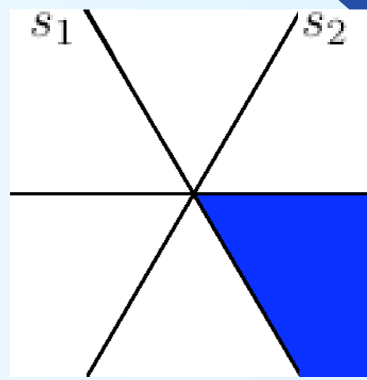

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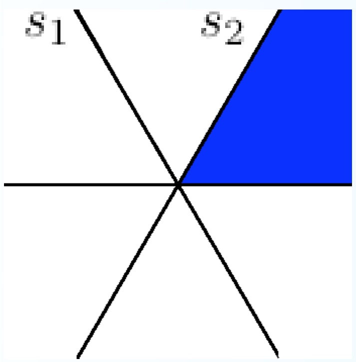

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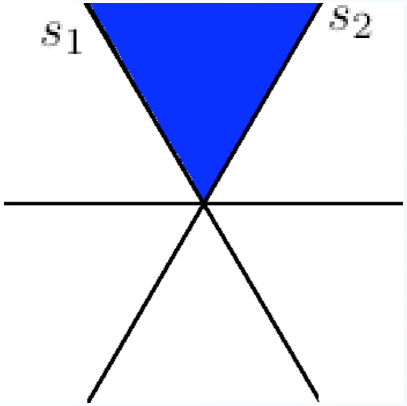

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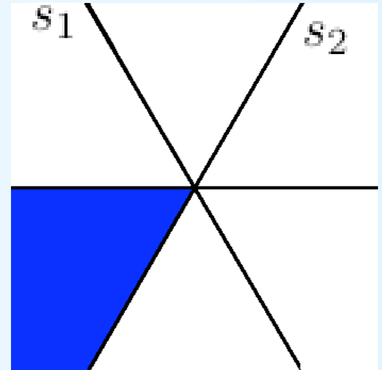

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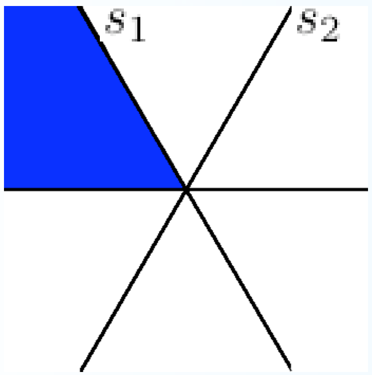

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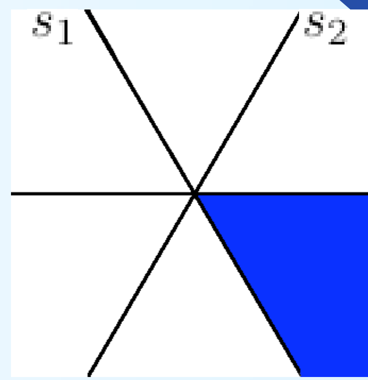

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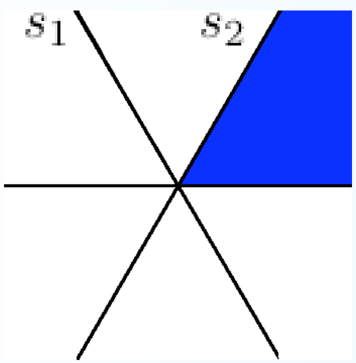

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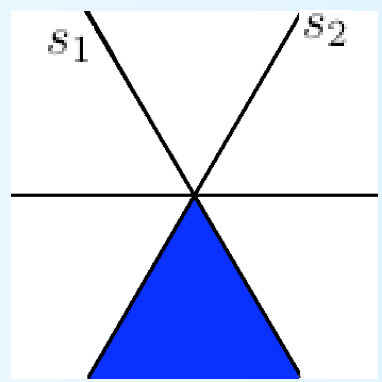

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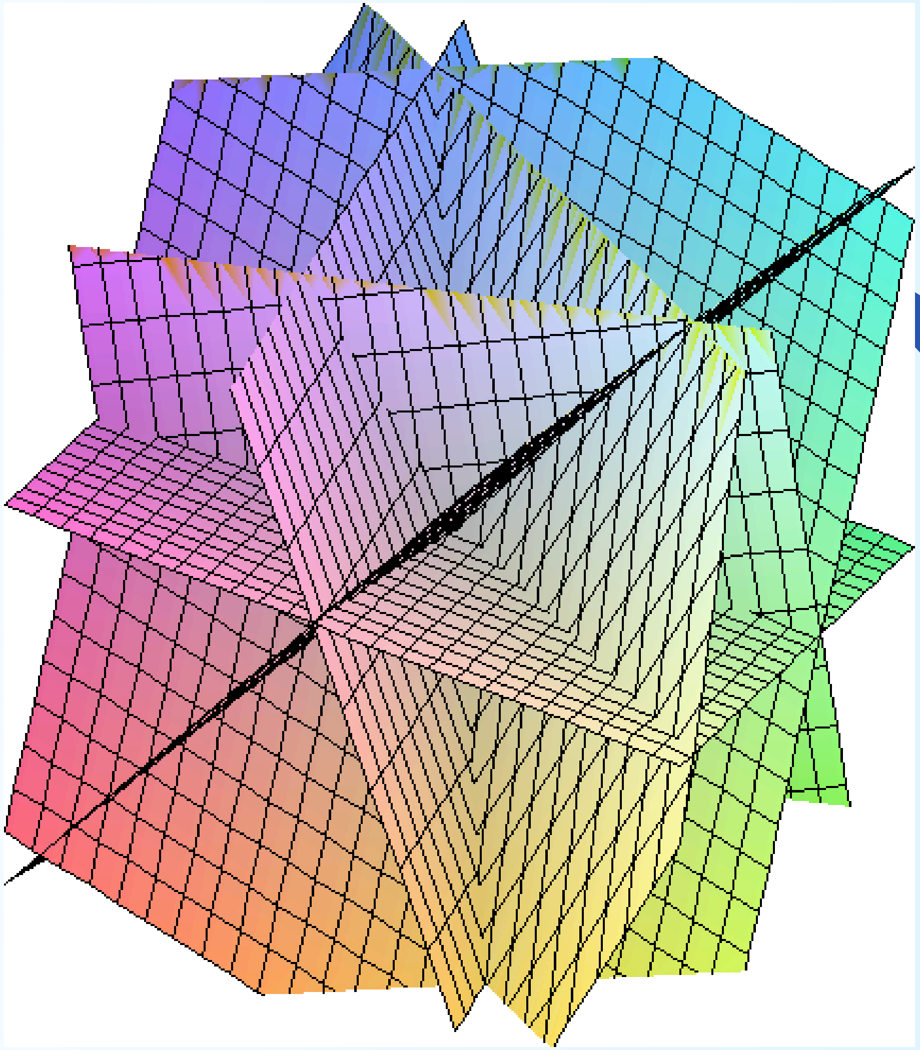
Hyperplanes of $\mathbb{R}^3$ representing S_4



From this perspective we have the notion of the length of a permutation.

The length of a permutation σ is the length of the smallest word of elements $s_i's$ that can be used to represent σ .

$$\ell(1) = 0 \qquad \ell(s_1) = \ell(s_2) = 1$$

$$\ell(s_1 s_2) = \ell(s_2 s_1) = 2 \qquad \ell(s_1 s_2 s_1) = 3$$

Consider:

$$\sum_{\sigma \in S_n} q^{\ell(\sigma)}$$

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Example:

$$\sum_{\sigma \in S_3} q^{\ell(\sigma)} = 1 + 2q + 2q^2 + q^3$$
$$= (1+q)(1+q+q^2)$$

$$\sum_{\sigma \in S_n} q^{\ell(\sigma)} = [n]!$$

where

$$[k] = 1 + q + q^2 + \cdots + q^{k-1}$$

and

$$[n]! = [n][n-1][n-2]\cdots[2][1]$$

$\ell(\sigma)$ is called an Eulerian 'statistic'

The length of a permutation is equal to the the number of inversions in the permutation.

$$inv(\sigma) = \{(i,j) : i < j \text{ and } \sigma(i) > \sigma(j)\}$$

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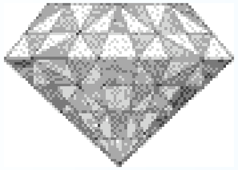

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$$367952184$$

number of inversions left of: 0 0 0 0 3 5 6 1 5

$$s_8 \ \ s_5 s_6 s_7 \ \ s_4 s_5 s_6 s_7 s_8 \ \ s_2 s_3 s_4 s_5 s_6 \ \ s_1 s_2 s_3 s_4 s_5 s_6$$

Weak order

$$\sigma < \tau \text{ if } \ell(\sigma) < \ell(\tau) \text{ and}$$
$$\sigma\pi = \tau \text{ for some permutation } \pi$$

We may 'draw' this with a graph (vertices and edges) so that the vertices are permutations and there is an edge between two permutations if $\sigma s_i = \tau$ for some i

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To create the following images we also put some additional restrictions

1. the level will depend on the length of the permutation
2. the color of the edge will determine the position that is changing so that every permutation will have one edge of each color.

Graph of Weak order for S_3

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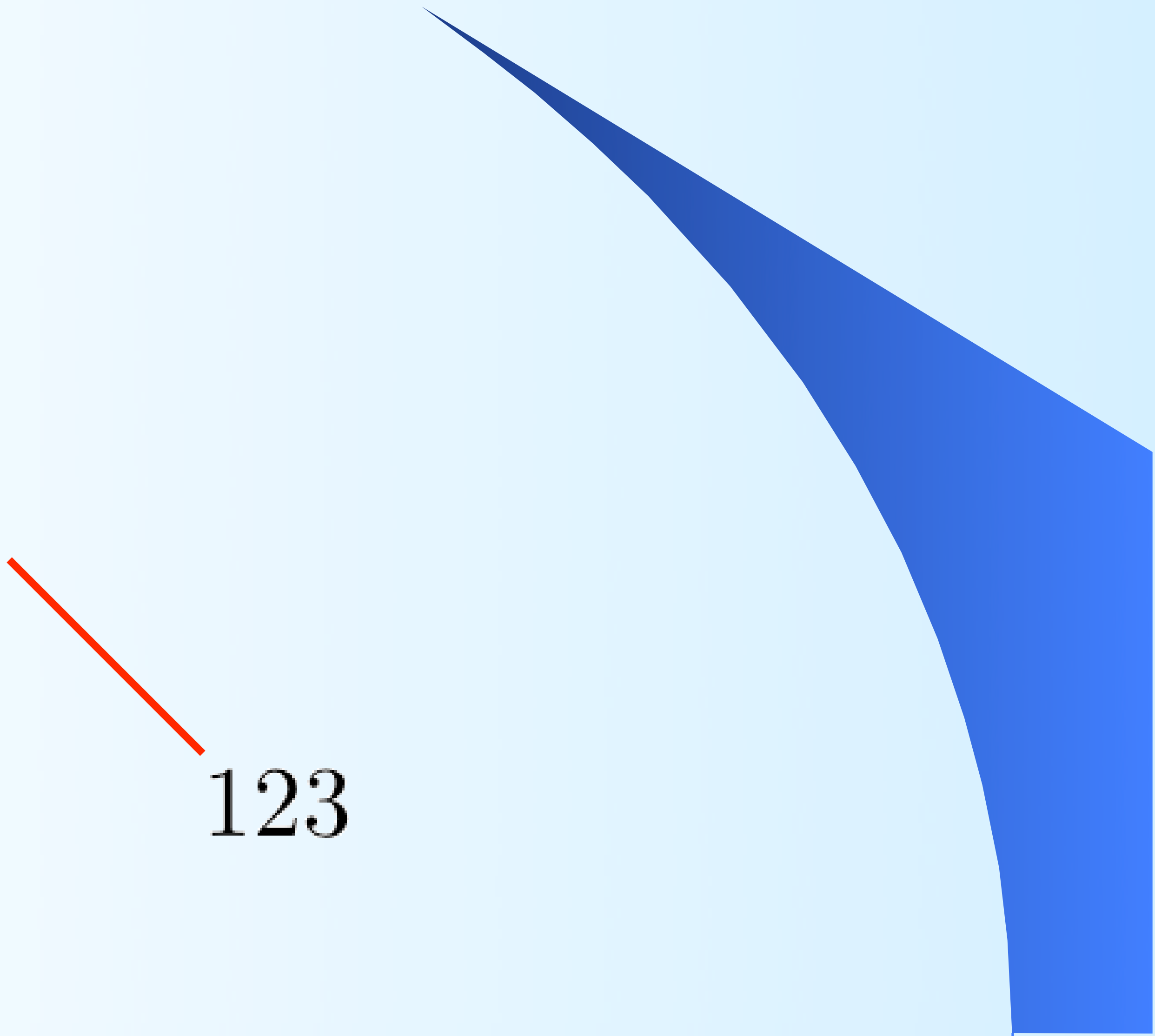
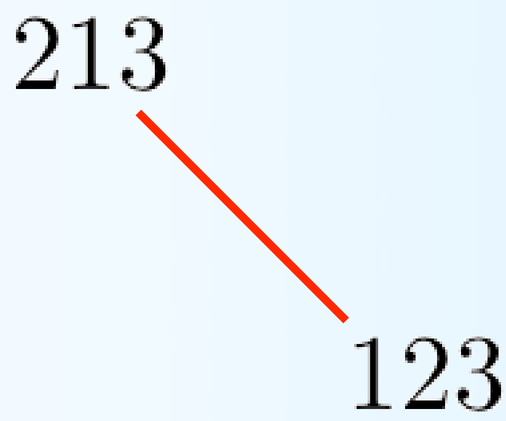


Graph of Weak order for S_3



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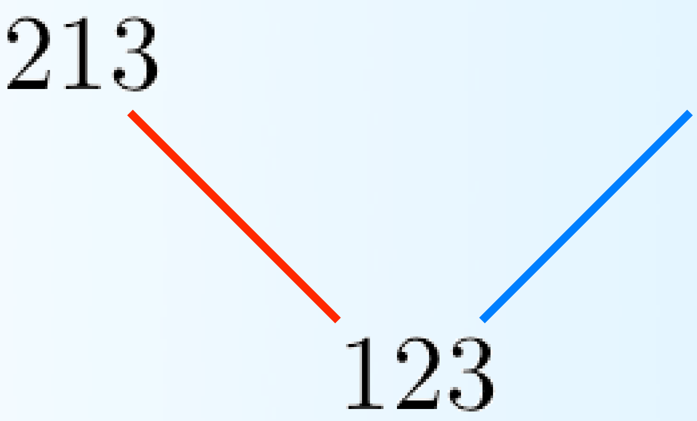
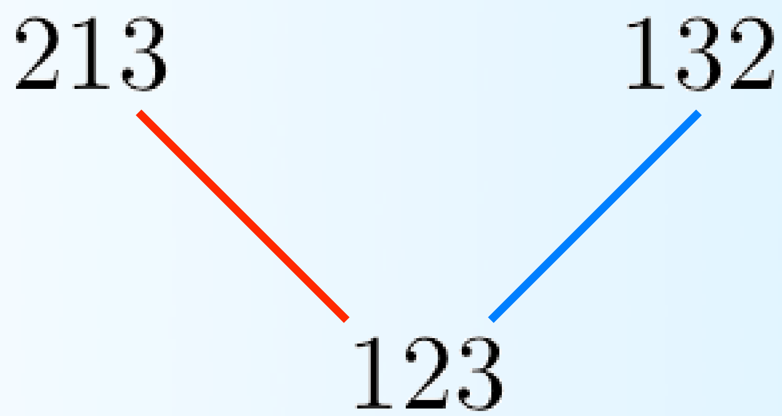
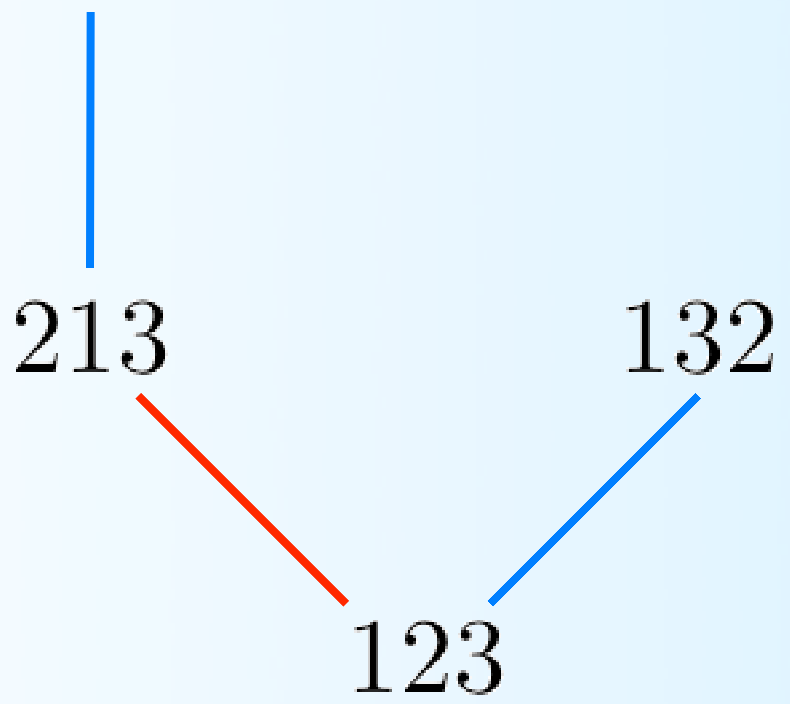


Graph of Weak order for S_3



Graph of Weak order for S_3



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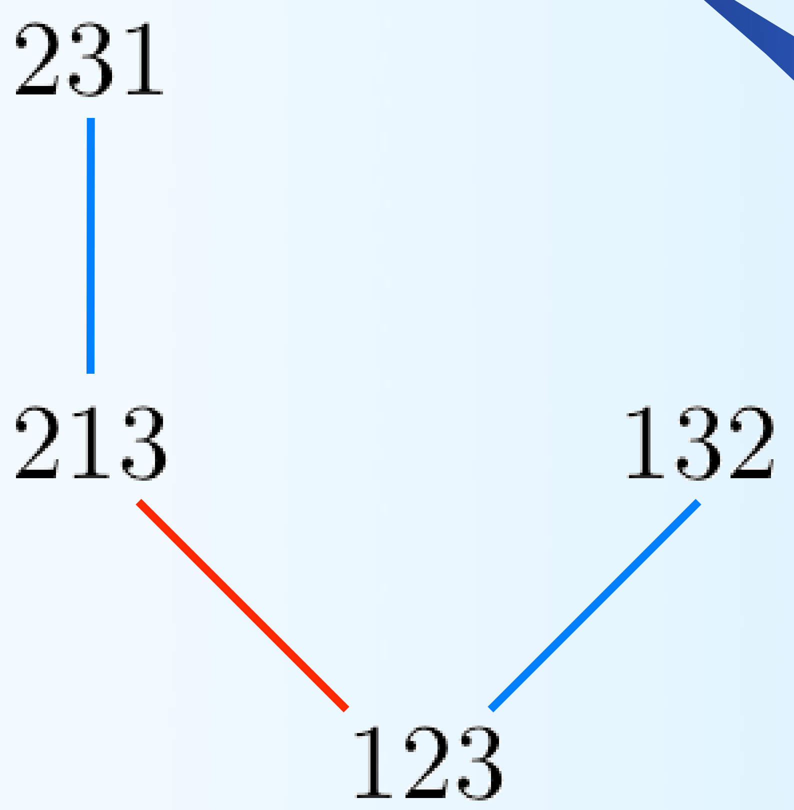

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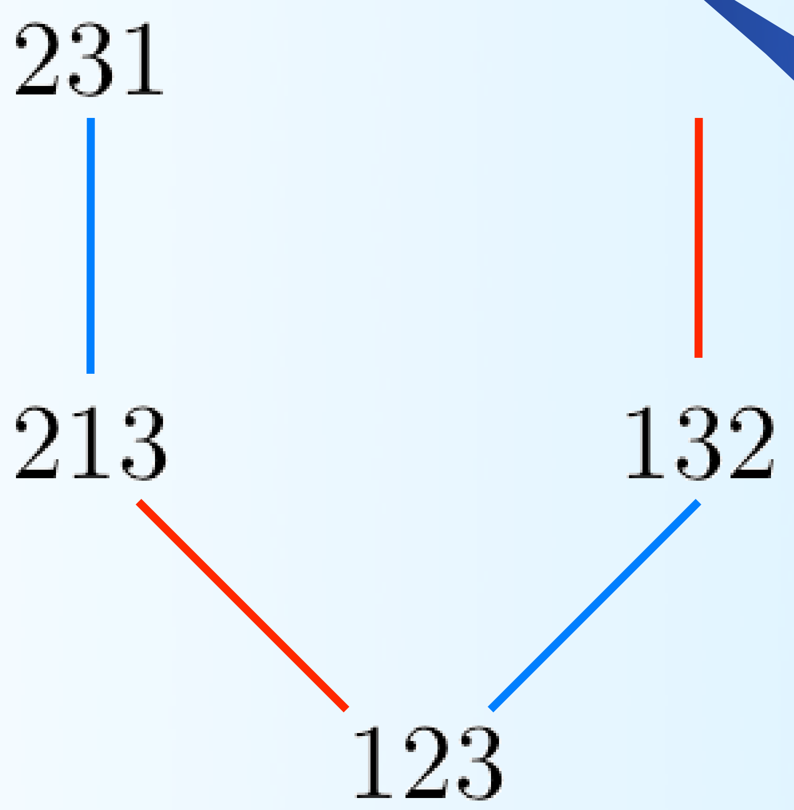

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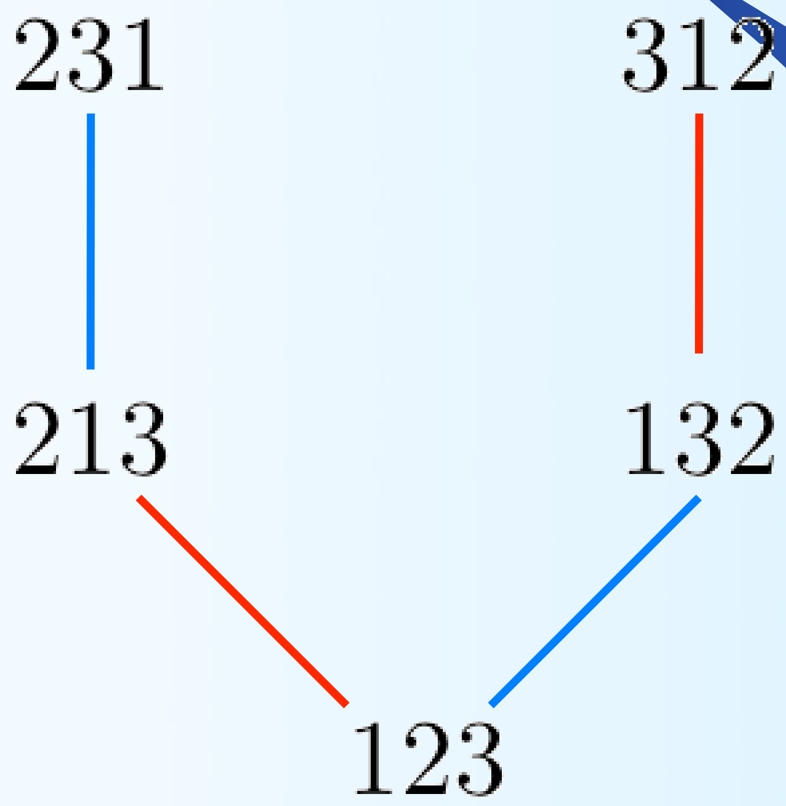

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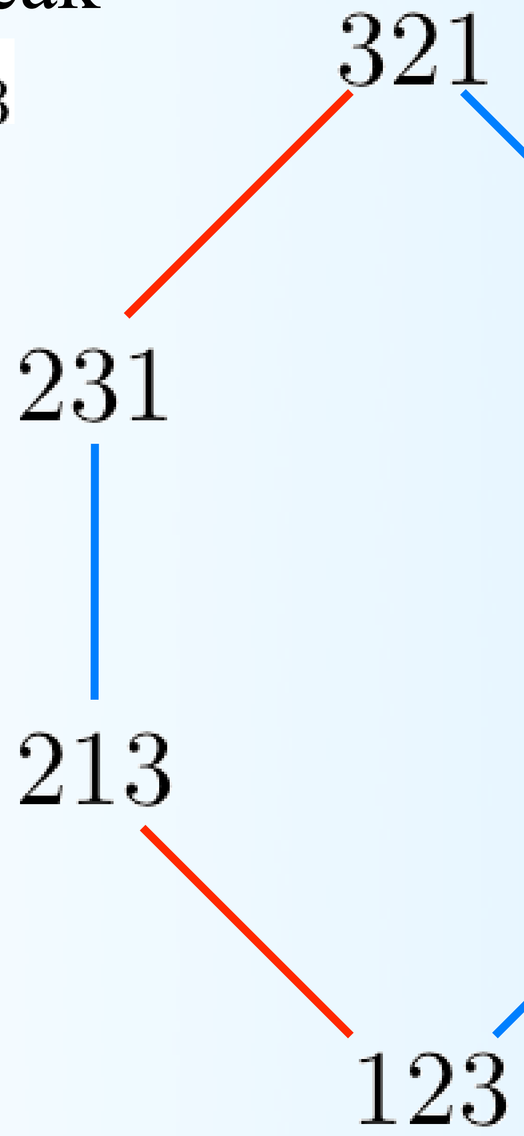
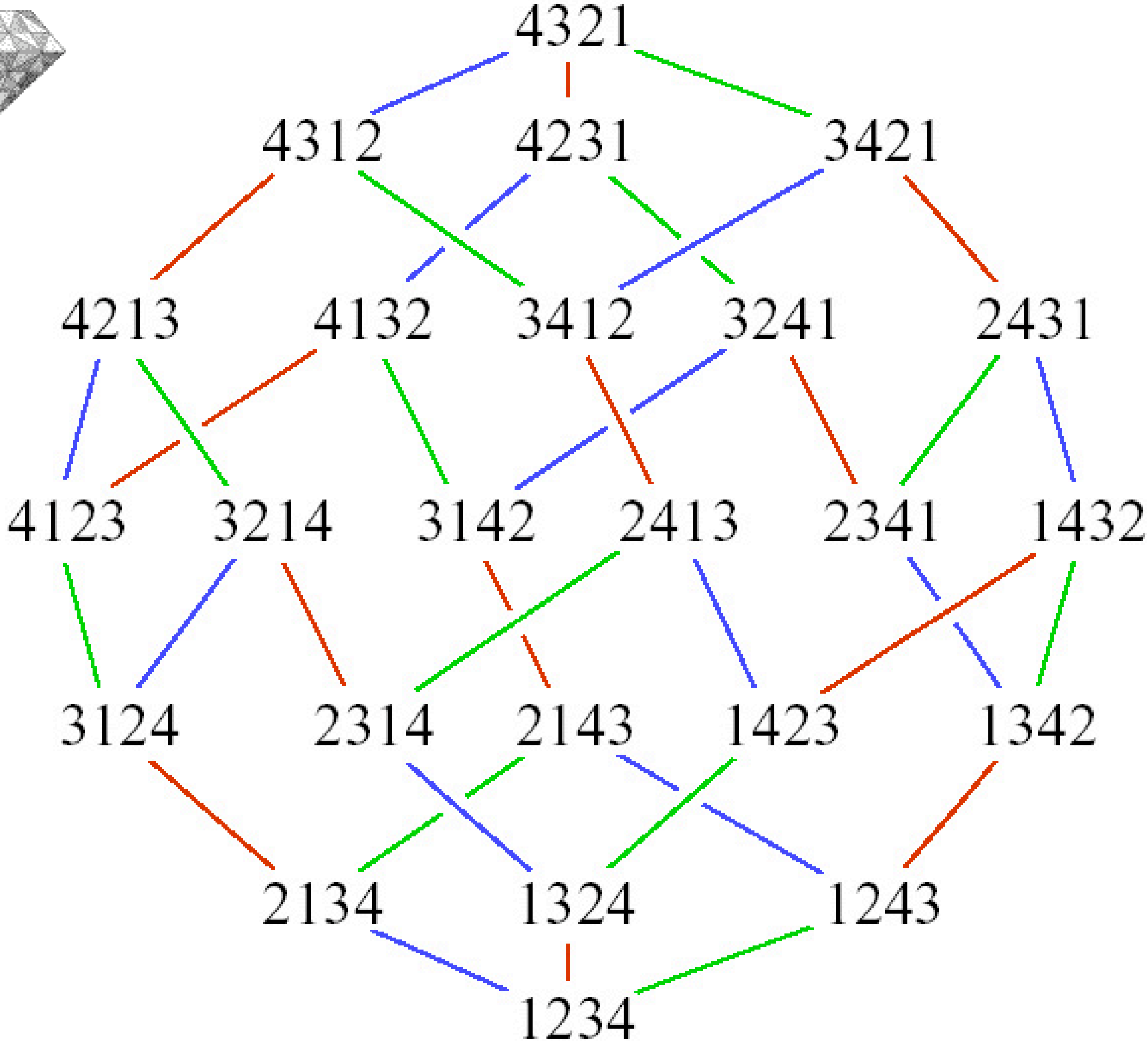
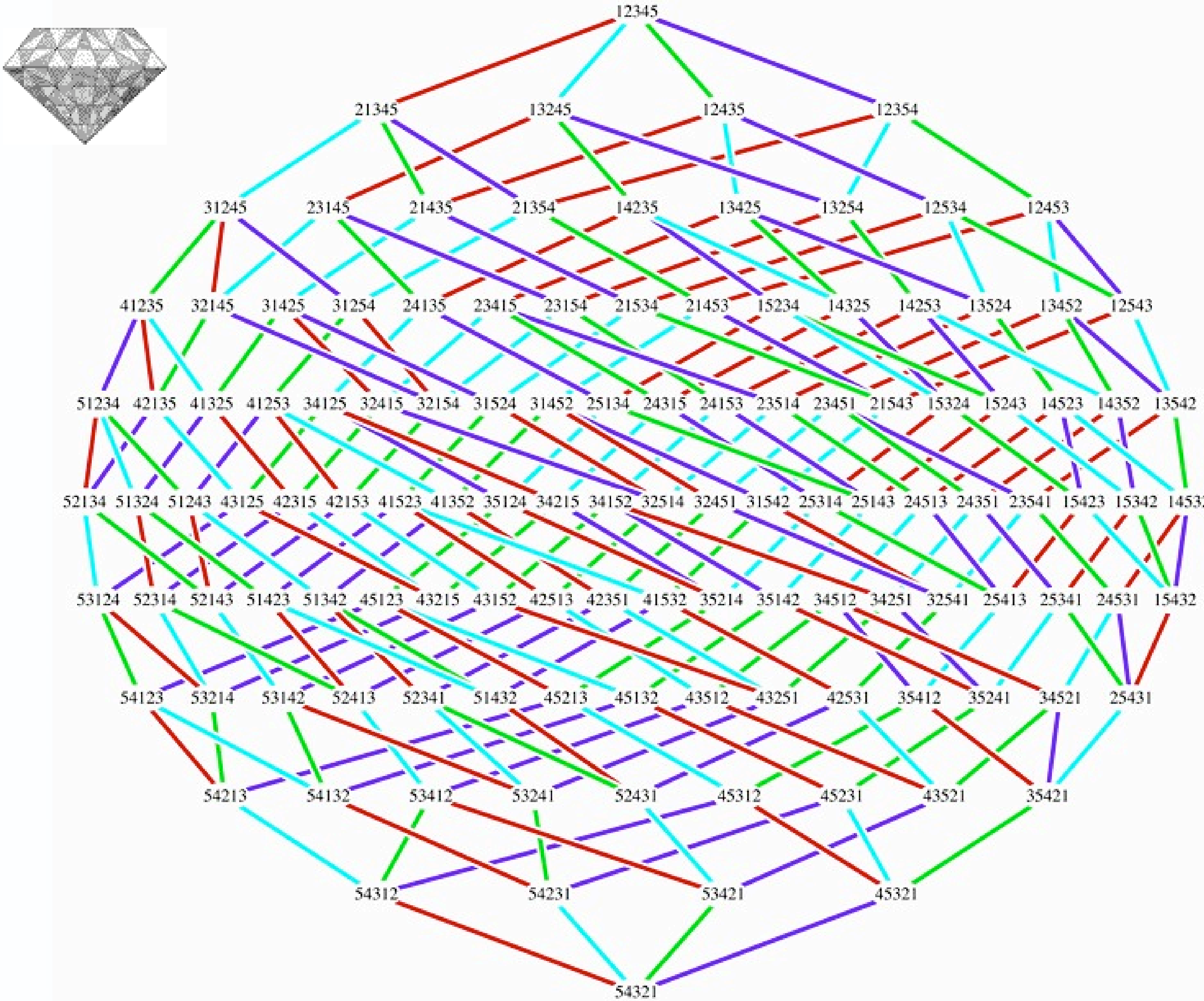
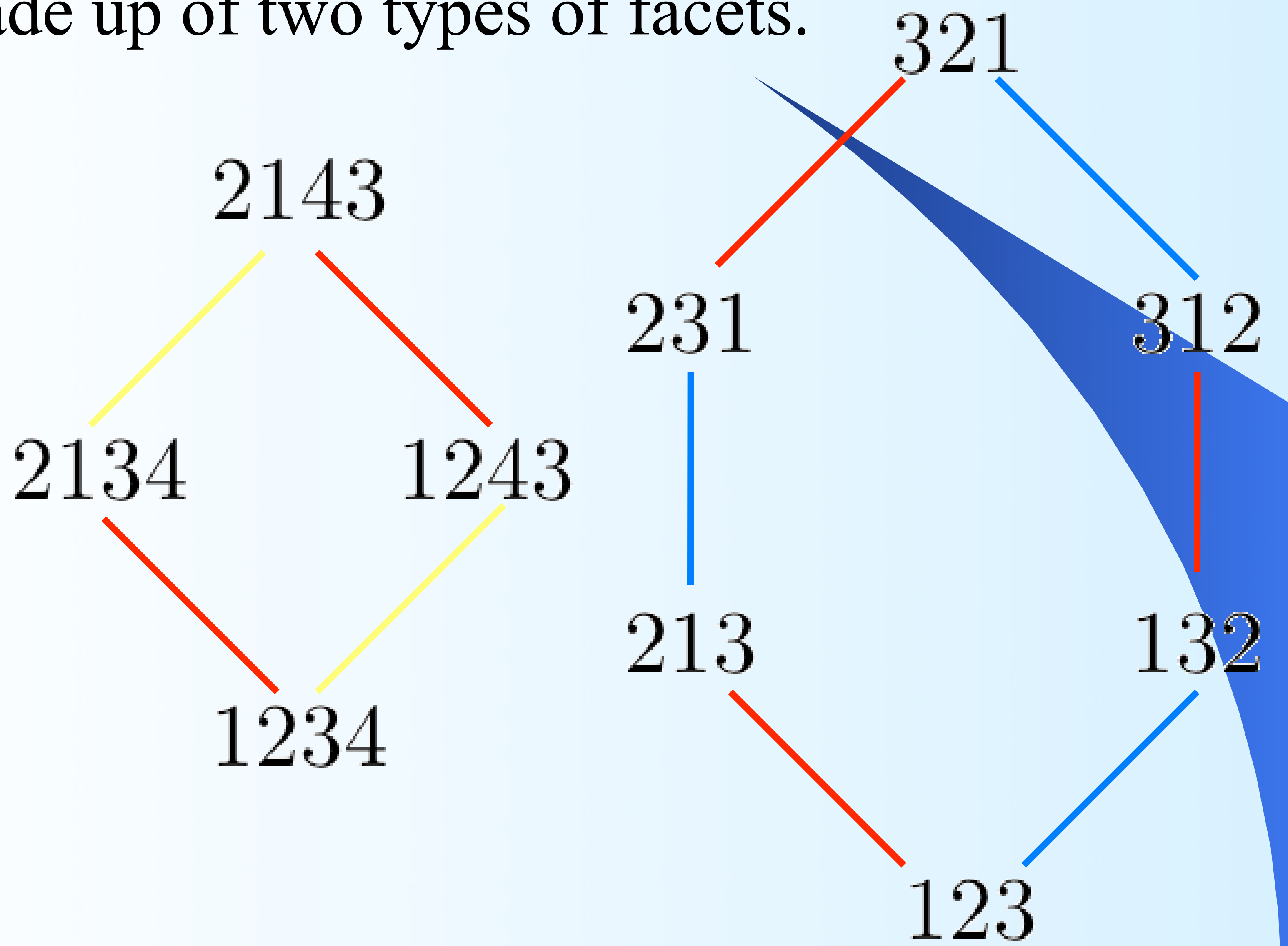


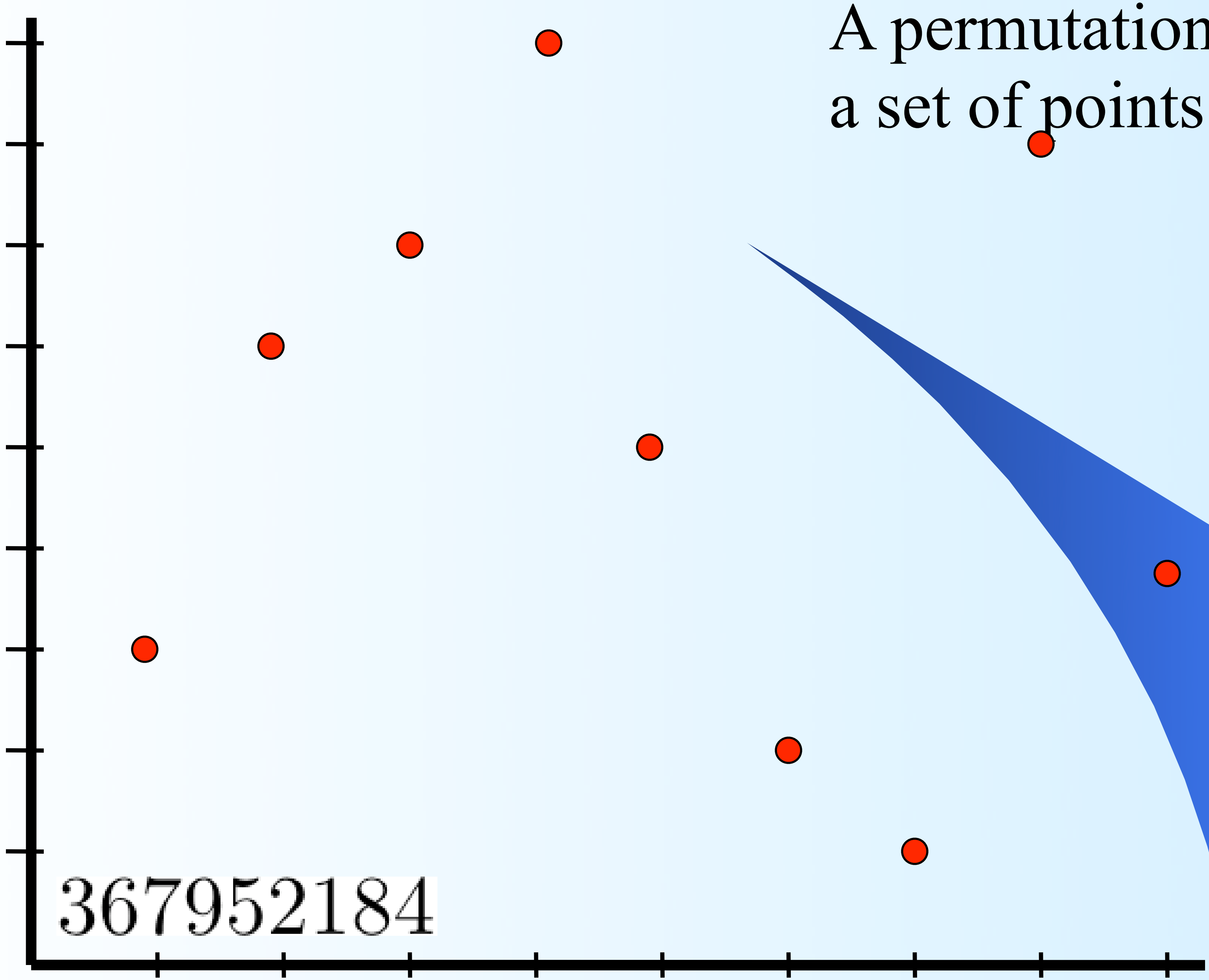

4321
4312
4231
3421
4213
4132
3412
3241
2431
4123
3214
3142
2413
2341
1432
3124
2314
2143
1423
1342
2134
1324
1243
1234


12345
21345 13245 12435 12354
31245 23145 21435 21354 14235 13425 13254 12534 12453
41235 32145 31425 31254 24135 23415 23154 21534 21453 15234 14325 14253 13524 13452 12543
51234 42135 41325 41253 34125 32415 32154 31524 31452 25134 24315 24153 23514 23451 21543 15324 15243 14523 14352 13542
52134 51324 51243 43125 42315 42153 41523 41352 35124 34215 34152 32514 32451 31542 25314 25143 24513 24351 23541 15423 15342 14532
53124 52314 52143 51423 51342 45123 43215 43152 42513 42351 41532 35214 35142 34512 34251 32541 25413 25341 24531 15432
54123 53214 53142 52413 52341 51432 45213 45132 43512 43251 42531 35412 35241 34521 25431
54213 54132 53412 53241 52431 45312 45231 43521 35421
54312 54231 53421 45321
54321

Note that all faces of the permuta-hedrons are made up of two types of facets.



A permutation as a set of points



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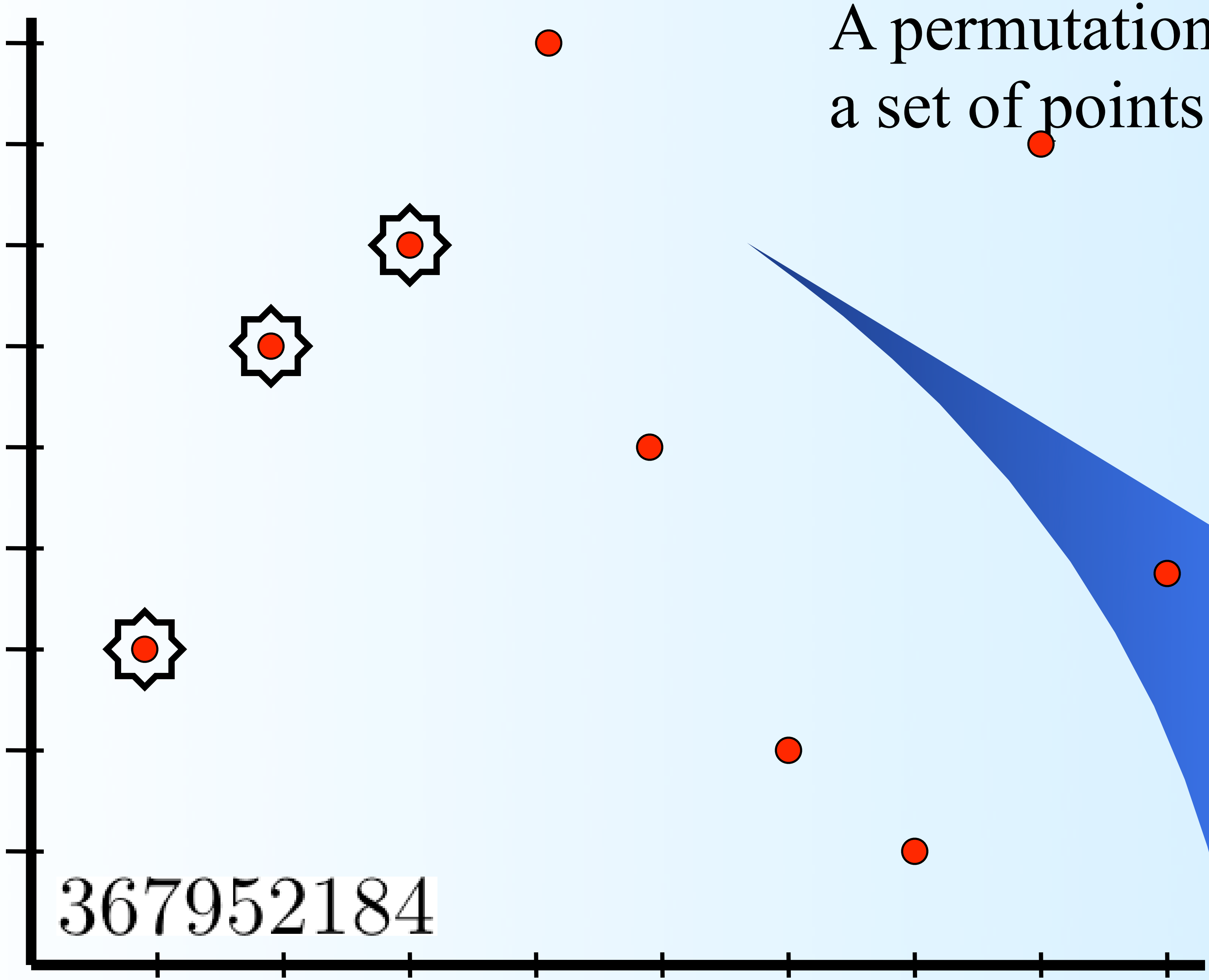
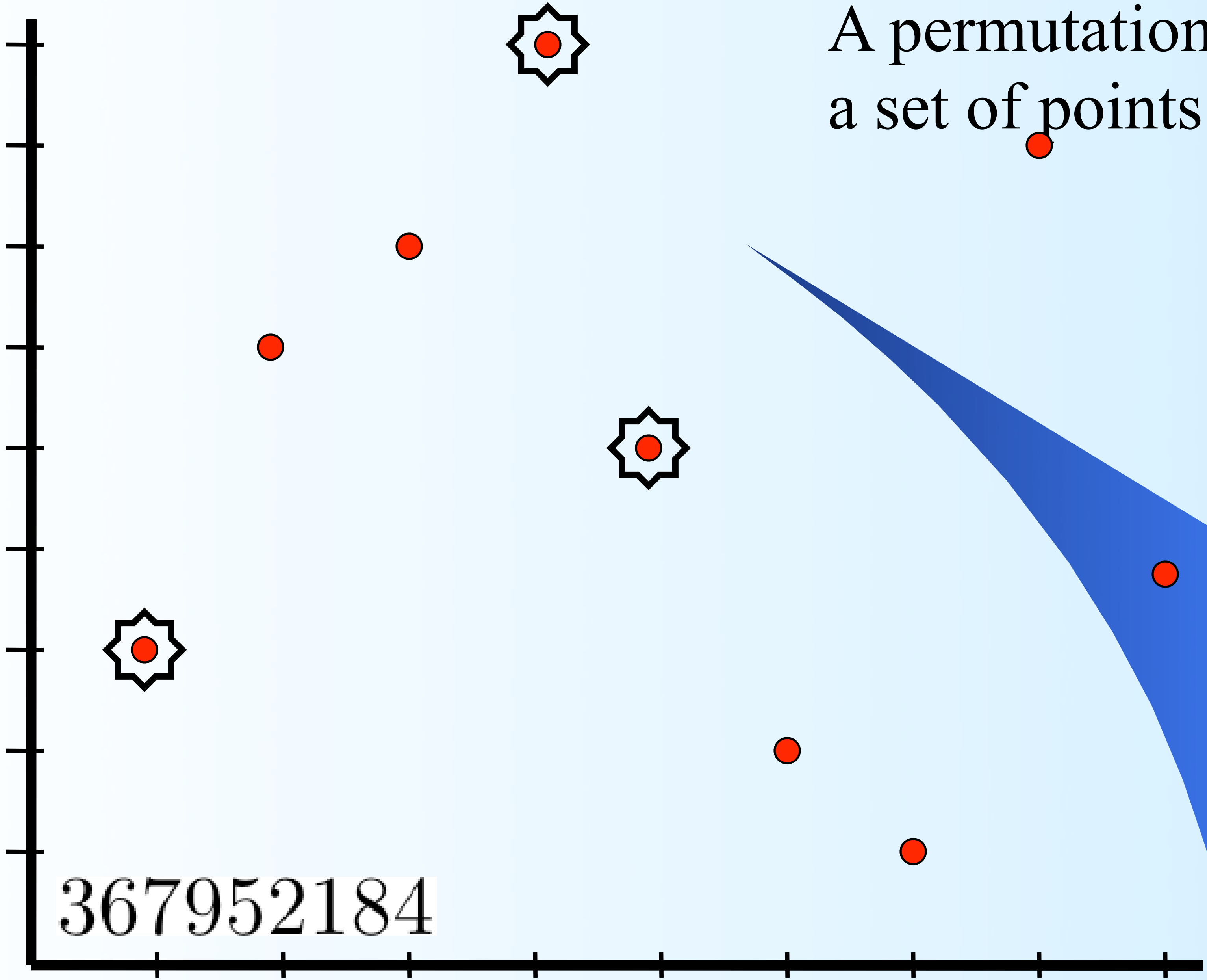


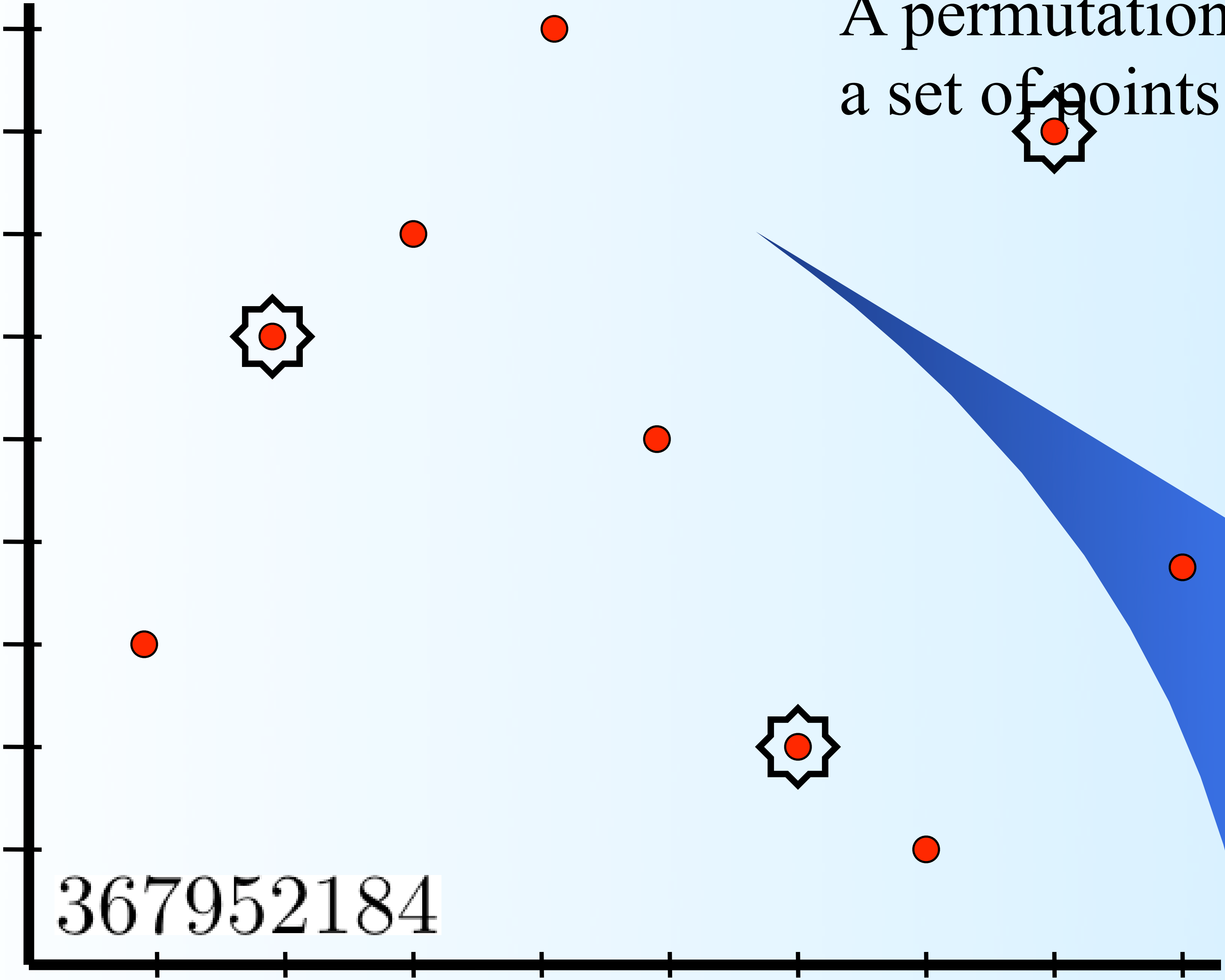
Contains the permutation 123

A permutation as a set of points



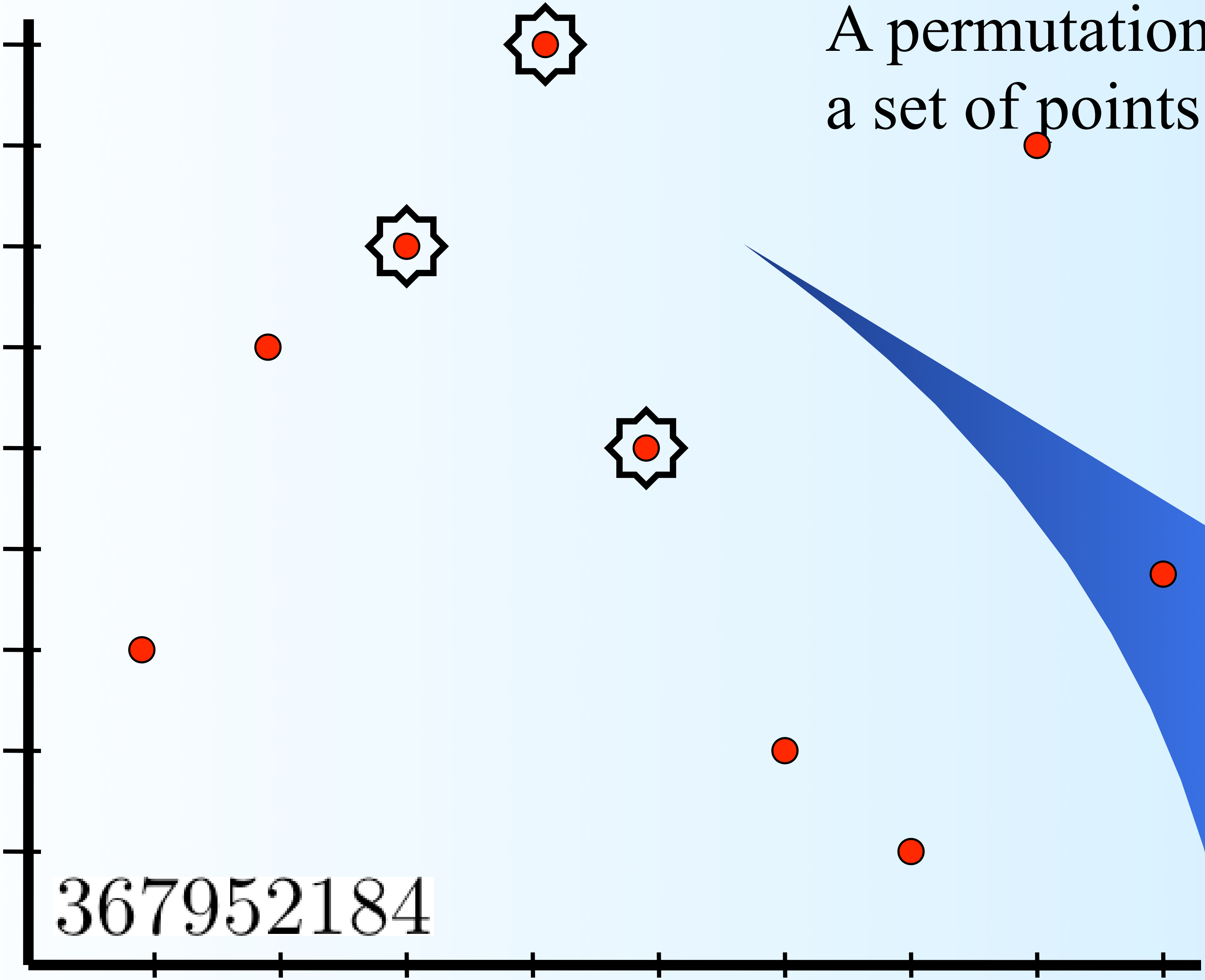
Contains the permutation 132

A permutation as a set of points



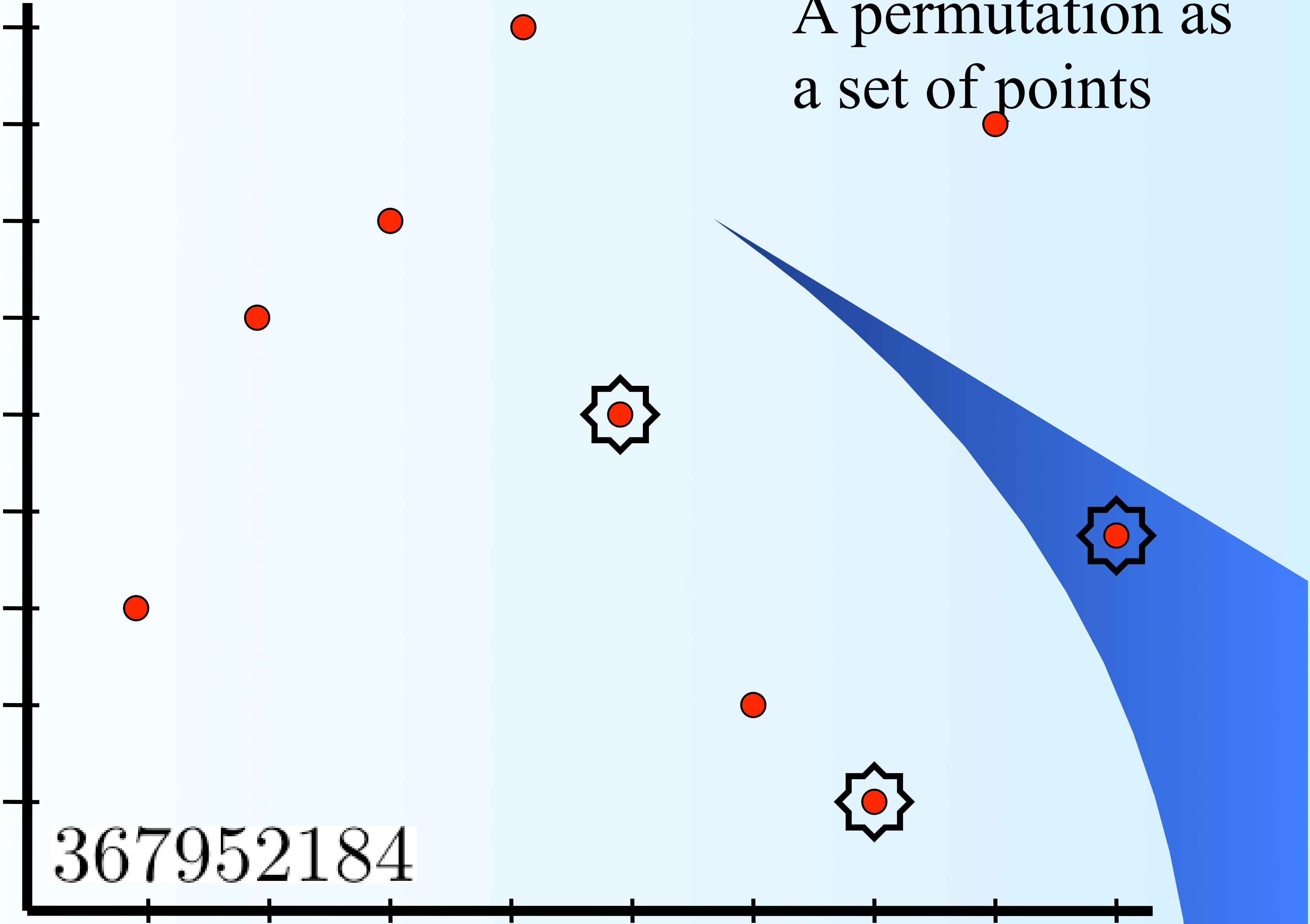
Contains the permutation 213

A permutation as a set of points



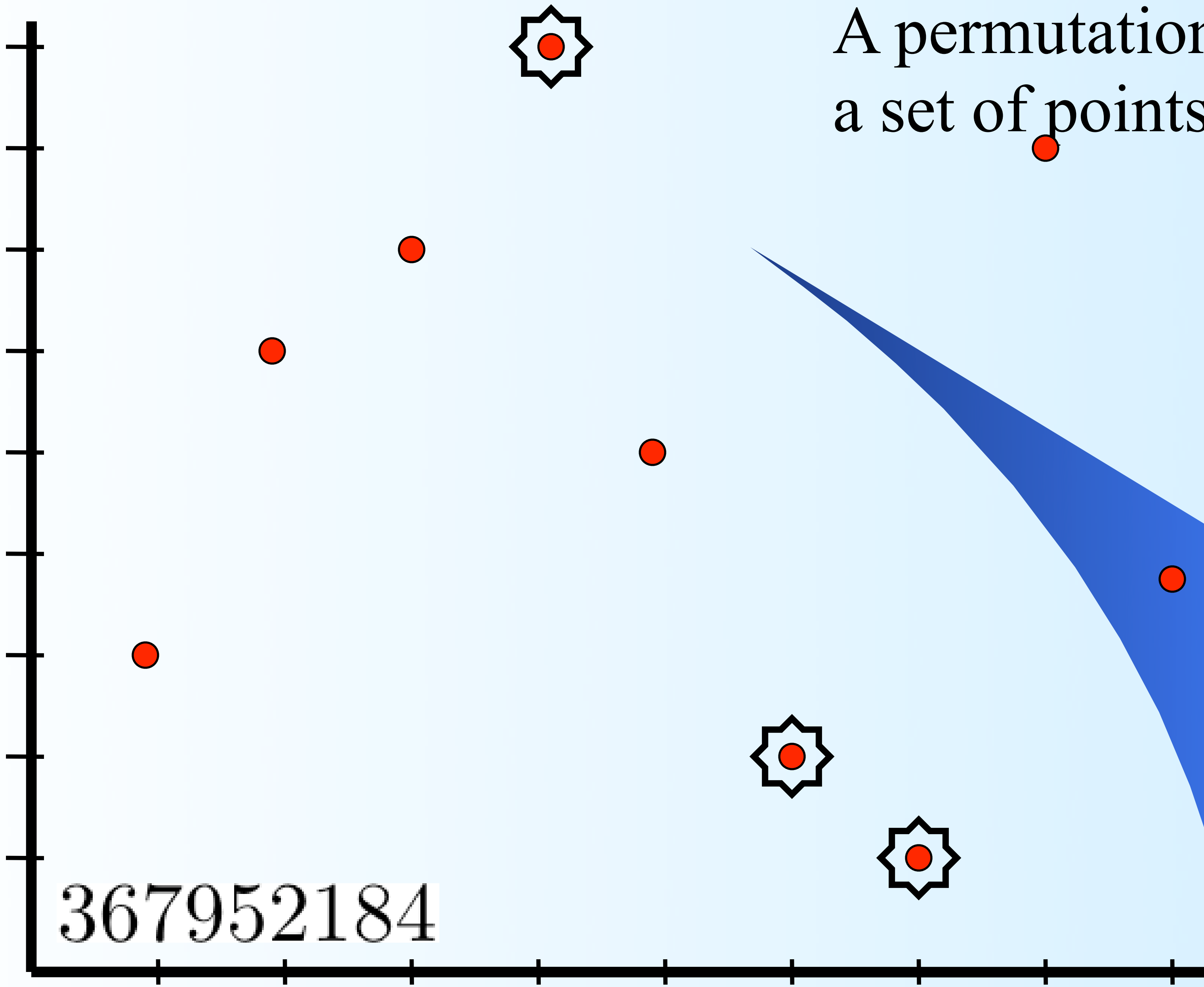
Contains the permutation 231

A permutation as a set of points



Contains the permutation 312

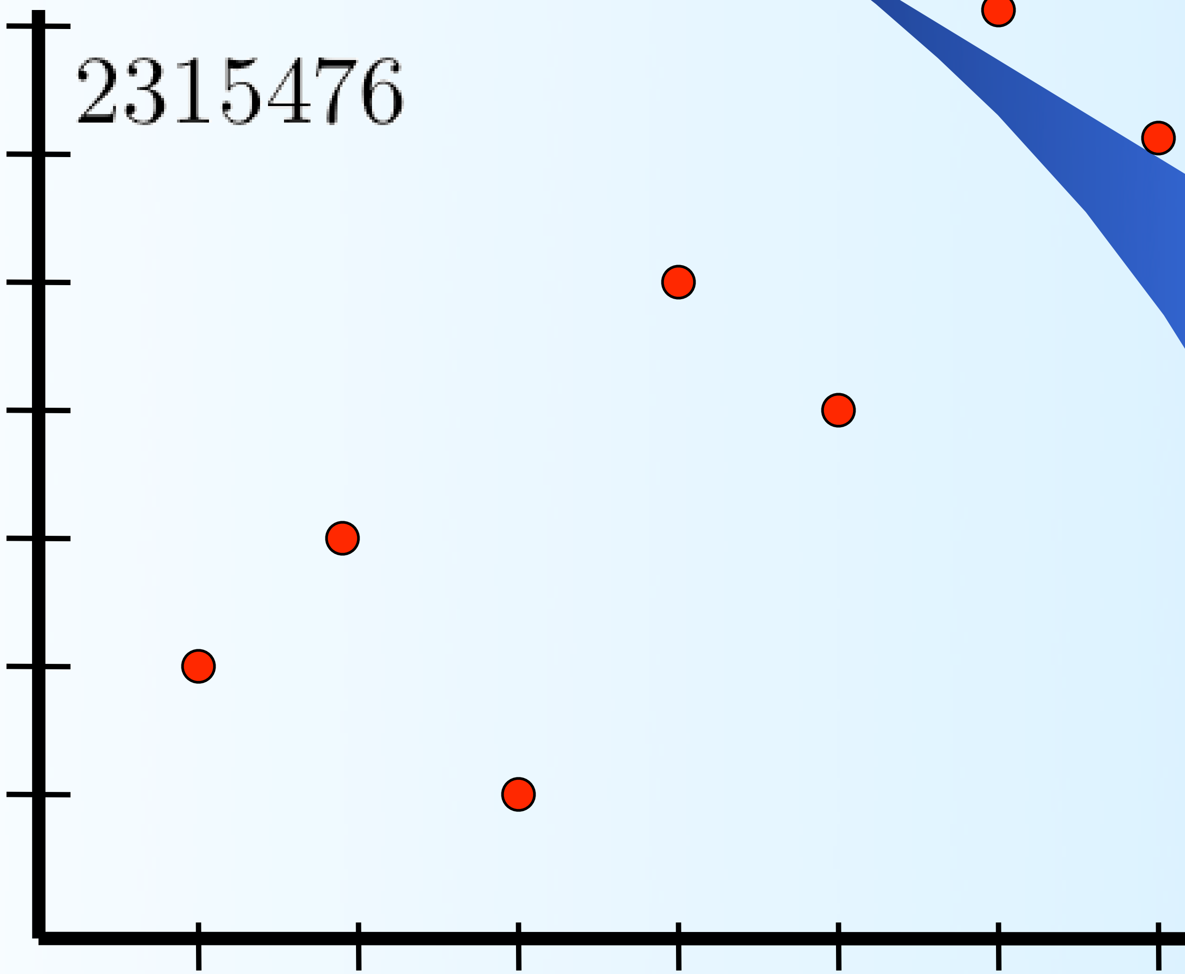
A permutation as a set of points



Contains the permutation 321

Define

$S_n(\pi_1\pi_2\cdots\pi_k)$ to be the set of partitions which do not contain the pattern $\pi = \pi_1\pi_2\cdots\pi_k$



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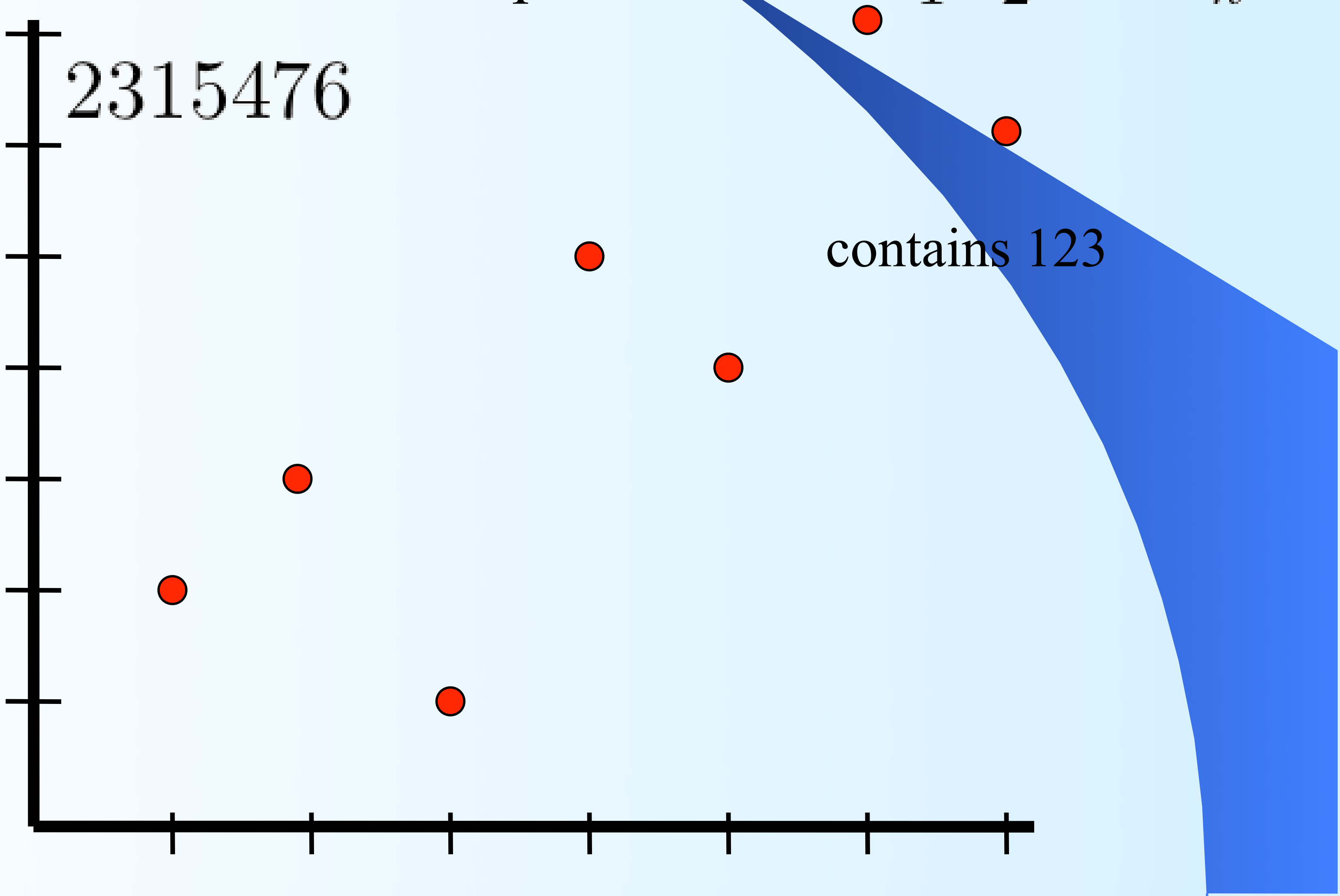

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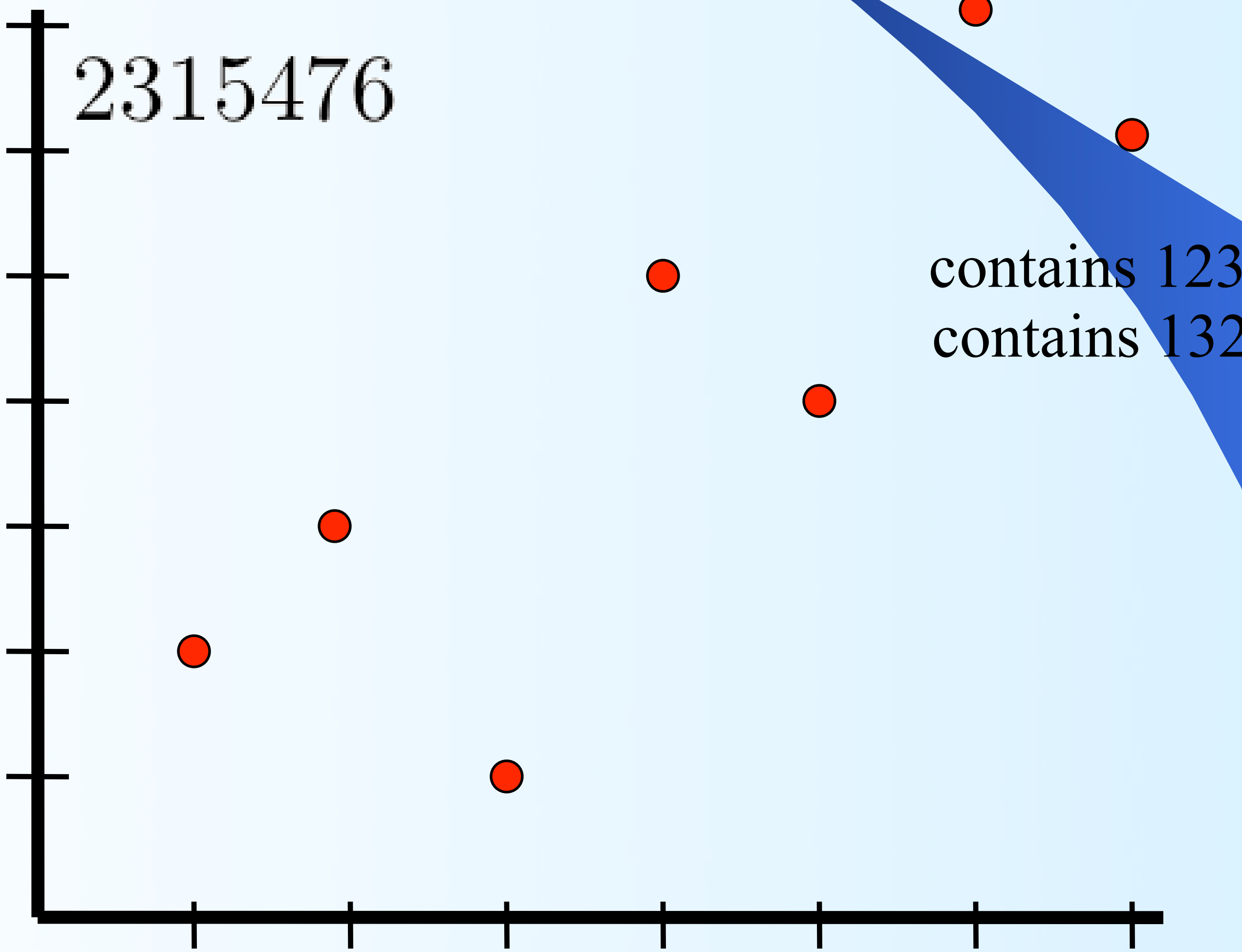

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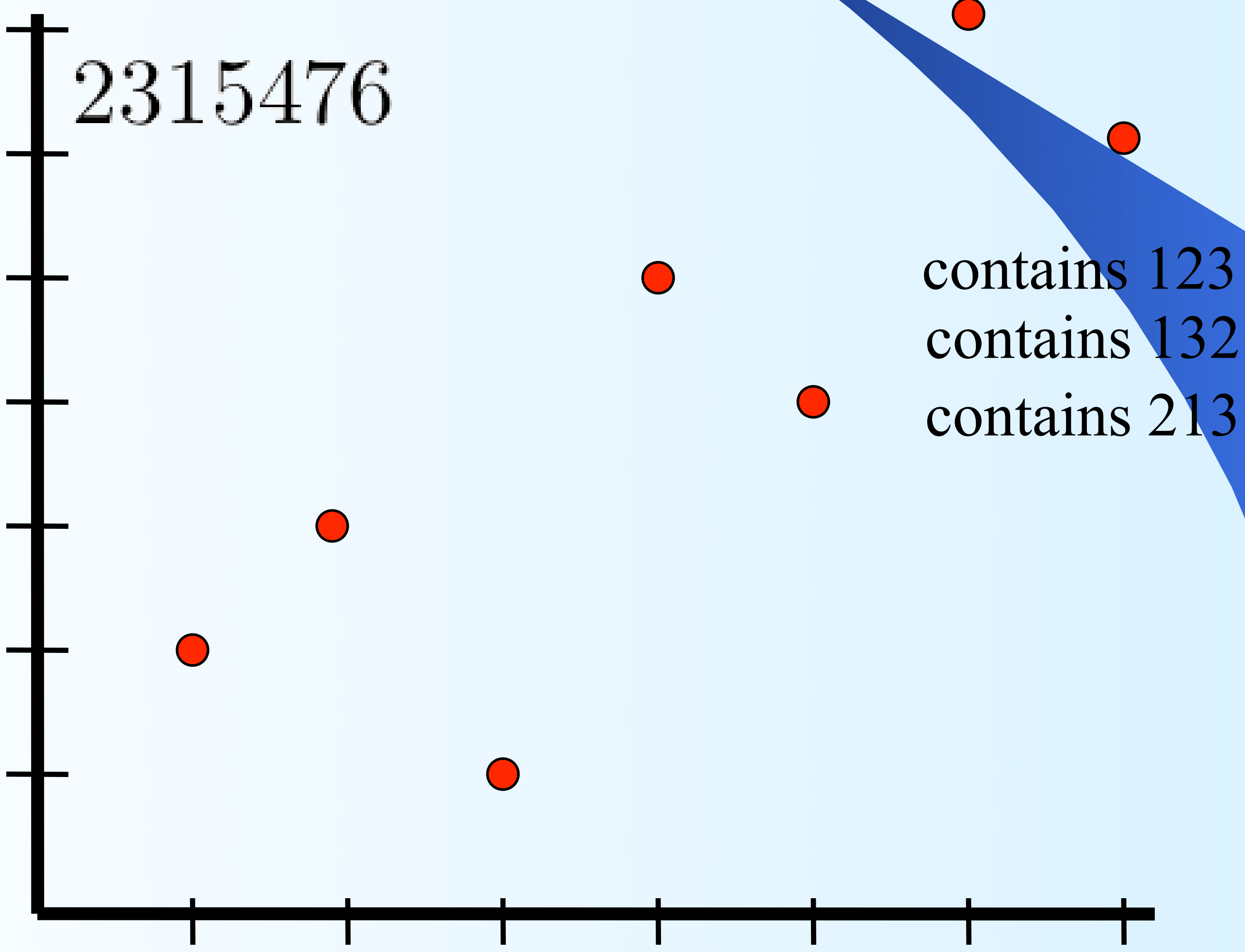

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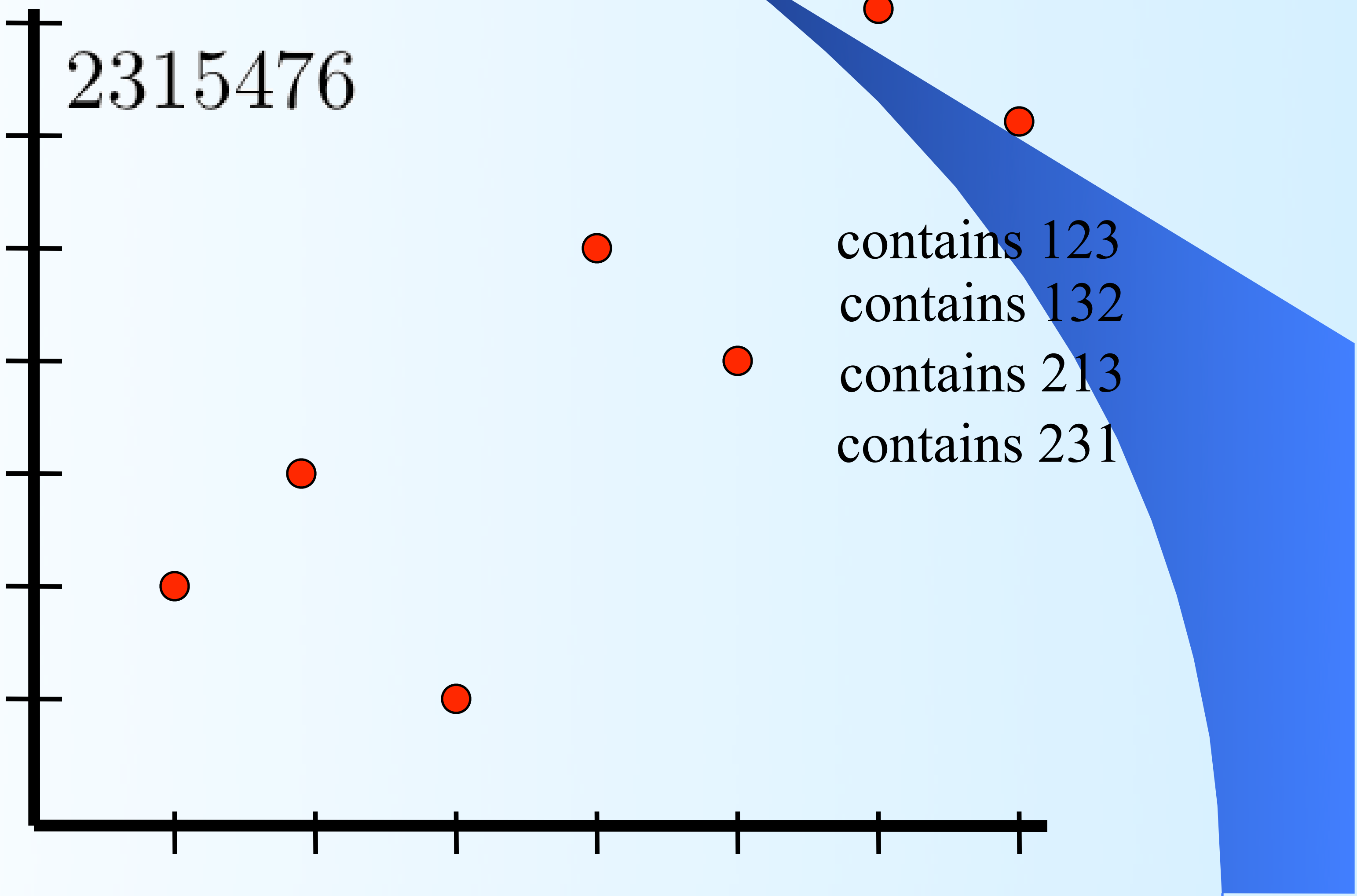

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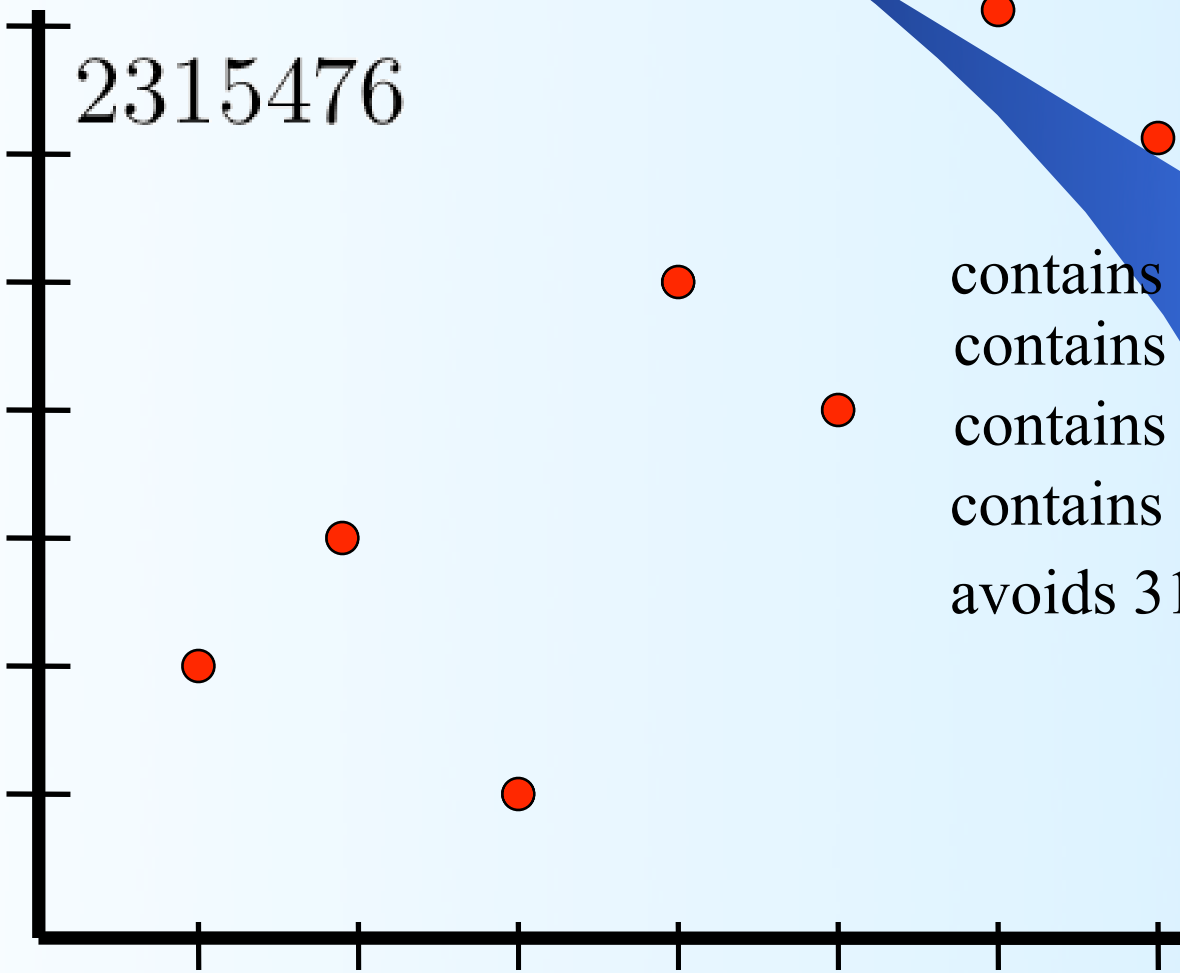

contains 123

contains 132

contains 213

contains 231

avoids 312

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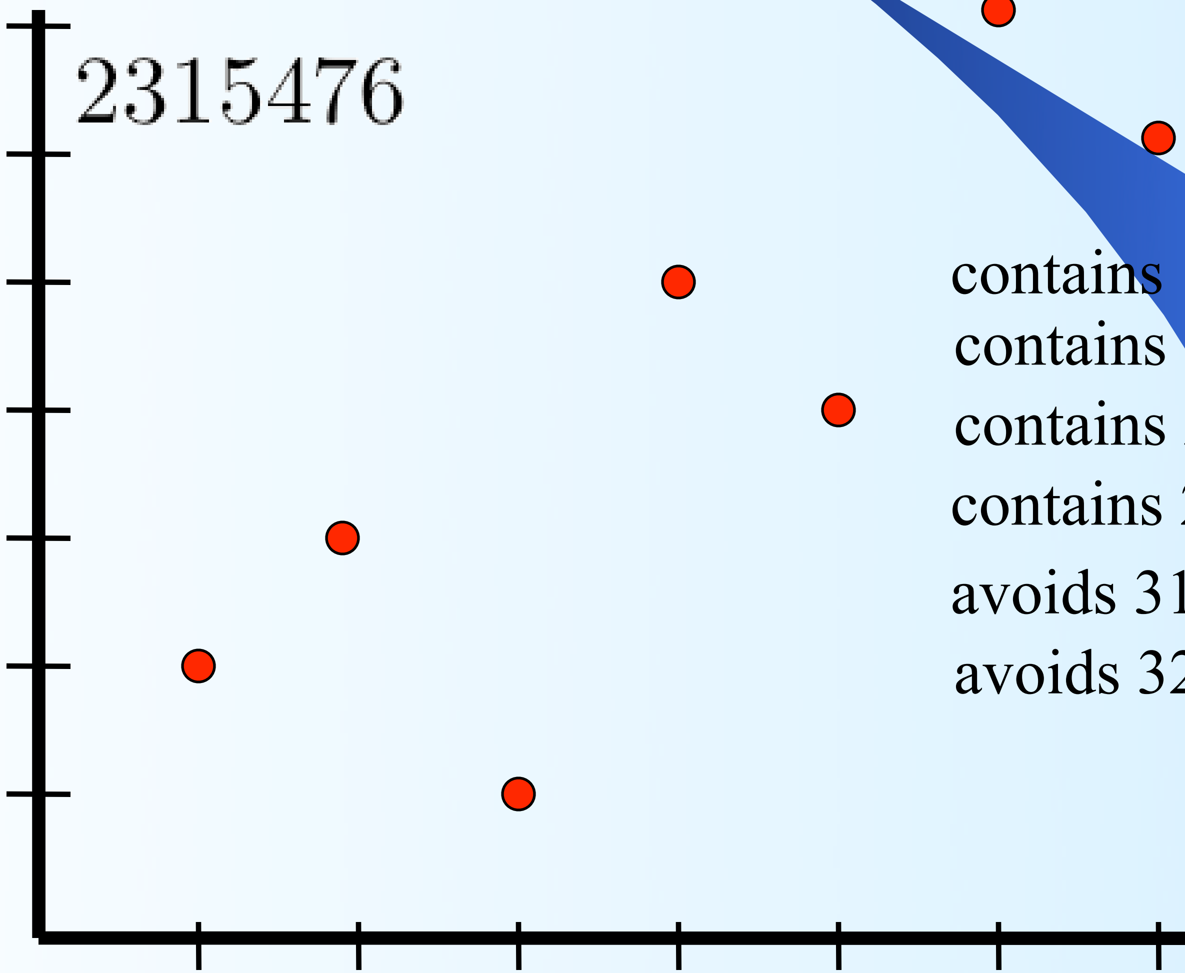

contains 123

contains 132

contains 213

contains 231

avoids 312

avoids 321

$$S_n(123) = S_n(132) = S_n(213) = S_n(231) = S_n(312) = S_n(321) = \frac{1}{n+1}\binom{2n}{n}$$

The following permutations of size n=3, 4

123	1234	2314
132	1243	2341
213	1324	2413
231	1342	3124
312	1423	3142
	2134	3412
	2143	4123

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231	1342	3124
312	1423	3142
	2134	3412
	2143	4123

$$|S_n(321)| = 1, 2, 5, 14, 42, 132, \ldots$$

A new way of looking at permutations?