Non-commutative symmetric functions III:

A representation theoretical approach

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Joint work with Nantel Bergeron, Mercedes Rosas, Christophe Hohlweg, and others

The structure of Sym

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- There are various bases of Sym that are linked with the representation theory of Sn/Gln: monomial, homogeneous, elementary, power, Schur, Hall-Littlewood, Macdonald, etc.

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- Hopf algebra of set partitions
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- non-commutative but co-commutative

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NCSym

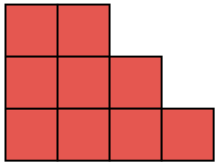
algebra of set partitions

$$\{\{1,4,9\},\{2,3,7,8\},\{5,6\}\}$$

vs

Sym

algebra of partitions



The join operation on set partitions

$$A \vdash [n], B \vdash [m]$$

$$A|B = \{A_1, A_2, \ldots, A_{\ell(A)}, B_1 + n, B_2 + n, \ldots, B_{\ell(B)} + n\}$$

$$\{\{1,4\},\{2\},\{3,5\}\}|\{\{1,3,4\},\{2\}\} = \{\{1,4\},\{2\},\{3,5\},\{6,8,9\},\{7\}\}$$

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Analogue of the power basis by Rosas-Sagan

$$p_A = \sum_{B \geq A} m_B$$

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Proposition: $p_A \cdot p_B = p_{A|B}$

NCSym is free

We say that a set partition A is *atomic* if

$A \neq B|C$ for non-empty set partitions B, C

or

$A_1 \uplus A_2 \uplus \cdots \uplus A_i \neq [k]$ for $k < n$

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Example: There are 5 set partitions of [3]

$$p_{\{\{1\},\{2\},\{3\}\}} = p_{\{\{1\}\}} \cdot p_{\{\{1\}\}} \cdot p_{\{\{1\}\}}$$

$$p_{\{\{1,2\},\{3\}\}} = p_{\{\{1,2\}\}} \cdot p_{\{\{1\}\}}$$

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$$p_{\{\{1,2,3\}\}}, p_{\{\{1,3\}\},\{2\}\}}$$

are atomic

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NCSym is freely generated by p_A, A atomic

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number of generators at each degree

1, 1, 2, 6, 22, 92, 426, 2146, ...

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$$\frac{1}{1-(t+t^2+2t^3+6t^4+22t^5+92t^6+\cdots)} = 1+t+2t^2+5t^3+15t^4+52t^5+203t^6+\cdots$$

The split of a set partition

For A set partition

Let $A^! = (A^{(1)}, A^{(2)}, \ldots, A^{(\ell)})$
where $A^{(i)}$ is atomic
and $A = A^{(1)} | A^{(2)} | \cdots | A^{(\ell)}$

$$p_A = p_{A^{(1)}} \cdot p_{A^{(2)}} \cdots p_{A^{(\ell)}}$$

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Fix an order on all atomic set partitions
We say A is *Lyndon* if
$A^! <_{lex} (A^{(i)}, A^{(i+1)}, \ldots, A^{(\ell)}) \quad \forall i > 1$

Example of a split of a Lyndon s.p.

$$A = \{\{1\}, \{2, 4\}, \{3\}, \{5\}, \{6, 7, 8\}\}$$

$$\{\{1\}\} < \{\{1, 2\}\} < \{\{1, 2\}, \{3\}\} < \{\{1, 2, 3\}\} < \cdots$$

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A is Lyndon $\qquad$ B is not Lyndon

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Basis for primitive elements is indexed by Lyndon set partitions

The number of primitive elements at each degree

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- NCSym is non-commutative and co-commutative.
- NCSym and the dual algebra are freely generated.
- (Rosas-Sagan) There are analogues of monomial, homogeneous, elementary, power as vector space bases, but not(?) a satisfactory Schur basis.

Open question:

Is there a representation theoretical model analogous to what happens with Sym?

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$$\bigoplus_{n\geq 0} \mathbf{k}\mathfrak{S}_n$$

is a graded algebra with irreps in 1-1 correspondence with partitions

$G_0(\mathbf{k}\mathfrak{S}, \circ)$ ring of isomorphism classes of representations of $\bigoplus_{n\geq 0} \mathbf{k}\mathfrak{S}_n$

Grothendeick ring of representations of $\bigoplus \mathbf{k}\mathfrak{S}_n$

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product $M \otimes N \mapsto Ind_{\mathbf{k}\mathfrak{S}_m \times \mathbf{k}\mathfrak{S}_n}^{\mathbf{k}\mathfrak{S}_{m+n}} M \otimes N$

coproduct $M \mapsto \bigoplus_{i=0}^{n} Res_{\mathbf{k}\mathfrak{S}_i \times \mathbf{k}\mathfrak{S}_{n-i}}^{\mathbf{k}\mathfrak{S}_n} M$

internal product diagonal action on $M \otimes N$

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$$G_0(\mathbf{k}\mathfrak{S}, \circ) \simeq Sym$$

Internal product on Sym?

$$f \mapsto f[XY]$$

is a 'natural' internal coproduct on Sym and

$$Sym \simeq Sym^*$$

by duality, this internal coproduct coincides with the internal coproduct on $G_0(\mathbf{k}\mathfrak{S}, \circ)$

$$G_0(\mathbf{k}\mathfrak{S}, \circ) \simeq Sym^*$$

Make this isomorphism explicit

irreducible $\mathfrak{S}_n$ module M^λ	s_λ
induction of $\mathfrak{S}_n$ module and an $\mathfrak{S}_m$ module to $\mathfrak{S}_{n+m}$	$f \cdot g$
restriction of an $\mathfrak{S}_n$ module	$\Delta(f)$
internal tensor product	$f \odot g$

Internal (co)product on NCSym

By ‘replacing one set of variables by two’ we define an internal coproduct on NCSym.

$$\Delta^{\odot}(m_A) = \sum_{B \wedge C = A} m_B \otimes m_C$$

where $B \wedge C$ is the set partition finer than both B and C

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Proposition:

$$\Delta^{\odot}(p_A) = p_A \otimes p_A$$

Is there a 'tower' of algebras whose representations are indexed by set partitions?

$(\mathbf{k}\Pi_n, \wedge)$ the algebra of set partitions of [n] with the meet product

$$\{\{1,3,4\},\{2,5\},\{6,7,8\}\} \wedge \{\{1,3,5,7\},\{2,4,6,8\}\}$$

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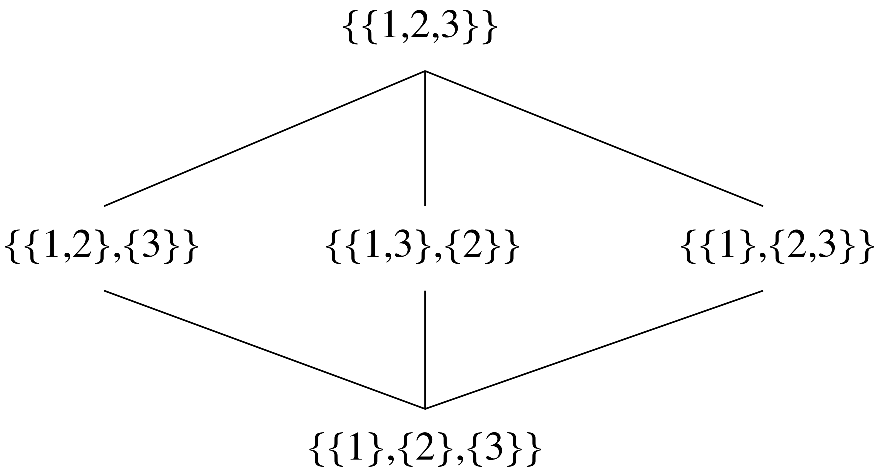
this algebra is semi-simple and, since it is commutative, all irreducible modules are of dimension 1 (hence there is one for every set partition).

Refinement order is a lattice



$A \leq B$

iff for each i, there is a j

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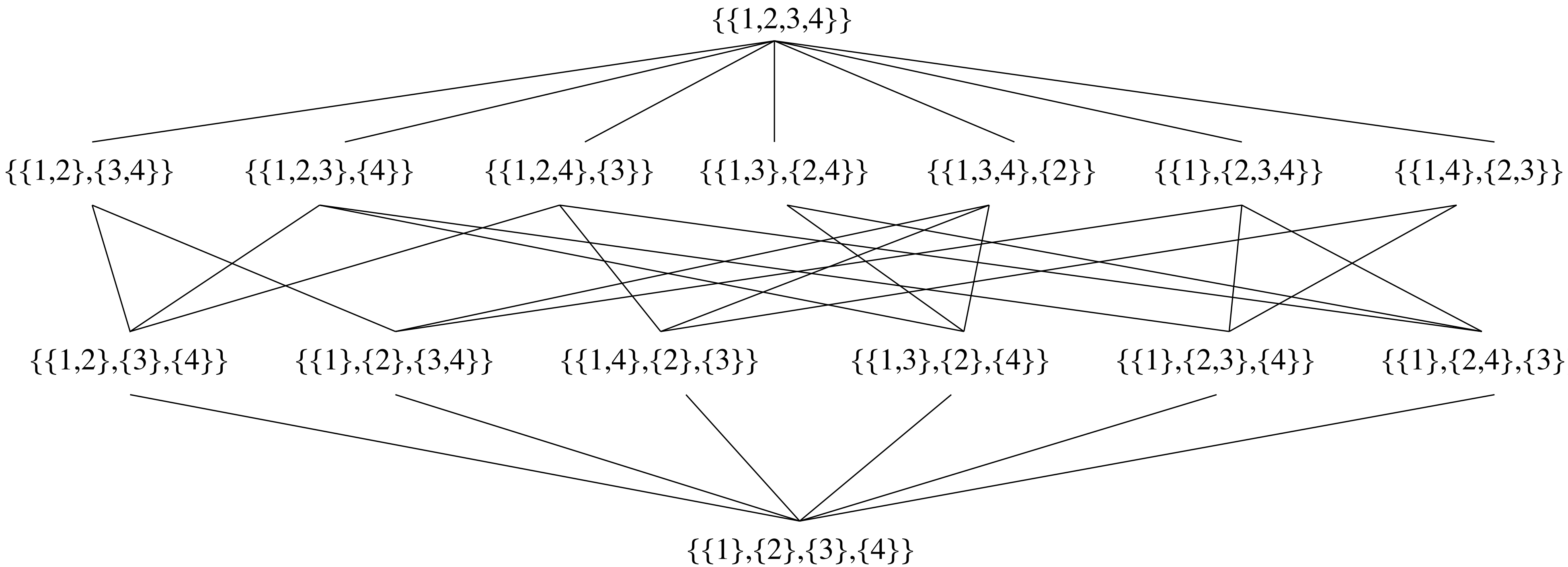

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$$Ind_{\mathbf{k}\Pi_n \times \mathbf{k}\Pi_m}^{\mathbf{k}\Pi_{n+m}} e_A \bigotimes e_B \simeq e_{A|B}$$

Is there a basis of NCSym* which matches this?

$$x_A x_B = x_{A|B}$$

$$\Delta^{\odot}(x_A) = \sum_{B \vee C = A} x_B \otimes x_C$$

Put this in the computer and solve

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there is essentially one solution to this...

$$x_A = \sum_{B \leq A} \mu(B, A) p_B$$

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- the commutative image of x_A is +/- something which is e-positive
- the coproduct is difficult to explain and has +/- signs, but still seems curious
- Restriction is dual to induction (and is commutative). No explanation for meaning of coproduct.

Directions and open problems

- Understand better the structure of particular bases. Product, coproduct, antipode. What role do these play with respect to representation theory of $(\mathbf{k}\Pi, \wedge)$

- Needs some applications. Lift structures from Sym to find new attacks on plethysm, inner tensor product, positivity questions.

- Understand better non-commutative invariants of finite reflection groups and relationship to commutative invariants.