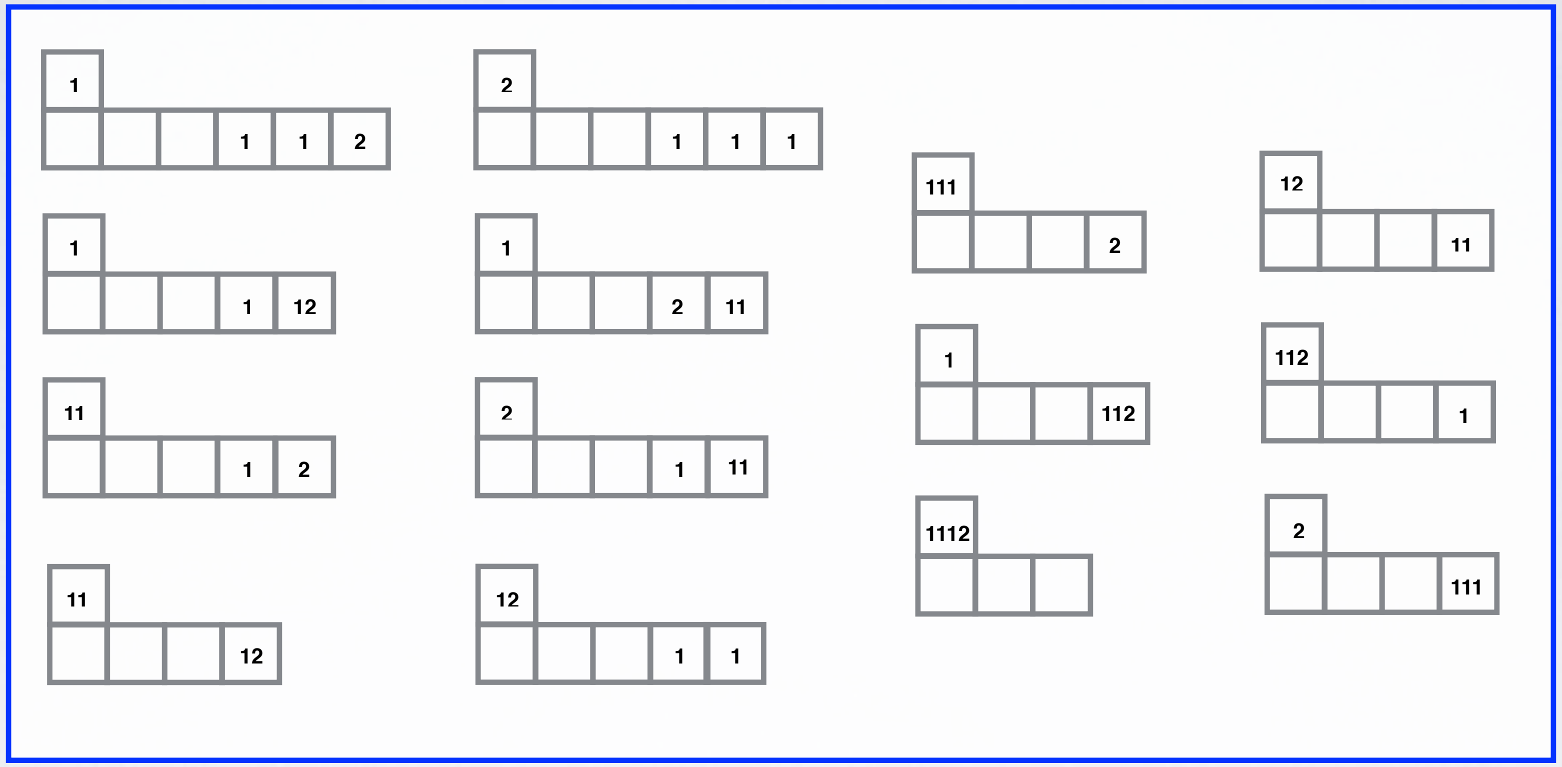

Irreducible characters of the symmetric group as symmetric functions

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$$S_n \subseteq Gl_n$$

permutation matrices contained in invertible matrices

Open problem:

irreducible Gl_n representation V^λ

how does V^λ decompose into irreducible symmetric group representations?

Answer using characters of Gl_n

character of V^λ is $s_\lambda(x_1, x_2, \ldots, x_n)$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \simeq A \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix} A^{-1}$$

$$\Xi_r := 1, \zeta_r, \zeta_r^2, \ldots, \zeta_r^{r-1} \qquad \zeta_r = e^{2\pi i/r}$$

$$\Xi_\mu := \Xi_{\mu_1}, \Xi_{\mu_2}, \ldots, \Xi_{\mu_{\ell(\mu)}}$$

eigenvalues of a permutation matrix with cycle structure μ

Frobenius image:

$$\phi_n(f) = \sum_{\mu \vdash n} f[\Xi_\mu] \frac{p_\mu}{z_\mu}$$

The multiplicity of an S_n irreducible M^γ

in the irreducible Gl_n module V^λ

is equal to the coefficient of s_γ in $\phi_n(s_\lambda)$

What are the irreducible characters of the symmetric group?

$$\tilde{s}_\lambda = \phi_n^{-1}(s_{(n-|\lambda|,\lambda)})$$

The irreducible characters of the symmetric group form a basis of the symmetric functions.

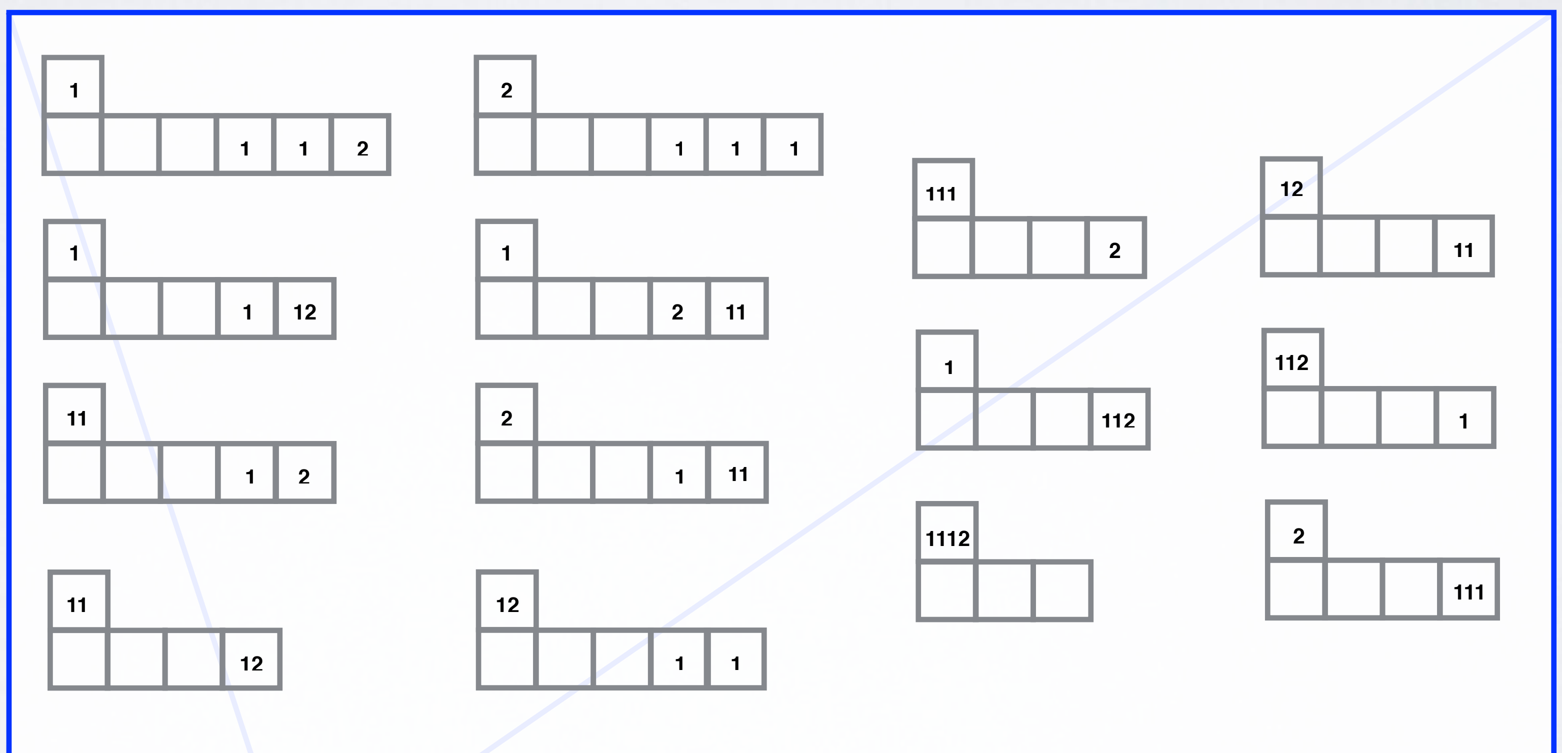
$$\tilde{s}_\lambda(\Xi_\mu) = \chi^{(|\mu|-|\lambda|,\lambda)}(\mu)$$

Theorem

The coefficient of $\tilde{s}_\lambda$ in h_μ

is the number of column strict tableaux of shape

(r, λ) and content μ whose entries are multisets



$$h_{31} = 7\tilde{s}_{()} + 14\tilde{s}_1 + 8\tilde{s}_{11} + \tilde{s}_{111} + 10\tilde{s}_2$$
$$+4\tilde{s}_{21} + 4\tilde{s}_3 + \tilde{s}_{31} + \tilde{s}_4$$

Intermediate basis- induced trivial characters

$$\tilde{h}_\lambda = \phi_n^{-1}(h_{(n-|\lambda|,\lambda)})$$

$$\tilde{h}_\lambda[\Xi_\mu] = \langle h_{(|\mu|-|\lambda|,\lambda)}, p_\mu \rangle$$

Combinatorial expansion

$$h_\lambda = \sum_{\pi \Vdash \{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_\ell}\}} \tilde{h}_{\tilde{m}(\pi)} \ .$$

$$h_{31} = \tilde{h}_1 + 3\tilde{h}_{11} + \tilde{h}_{111} + \tilde{h}_{21} + \tilde{h}_{31}$$

$$\{1112\}, \{111|2\}, \{112|1\}, \{11|12\}, \{11|1|2\}, \{12|1|1\}, \{1|1|1|2\}$$

There exists a Hopf algebra of multi-set partitions…

- Contains Hopf algebra of set partitions (NCSym) symmetric functions in non-commuting variables
- monomial and power sum bases defined exactly as in NCSym. Commutative image of monomial is induced trivial character, power is complete basis
- product and coproduct give interesting combinatorial interpretation to the Kronecker product of complete symmetric functions

Structure coefficients are Kronecker

For $\lambda, \mu \vdash n$ where n is sufficiently large

$$\tilde{s}_{\overline{\lambda}} \tilde{s}_{\overline{\mu}} = \sum_{\nu \vdash n} k_{\lambda\mu\nu} \tilde{s}_{\overline{\nu}}$$

where the $k_{\lambda\mu\nu}$ are the coefficients in the Kronecker product

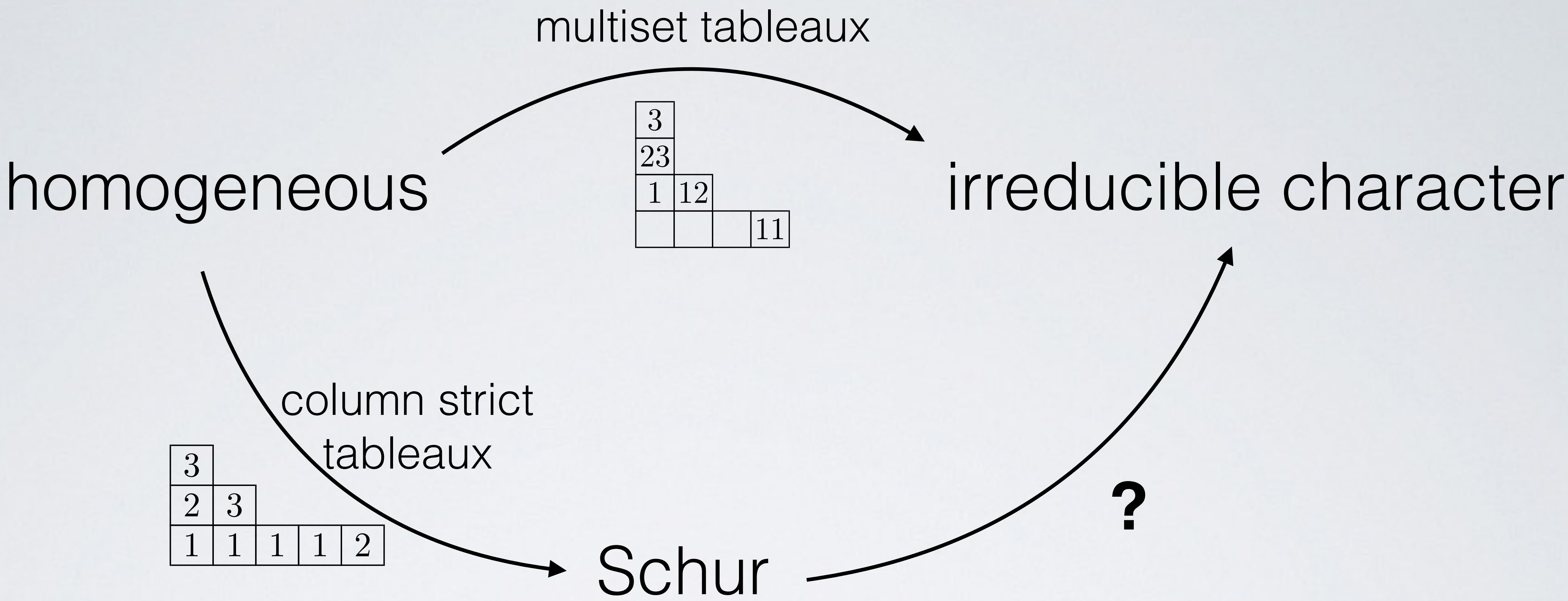
$$s_\lambda * s_\mu = \sum_{\nu \vdash n} k_{\lambda\mu\nu} s_\nu$$

Sales pitch

- Positive structure coefficients (reduced Kronecker)
- Positive coproduct structure coefficients
- Combinatorial tools for working with symmetric group and partition algebra characters
- Orthonormal basis with respect to character scalar product

$$\langle f, g\rangle_{@} = \sum_{\mu \vdash n} \frac{f[\Xi_\mu] g[\Xi_\mu]}{z_\mu} = \langle \phi_n(f), \phi_n(g)\rangle$$

  - $\{\tilde{s}_\lambda\}_\lambda$ is an orthonormal basis with respect to $\langle -, -\rangle_{@}$
    $\tilde{s}_\lambda = s_\lambda +$ terms of lower degree



column strict tableaux $\times$ **?** $\longleftrightarrow$ multiset tableaux

Implementation in Sage

```
sage: Sym = SymmetricFunctions(QQ)
sage: st = Sym.irreducible_symmetric_group_character()
sage: st
Symmetric Functions over Rational Field in the irreducible symmetric group character basis
sage: s = Sym.Schur()
```

```
sage: s(st[3,2])
3*s[1] - 6*s[1, 1] - 6*s[2] + 3*s[1, 1, 1] + 8*s[2, 1] + 4*s[3] - s[2, 1, 1] - 2*s[2, 2] - 3*s[3, 1]
- s[4] + s[3, 2]
```

```
sage: st(s[3,2])
4*st[] + 10*st[1] + 8*st[1, 1] + 11*st[2] + 2*st[1, 1, 1] + 8*st[2, 1] + 6*st[3] + st[2, 1, 1]
+ 2*st[2, 2] + 3*st[3, 1] + st[4] + st[3, 2]
```

```
sage: st[2]*st[2,1]
st[1] + 2*st[1, 1] + 2*st[2] + 2*st[1, 1, 1] + 4*st[2, 1] + 2*st[3] + st[1, 1, 1, 1] + 3*st[2, 1, 1]
+ 2*st[2, 2] + 3*st[3, 1] + st[4] + st[2, 2, 1] + st[3, 1, 1] + st[3, 2] + st[4, 1]
```

```
sage: s[7,2].kronecker_product(s[6,2,1])
s[8, 1] + 2*s[7, 2] + 2*s[7, 1, 1] + 2*s[6, 1, 1, 1] + 4*s[6, 2, 1] + 2*s[6, 3] + s[5, 1, 1, 1, 1]
+ 3*s[5, 2, 1, 1] + 2*s[5, 2, 2] + 3*s[5, 3, 1] + s[5, 4] + s[4, 2, 2, 1] + s[4, 3, 1, 1]
+ s[4, 3, 2] + s[4, 4, 1]
```