From Robinson–Schensted–Knuth correspondence to Schur-Weyl duality

Mike Zabrocki
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joint work with Rosa Orellana

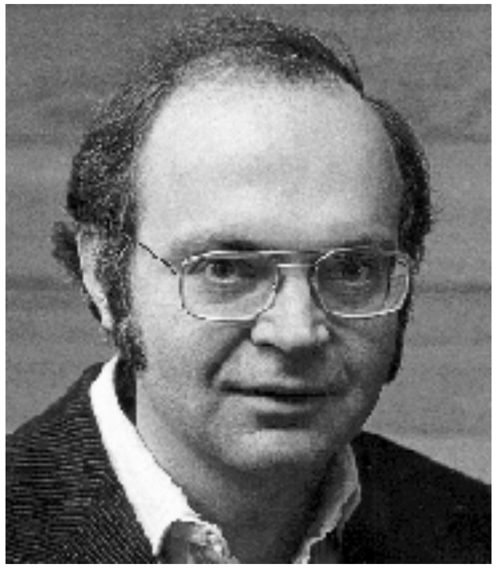
Gilbert Robinson

(1938) On the representations
of the symmetric group

(formerly) Craige Schensted

also: (formerly) Ea, (now) Ea Ea

(1961) Longest Increasing and
Decreasing Subsequences

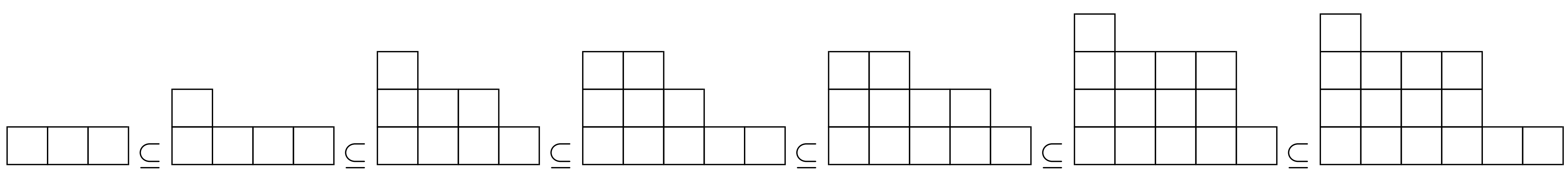


Donald Knuth

(1969) Permutations, Matrices and
Generalized Young Tableaux

column strict tableaux

6					
3	4	6	6		
2	3	3	5		
1	1	1	2	4	7

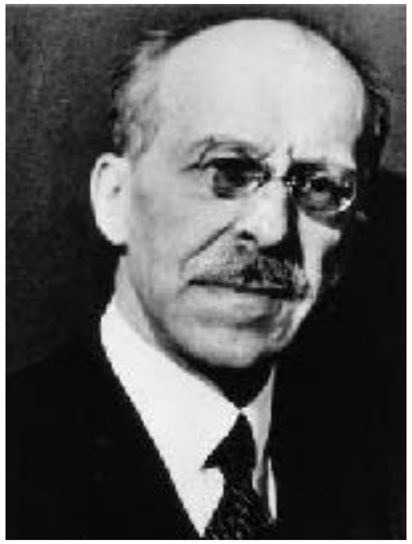


$$\begin{pmatrix} 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 6 \\ 3 & 6 & 6 & 1 & 2 & 3 & 4 & 6 & 3 & 5 & 1 & 1 & 2 & 4 & 7 \end{pmatrix}$$

$$\left(\begin{array}{llllll} 6 & & & & & \\ 3 & 4 & 6 & 6 & & \\ 2 & 3 & 3 & 5 & & \\ 1 & 1 & 1 & 2 & 4 & 7 \end{array} \quad \begin{array}{llllll} 5 & & & & & \\ 4 & 5 & 5 & 5 & & \\ 3 & 3 & 3 & 4 & & \\ 1 & 2 & 2 & 3 & 3 & 6 \end{array} \right)$$

$$n! = \text{number of permutations of } \{1, 2, \ldots, n\} = \sum_{\lambda \vdash n} \left(\begin{array}{c} \text{\# of standard} \\ \text{tableaux of} \\ \text{shape } \lambda \end{array} \right)^2$$

$$n^k = \text{number of words in } \{1, 2, \ldots, n\} \text{ of length } k = \sum_{\lambda \vdash k} \left(\begin{array}{c} \text{\# of column} \\ \text{strict tableaux} \\ \text{of shape } \lambda \end{array} \right) \left(\begin{array}{c} \text{\# of standard} \\ \text{tableaux of} \\ \text{shape } \lambda \end{array} \right)$$



Issai Schur

representations of the
general linear group
thesis - 1901

Ferdinand Georg
Frobenius

representations of
finite groups - 1904

Alfred Young

representations of
symmetric groups
1927

Hermann Weyl

representation theory
classical groups
book - 1937

$$\mathbb{C}Sym_n \simeq \bigoplus_{\lambda \vdash n} \left(\begin{array}{c}\text{irreducible} \\ \text{subspace}\end{array} \ \lambda \right)^{\oplus m_\lambda}$$

$$n! = \begin{array}{c}\text{number of permutations} \\ \text{of } \{1, 2, \ldots, n\}\end{array} = \sum_{\lambda \vdash n} \left(\begin{array}{c}\text{\# of standard} \\ \text{tableaux of} \\ \text{shape } \lambda\end{array} \right)^2$$

$$\mathbb{C}Sym_3 = \mathcal{L}_{\mathbb{C}}\{123, 132, 213, 231, 312, 321\}$$

$$E_{\begin{smallmatrix}1&2&3\end{smallmatrix},\ \begin{smallmatrix}1&2&3\end{smallmatrix}} = 123 + 132 + 213 + 231 + 312 + 321$$

$$E_{\begin{smallmatrix}3&\\1&2\end{smallmatrix}\ \begin{smallmatrix}3&\\1&2\end{smallmatrix}} = 123 + 213 - 321 - 231$$

$$E_{\begin{smallmatrix}3&\\1&2\end{smallmatrix}\ \begin{smallmatrix}2&\\1&3\end{smallmatrix}} = 132 + 231 - 312 - 213$$

$$E_{\begin{smallmatrix}2&\\1&3\end{smallmatrix}\ \begin{smallmatrix}3&\\1&2\end{smallmatrix}} = 132 + 312 - 231 - 321$$

$$E_{\begin{smallmatrix}2&\\1&3\end{smallmatrix}\ \begin{smallmatrix}2&\\1&3\end{smallmatrix}} = 123 + 321 - 213 - 312$$

$$E_{\begin{smallmatrix}3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}} = 123 + 231 + 312 - 132 - 213 - 321$$

$V^{\otimes k} \simeq$ non commutative polynomials of degree k in variables $\{x_1, x_2, \ldots, x_n\}$

$$V^{\otimes k} \simeq \bigoplus_{\lambda \vdash k} \begin{pmatrix} \text{irreducible} & Gl_n \\ \text{subspace} & \lambda \end{pmatrix} \otimes \begin{pmatrix} \text{irreducible} & Sym_k \\ \text{subspace} & \lambda \end{pmatrix}$$

$$n^k = \begin{matrix} \text{number of words in} \\ \{1, 2, \ldots, n\} \text{ of length } k \end{matrix} = \sum_{\lambda \vdash k} \begin{pmatrix} \text{\# of column} \\ \text{strict tableaux} \\ \text{of shape } \lambda \end{pmatrix} \begin{pmatrix} \text{\# of standard} \\ \text{tableaux of} \\ \text{shape } \lambda \end{pmatrix}$$

$$V^{\otimes 3} = \mathcal{L}_{\mathbb{C}}\{x_1x_1x_1, x_1x_1x_2, x_1x_2x_1, x_2x_1x_1, x_1x_2x_2, x_2x_1x_2, x_2x_2x_1, x_2x_2x_2\}$$

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \qquad x_1x_1x_1 E_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array},\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} = 6x_1x_1x_1$$

$$\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \qquad x_1x_1x_2 E_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array},\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} = 2x_1x_1x_2 + 2x_1x_2x_1 + 2x_2x_1x_1$$

$$\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \qquad x_1x_2x_2 E_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array},\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} = 2x_1x_2x_2 + 2x_2x_1x_2 + 2x_2x_2x_1$$

$$\begin{array}{|c|c|c|}\hline 2&2&2\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \qquad x_2x_2x_2 E_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array},\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} = 6x_2x_2x_2$$

$$\begin{array}{|c|c|}\cline{1-1} 2\\\hline 1&1\\\hline\end{array} \quad \begin{array}{|c|c|}\cline{1-1} 3\\\hline 1&2\\\hline\end{array} \qquad x_1x_1x_2 E_{\begin{array}{|c|c|}\cline{1-1} 3\\\hline 1&2\\\hline\end{array}\,\begin{array}{|c|c|}\cline{1-1} 3\\\hline 1&2\\\hline\end{array}} = 2x_1x_1x_2 - x_1x_2x_1 - x_2x_1x_1$$

$$\begin{array}{|c|c|}\cline{1-1} 2\\\hline 1&2\\\hline\end{array} \quad \begin{array}{|c|c|}\cline{1-1} 3\\\hline 1&2\\\hline\end{array} \qquad x_1x_2x_2 E_{\begin{array}{|c|c|}\cline{1-1} 3\\\hline 1&2\\\hline\end{array}\,\begin{array}{|c|c|}\cline{1-1} 3\\\hline 1&2\\\hline\end{array}} = x_1x_2x_2 + x_2x_1x_2 - 2x_2x_2x_1$$

$$\begin{array}{|c|c|}\cline{1-1} 2\\\hline 1&1\\\hline\end{array} \quad \begin{array}{|c|c|}\cline{1-1} 2\\\hline 1&3\\\hline\end{array} \qquad x_1x_1x_2 E_{\begin{array}{|c|c|}\cline{1-1} 3\\\hline 1&2\\\hline\end{array}\,\begin{array}{|c|c|}\cline{1-1} 2\\\hline 1&3\\\hline\end{array}} = 2x_1x_2x_1 - x_1x_1x_2 - x_2x_1x_1$$

$$\begin{array}{|c|c|}\cline{1-1} 2\\\hline 1&2\\\hline\end{array} \quad \begin{array}{|c|c|}\cline{1-1} 2\\\hline 1&3\\\hline\end{array} \qquad x_1x_2x_2 E_{\begin{array}{|c|c|}\cline{1-1} 3\\\hline 1&2\\\hline\end{array}\,\begin{array}{|c|c|}\cline{1-1} 2\\\hline 1&3\\\hline\end{array}} = x_1x_2x_2 + x_2x_2x_1 - 2x_2x_1x_2$$

$$Gl_n \circlearrowright V^{\otimes k} \circlearrowleft Sym_k$$

$$(A(x_{i_1} x_{i_2} \cdots x_{i_k}))\sigma = A((x_{i_1} x_{i_2} \cdots x_{i_k})\sigma)$$

Gl_n Sym_k Schur-Weyl duality 1901-1937

Sp_n

O_n

Brauer algebra

perfect matchings

Richard Brauer

On Algebras Which are Connected with the Semisimple Continuous Groups (1937)

Sym_n

Partition algebra

Vaughan Jones

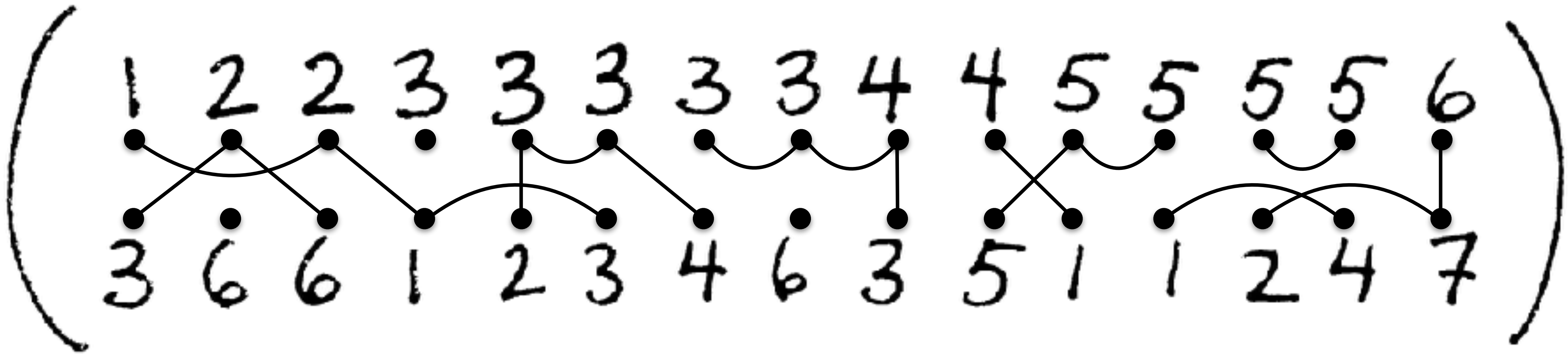
The Potts model and the symmetric group (1994)

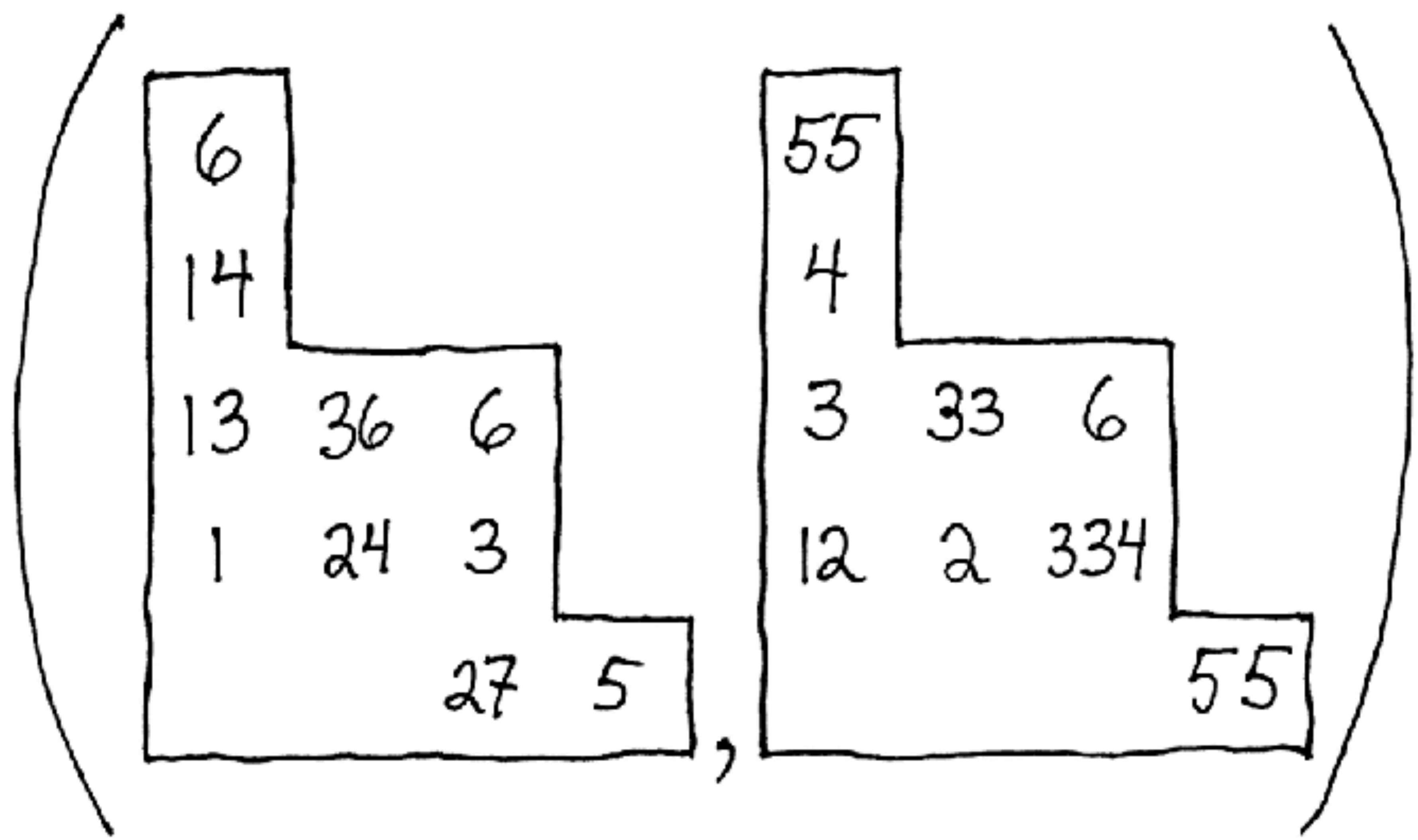
Paul Martin

Representations of the Temperly-Lieb algebras (1990)
Potts Models and Related Problems in Statistical Mechanics (1991)
Temperley-Lieb algebras … The partition algebra construction (1994)
The structure of the partition algebras (1996)

$$\begin{pmatrix} 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 6 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 3 & 6 & 6 & 1 & 2 & 3 & 4 & 6 & 3 & 5 & 1 & 1 & 2 & 4 & 7 \end{pmatrix}$$

$$\left(\begin{array}{|llllll} \hline 6 & & & & & \\ 3 & 4 & 6 & 6 & & \\ 2 & 3 & 3 & 5 & & \\ 1 & 1 & 1 & 2 & 4 & 7 \\ \hline \end{array} \quad \begin{array}{|llllll} \hline 5 & & & & & \\ 4 & 5 & 5 & 5 & & \\ 3 & 3 & 3 & 4 & & \\ 1 & 2 & 2 & 3 & 3 & 6 \\ \hline \end{array} \right)$$


1 2 2 3 3 3 3 3 4 4 5 5 5 5 6
3 6 6 1 2 3 4 6 3 5 1 1 2 4 7


6
14
13 36 6
1 24 3
27 5
55
4
3 33 6
12 2 334
55

$$n^k = \sum_{\lambda \vdash n} \begin{pmatrix} \text{\# of standard} \\ \text{tableaux shape} \\ \lambda \end{pmatrix} \begin{pmatrix} \text{\# of standard} \\ \text{set tableaux} \\ \text{in } \{1,2,\ldots,k\} \text{ of} \\ \text{shape } \lambda \end{pmatrix}$$

$$\begin{matrix} \text{\# of set partitions} \\ \text{of } \{1,2,\ldots,2n\} \end{matrix} = \sum_{|\lambda| \leq n} \begin{pmatrix} \text{\# of standard set} \\ \text{valued tableaux in} \\ \{1,2,\ldots,n\} \text{ of shape } (k,\lambda) \end{pmatrix}^2$$

Paul Martin
G. Rollet

The Potts model representation and a Robinson-Schensted correspondence for the partition algebra (1998)

Partition	Walk	
$\{\{1,3,4\},\{2,5,6,7\}\}$		$(0,0^+,\square,0^+,\square,0^+,0,0^+,\square,0^+,0,0^+,0,0^+,0)$

Tim Lewandowski
Tom Halverson

RSK Insertion for Set Partitions and Diagram Algebras (2005)