Symmetric Group Representations and Howe Duality

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(joint work with Rosa Orellana)

3 identities

$$n! = \sum_{\lambda \vdash n} f_\lambda^2$$

$$n^k = \sum_{\lambda \vdash k} F_{n,\lambda} f_\lambda$$

$$\binom{nk+r-1}{r} = \sum_{\lambda \vdash r} F_{n,\lambda} F_{k,\lambda}$$

f_λ = # of standard tableaux shape λ

$F_{n,\lambda}$ = # of column strict tableaux of shape λ in $\{1,2,\dots,n\}$

3 identities:

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3 proofs:

combinatorial - Robinson-Schensted-Knuth

symmetric functions - Cauchy kernel coefficient

representation theory - Schur-Weyl and Howe duality

Robinson-Schensted-Knuth

generalized permutation

$$\begin{pmatrix} 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 6 \\ 3 & 6 & 6 & 1 & 2 & 3 & 4 & 6 & 3 & 5 & 1 & 1 & 2 & 4 & 7 \end{pmatrix}$$

$$\left(\begin{array}{llllll} 6 & & & & & \\ 3 & 4 & 6 & 6 & & \\ 2 & 3 & 3 & 5 & & \\ 1 & 1 & 1 & 2 & 4 & 7 \end{array} \quad \begin{array}{llllll} 5 & & & & & \\ 4 & 5 & 5 & 5 & & \\ 3 & 3 & 3 & 4 & & \\ 1 & 2 & 2 & 3 & 3 & 6 \end{array} \right)$$

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permutations in bijection with pairs of standard tableaux of same shape.

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Words in $\{1,2,\ldots,n\}$ of length k in bijection with pairs consisting of a column strict tableau and a standard tableau both of the same shape $\lambda \vdash k$

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"generalized permutations" in $\{1,2,\ldots,n\}$ top row and $\{1,2,\ldots,k\}$ bottom row of length r is in bijection with pairs of column strict tableaux of the same shape $\lambda \vdash r$ with content in $\{1,2,\ldots,n\}$ and $\{1,2,\ldots,k\}$ respectively

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f_λ = coefficient of $\frac{p_{1^n}[X]}{n!}$ in $s_\lambda[X]$

$$h_n[XY] = \sum_{\lambda \vdash n} s_\lambda[X] s_\lambda[Y]$$

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$F_{n,\lambda} = s_\lambda(\underbrace{1,1,\ldots,1}_{n \text{ times}})$

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$$F_{n,\lambda} = s_\lambda(\underbrace{1,1,\ldots,1}_{n \text{ times}})$$

$$h_r[XY] = \sum_{\lambda \vdash r} s_\lambda[X] s_\lambda[Y]$$

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$\mathcal{S}^\lambda$ irreducible S_n module
$\lambda \vdash n$ $\quad \dim \mathcal{S}^\lambda = f_\lambda$

$$S_n \circlearrowright \mathbb{C}S_n \circlearrowleft S_n \cong \bigoplus_{\lambda \vdash n} \mathcal{S}^\lambda \otimes \mathcal{S}^\lambda$$

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M_n^λ irreducible Gl_n module
$\lambda \vdash k$ $\dim M_n^\lambda = F_{n,\lambda}$

$$Gl_n \circlearrowright V_n^{\otimes k} \circlearrowleft S_k \cong \bigoplus_{\lambda \vdash k} M_n^\lambda \otimes \mathcal{S}^\lambda$$

$V_n = \mathrm{span}\{x_1, x_2, \ldots, x_n\}$

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M_n^λ irreducible Gl_n module

$\lambda \vdash k$ $\quad \dim M_n^\lambda = F_{n,\lambda}$

$$Gl_n \circlearrowright S^r(V_n \otimes V_k) \circlearrowleft Gl_k \cong \bigoplus_{\lambda \vdash r} M_n^\lambda \otimes M_k^\lambda$$

Orellana-Zabrocki

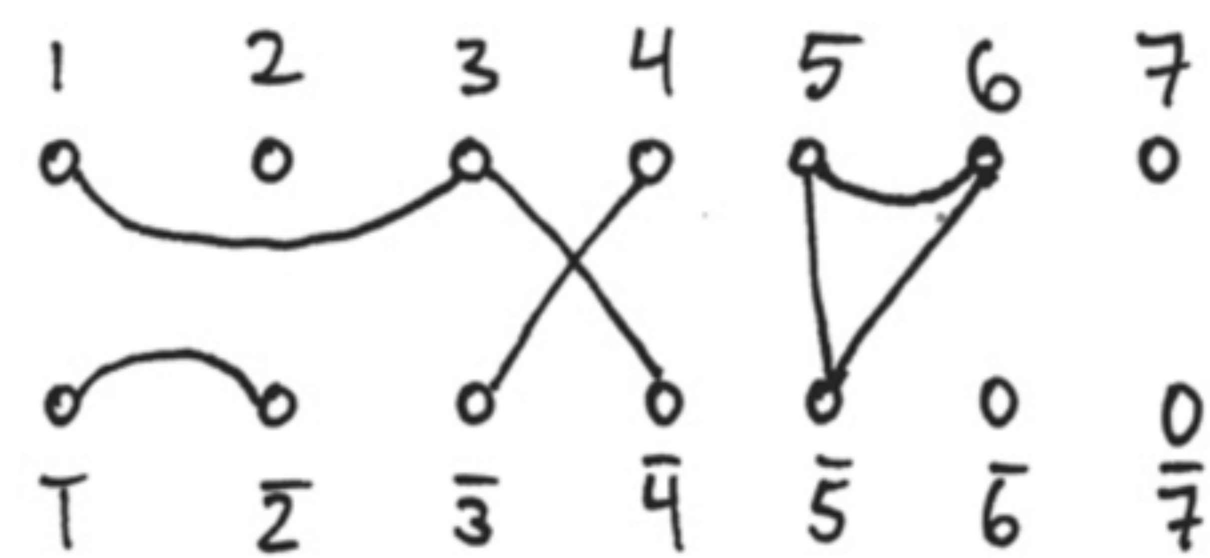
There is an inhomogeneous basis of the symmetric functions $\tilde{S}_\lambda$ that are the characters of the permutation matrices $S_n \subseteq Gl_n$

$$\tilde{S}_\lambda[\text{eigenvals of } \sigma \in Gl_n] = \chi^{(n-|\lambda|,\lambda)}(\sigma)$$

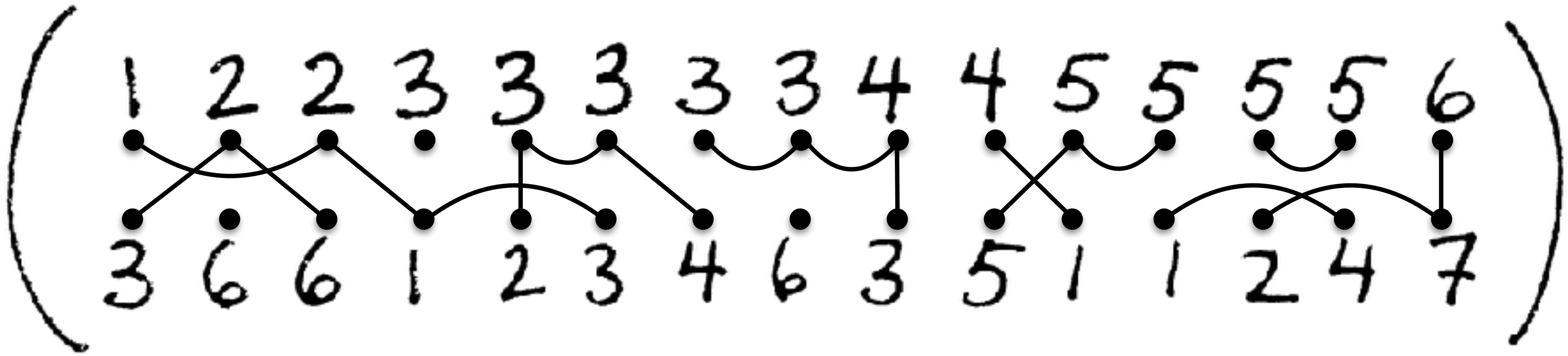
Martin/Jones 1990's: The symmetric group of permutations realized as matrices in Gl_n acts on $V^{\otimes k}$ and there is an algebra indexed by set partitions of $\{1, 2, \ldots, k, \bar{1}, \bar{2}, \ldots, \bar{k}\}$ which commutes with the action of S_n on $V_n^{\otimes k}$

partition algebra

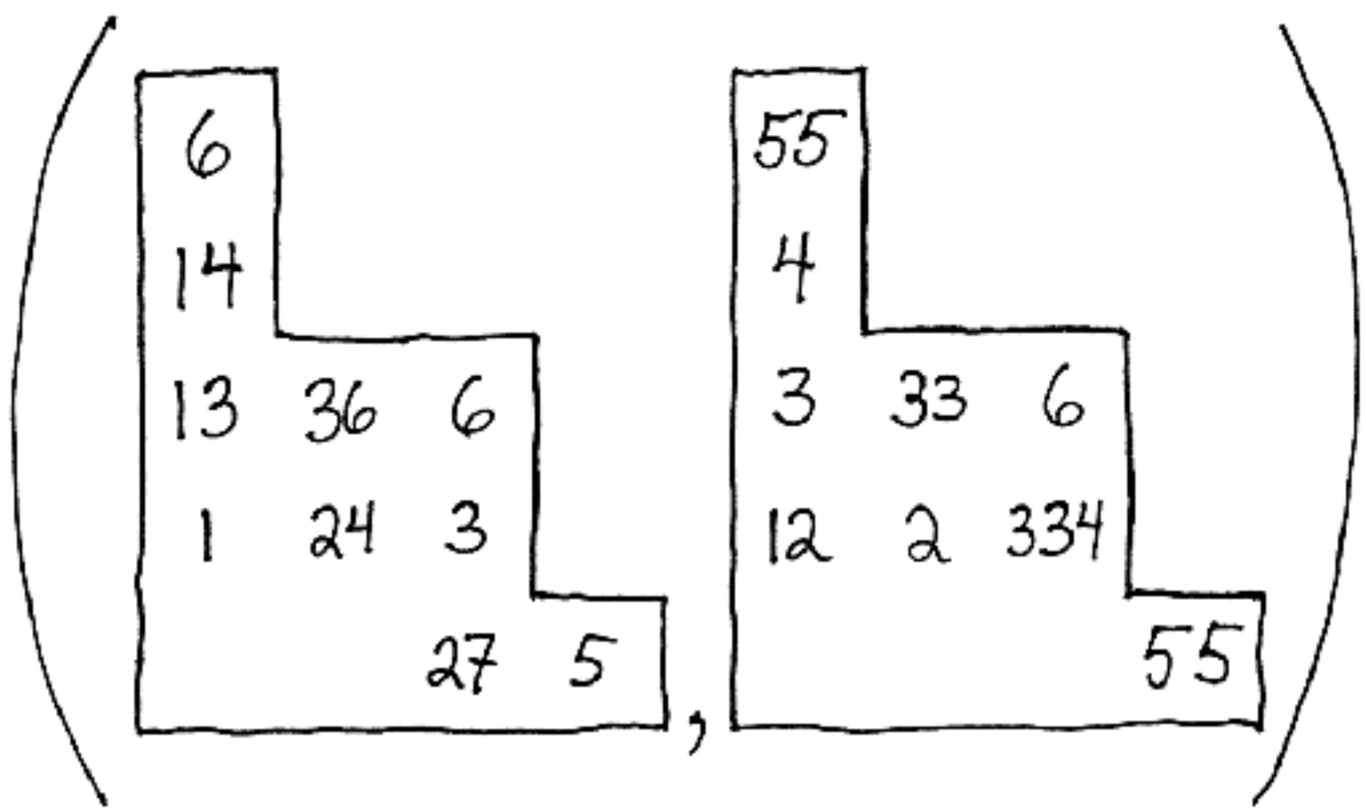
$P_k(n)$



Orellana-Zabrocki extension of RSK



$$\begin{pmatrix} \emptyset & \emptyset & \emptyset & 1,2 & 2 & 3 & 3,3 & 3,3,4 & 4 & 5,5 & 5,5 & 6 \\ 14 & 6 & 6 & 1,3 & 3,6 & \emptyset & 1,4 & 3 & 1 & \emptyset & 5 & 2,7 \end{pmatrix}$$

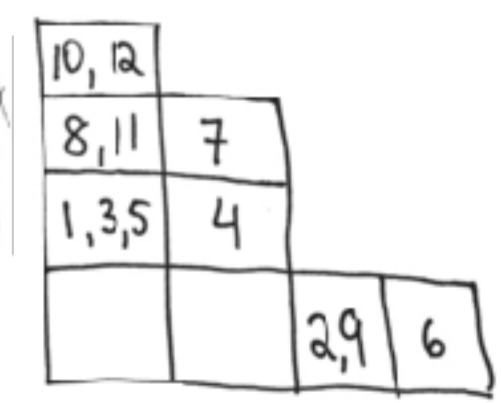


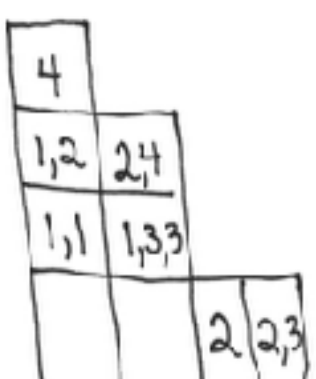
$$B(2n) = \sum_{\lambda : |\lambda| \leq n} SST_{n,\lambda}^{2}$$

$$n^{k} = \sum_{\lambda : |\lambda| \leq k} SST_{k,\lambda} f_{(n-|\lambda|,\lambda)}$$

$$\binom{nk+r-1}{r} = \sum_{\lambda : |\lambda| \leq r} MST_{\lambda}^{r,n} f_{(k-|\lambda|,\lambda)}$$

$SST_{n,\lambda}$ = # of standard set valued tableaux of shape $(a,\lambda)/(\lambda_1)$ of content $\{1..n\}$





$MST_{\lambda}^{r,n}$ = # of multiset tableaux of shape $(a,\lambda)/(\lambda_1)$ with r values taken from $\{1,2,\ldots,n\}$

Theorem (Orellana-Zabrocki)

The commutant algebra of S_n acting on $S^r(V_n \otimes V_k)$ has a basis indexed by multiset partitions of size r and values in $\{1, 2, \dots, k\}$.

multiset partition algebra