A multiset partition algebra

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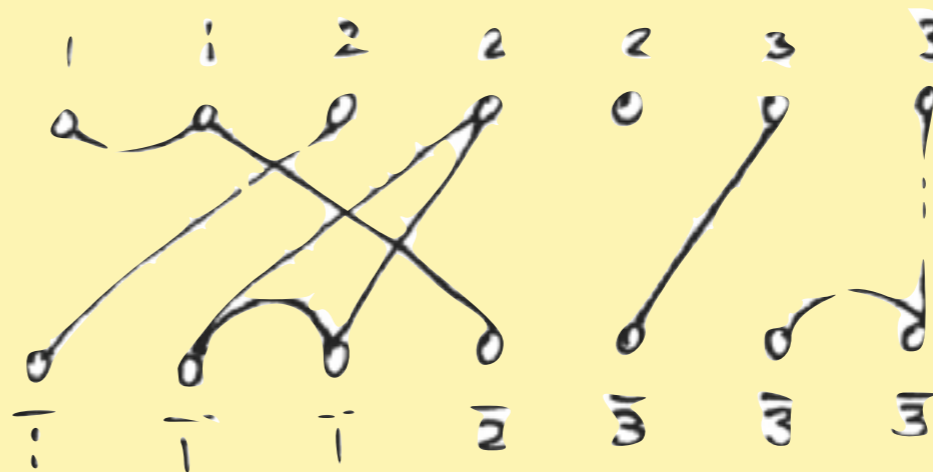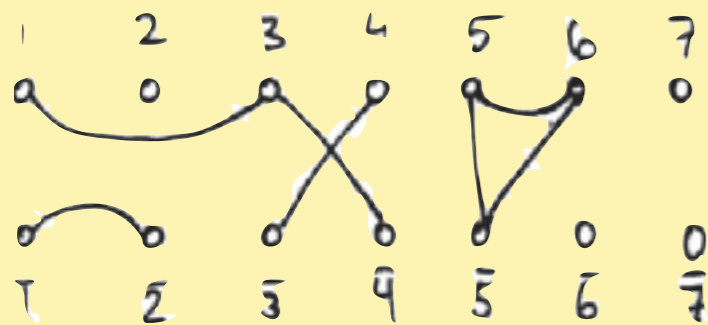
Schur-Weyl duality

$$V_n = \text{span}\{v_1, v_2, \ldots, v_n\} \qquad A(v_i) = \sum_{j=1}^{n} a_{ij} v_j$$

$$Gl_n \circlearrowright V_n^{\otimes k} \circlearrowright S_k \simeq \bigoplus_{\lambda \vdash k} W^{\lambda, n} \otimes \mathbb{S}^\lambda$$

irreducible Gl_n/S_k module

$$v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$$
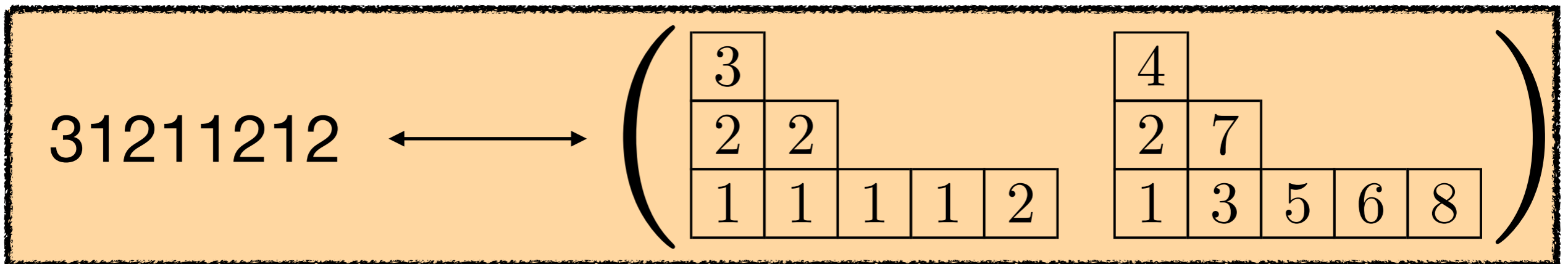
The actions of Gl_n/S_k are commutants of each other and as a consequence, $V_n^{\otimes k}$ decomposes into tensors of irreducibles with multiplicity 1.

$$V_n^{\otimes k} \simeq \bigoplus_{\lambda \vdash k} W^{\lambda,n} \otimes \mathbb{S}^\lambda$$

dim = n^k

dim = # of column strict tableaux

dim = # of standard tableaux

$$n^k = \sum_{\lambda \vdash k} \left(CST^\lambda_{\{1,2,\ldots,n\}} \right) \left(SYT^\lambda \right)$$

Robinson-Schenstead $n = 3, k = 8$

$$31211212 \longleftrightarrow \left(\begin{array}{c} \boxed{3} \\ \boxed{2}\,\boxed{2} \\ \boxed{1}\,\boxed{1}\,\boxed{1}\,\boxed{1}\,\boxed{2} \end{array} \quad \begin{array}{c} \boxed{4} \\ \boxed{2}\,\boxed{7} \\ \boxed{1}\,\boxed{3}\,\boxed{5}\,\boxed{6}\,\boxed{8} \end{array} \right)$$

Howe duality

$$^{Gl_n\circlearrowleft}S^r\left(V_n\otimes V_k\right)^{\circlearrowright Gl_k}\simeq\bigoplus_{\lambda\vdash r}W^{\lambda,n}\otimes W^{\lambda,k}$$

$$\binom{nk+r-1}{r}=\sum_{\lambda\vdash r}\left(CST^{\lambda}_{\{1,2,\ldots,n\}}\right)\left(CST^{\lambda}_{\{1,2,\ldots,k\}}\right)$$
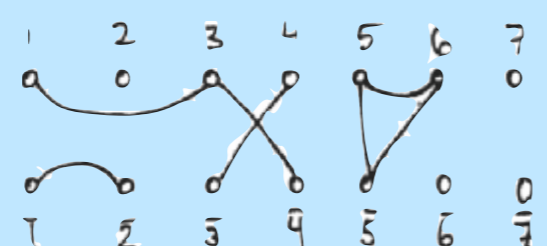
$$S^r\left(V_n\otimes V_k\right)\simeq\mathbb{C}[x_{ij}:1\leq i\leq n,1\leq j\leq k]\text{ of degree }r$$

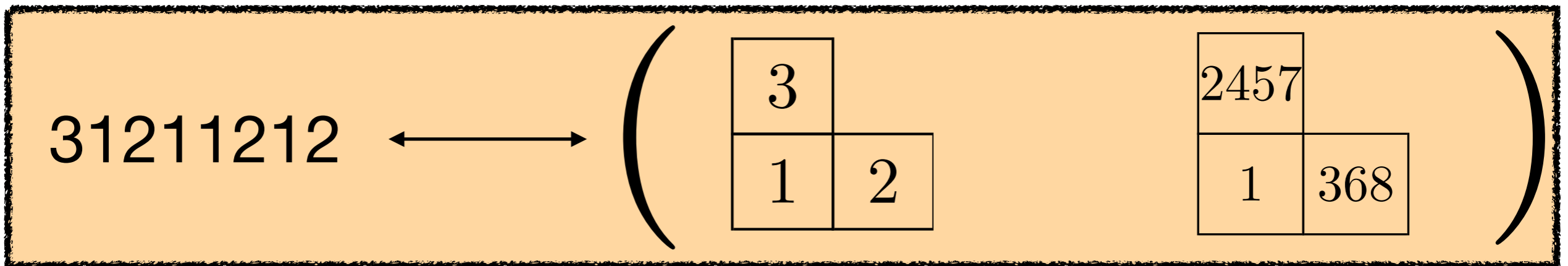Robinson-Schenstead-Knuth $\qquad n=3, k=5, r=8$
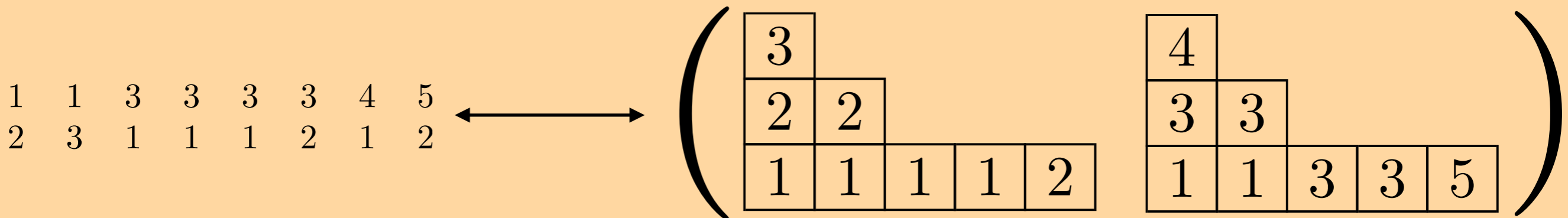
Partition Algebra

$${}^{S_n \circlearrowleft} V_n^{\otimes k \circlearrowright P_k(n)} \simeq \bigoplus_{|\lambda| \leq k} \mathbb{S}^{(n-|\lambda|,\lambda)} \otimes U^\lambda$$

irreducible $P_k(n)$ **module**

$$n^k = \sum_{|\lambda| \leq k} \left(SYT^{(n-|\lambda|,\lambda)} \right) \left(SSetTab^{(n-|\lambda|,\lambda)}_{\{1,2,\ldots,k\}} \right)$$

insertion algorithm $n = 3, k = 8$

$$31211212 \longleftrightarrow \left(\begin{array}{cc} \boxed{3} & \\ \boxed{1} & \boxed{2} \end{array} , \begin{array}{cc} \boxed{2457} & \\ \boxed{1} & \boxed{368} \end{array} \right)$$

insertion algorithm

$n = 3, k = 5, r = 8$

$$
\begin{matrix}
1 & 1 & 3 & 3 & 3 & 3 & 4 & 5 \\
2 & 3 & 1 & 1 & 1 & 2 & 1 & 2
\end{matrix}
\longleftrightarrow
\left(
\begin{array}{c}
\boxed{3} \\
\boxed{2} \\
\boxed{1}
\end{array}
\qquad
\begin{array}{c}
\boxed{3334} \\
\boxed{135} \\
\boxed{1}
\end{array}
\right)
$$

$$
\binom{nk + r - 1}{r} = \sum_{|\lambda| \leq k} \left(SYT^{(n-|\lambda|,\lambda)} \right) \left(MST_{\{1,2,\ldots,k\}^r}^{(n-|\lambda|,\lambda)} \right)
$$

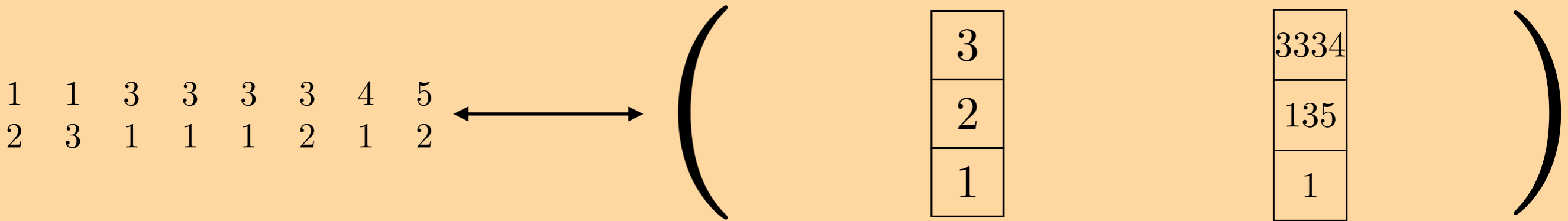# A multiset partition algebra

$$
{}^{S_n \circlearrowleft} S^r \left(V_n \otimes V_k \right)^{\circlearrowleft} \textbf{Multiset partition algebra } MSP_r(n,k)
$$