The multiset partition algebra and Kronecker product

Mike Zabrocki (York Univeristy, Canada)
joint work with Rosa Orellana

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Goal: develop combinatorics of set valued and multiset valued tableaux to understand the Kronecker and restriction/branching from Gln to Sn.

Part I - The multiset partition algebra and its irreducible representations

Schur-Weyl duality

$$Gl_n \circlearrowright V^{\otimes k} \circlearrowleft S_k$$

$$V^{\otimes k} \simeq \bigoplus_{\lambda \vdash k} W^{\lambda}_{Gl_n} \otimes W^{\lambda}_{S_k}$$

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Howe duality

$$Gl_n \circlearrowright \mathbb{C}[X_{n\times k}]_{deg\ r} \circlearrowleft Gl_k$$

$$X_{n\times k} := \{x_{ij} : i \in [n], j \in [k]\}$$

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dimension of $W^{\lambda}_{Gl_n}$ number of column strict tableaux of shape λ entries in 1,2, …, n

3	3				
2	2	3	4		
1	1	1	2	3	4

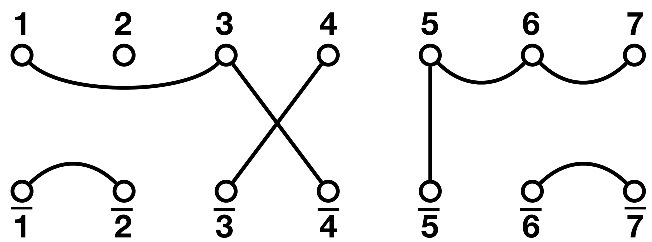
dimension of $W^{\lambda}_{S_k}$ standard tableaux of shape λ

5	7				
3	4	8	11		
1	2	6	9	10	12

$$S_n \subseteq Gl_n$$

Partition algebra

$$S_n \circlearrowright \boldsymbol{V^{\otimes k}} \circlearrowleft P_k(n)$$



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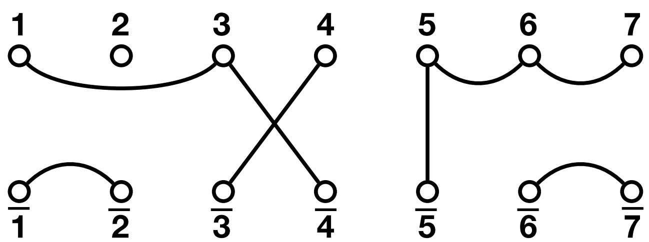
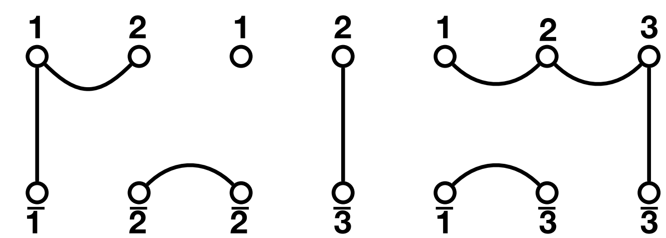


$$V^{\otimes k} \simeq \bigoplus_{\lambda \vdash n} W^{\lambda}_{S_n} \otimes W^{\lambda}_{P_k(n)}$$

multiset partition algebra

$$S_n \circlearrowright \mathbb{C}[X_{n\times k}]_{deg\ r} \circlearrowleft MP_{r,k}(n)$$



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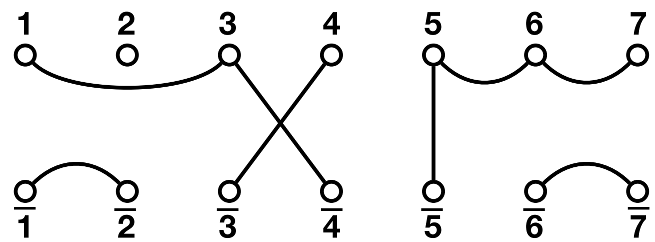

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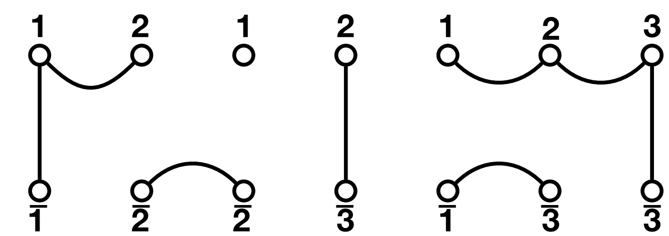


$$\mathbb{C}[X_{n\times k}]_{deg\ r} \simeq \bigoplus_{\lambda \vdash n} W^{\lambda}_{S_n} \otimes W^{\lambda}_{MP_{r,k}(n)}$$

dimension of $W^{\lambda}_{P_k(n)}$ number of set valued tableaux of shape λ entries in 1,2, …, k

3	17				
2	6	49	10		
				5	8,11

dimension of $W^{\lambda}_{MP_{r,k}(n)}$ number of multiset valued tableaux of shape λ

with r values from 1, 2, …, k

2	12				
1	1	11	112		
				11	112

Partition algebra orbit basis

Theorem 4.14. *Multiplication in $\mathsf{P}_k(n)$ in terms of the orbit basis $\{x_\pi\}_{\pi\in\Pi_{2k}(n)}$ is given by*

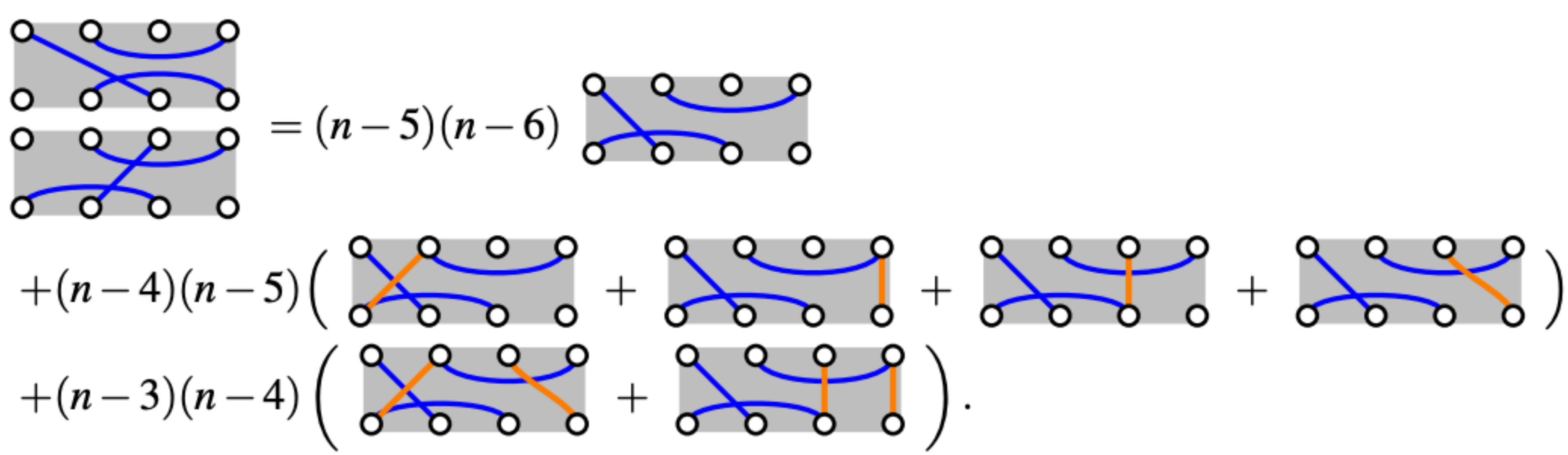
$$x_{\pi_1}x_{\pi_2} = \begin{cases} \sum_{\rho}(n-|\rho|)_{[\pi_1*\pi_2]}\,x_\rho, & \textit{if } \pi_1*\pi_2 \textit{ exactly matches in the middle,} \\ 0 & \textit{otherwise,} \end{cases}$$

where the sum is over all coarsenings ρ of $\pi_1*\pi_2$ obtained by connecting blocks that lie entirely in the top row of π_1 to blocks that lie entirely in the bottom row of π_2.

Benkart-Halverson 2017

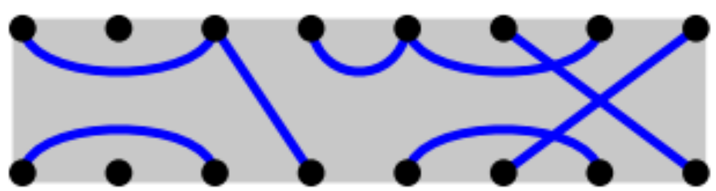
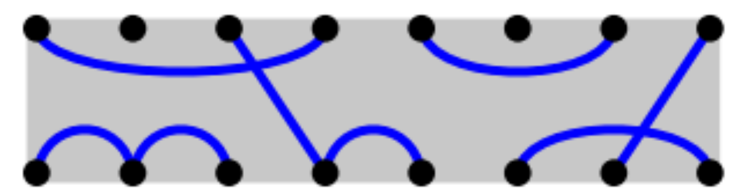
Partition Algebras and the Invariant Theory of the Symmetric Group

Example 4.16. Here $k=4$, $n\geq 5$, and $[\pi_1*\pi_2]=2$ (two blocks are removed upon concatenation of π_1 and π_2).

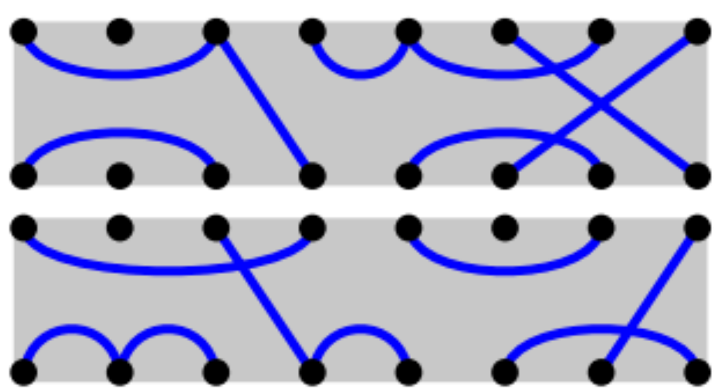
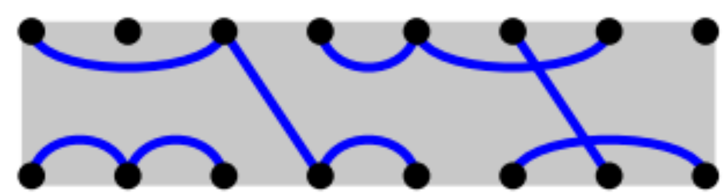


Partition Algebra diagram basis

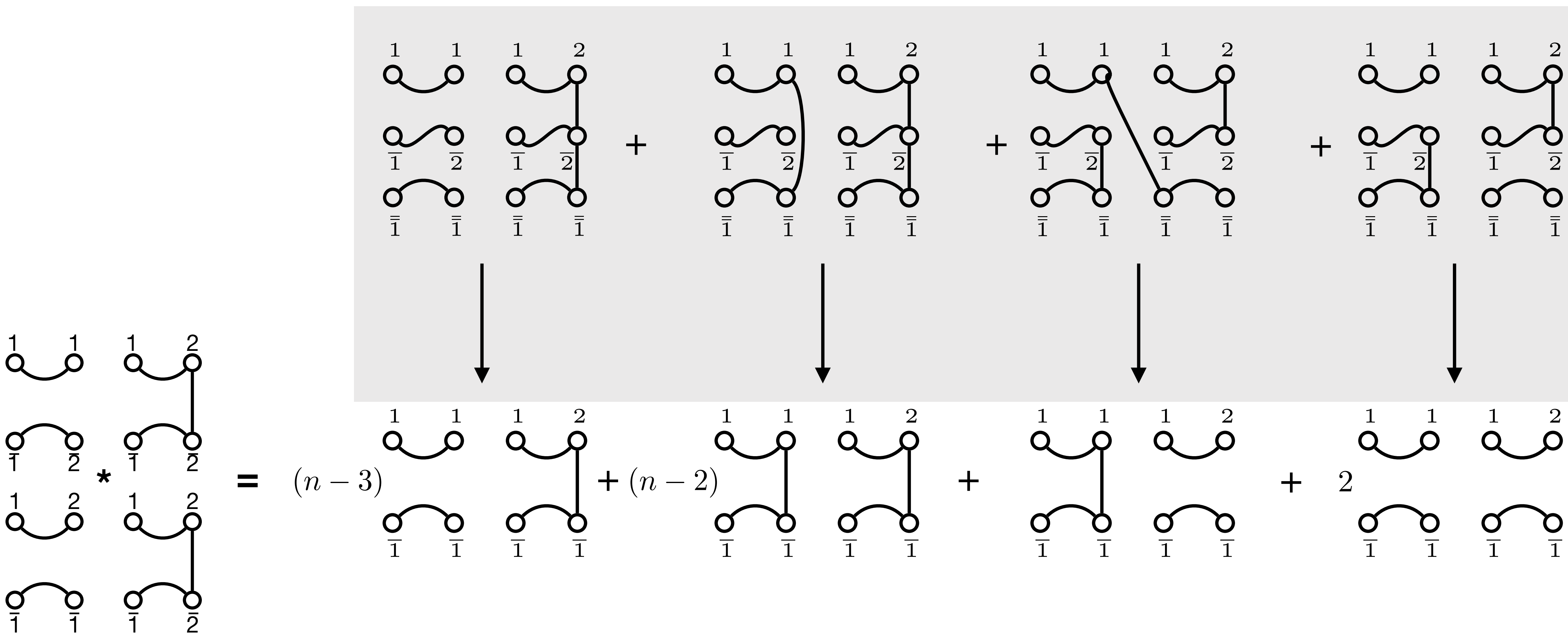
$$d_\pi = \sum_{\pi \preceq \rho} x_\rho \qquad\qquad d_{\pi_1} d_{\pi_2} = n^{[\pi_1 * \pi_2]} d_{\pi_1 * \pi_2}$$

$d_{\pi_1} =$  and $d_{\pi_2} =$ 

then

$d_{\pi_1} d_{\pi_2} =$  $= n^2$  $= n^2 d_{\pi_1 * \pi_2}$,

Multiset partition algebra orbit basis



generating function for the dimensions of the algebra

$$\sum_{r\geq 0}\sum_{n\geq 0} z^n q^r t^r \dim\ M\!P_{r,k}(n) + \text{other terms} = \prod_{i\geq 0}\prod_{j\geq 0} \frac{1}{(1 - zq^i t^j)^{\binom{k+i-1}{i}\binom{k+j-1}{j}}}$$

generating function for the dimensions of the irreducibles

$$\sum_{r\geq 0} q^r \dim\ W^{\lambda}_{M\!P_{r,k}(n)} = \sum_{\gamma\vdash n} \frac{\chi^{W^{\lambda}_{S_n}}(\gamma)}{z_\gamma} \prod_{i=1}^{\ell(\gamma)} \frac{1}{(1-q^{\gamma_i})^k}$$

$$= s_\lambda\left[\frac{1}{(1-q)^k}\right]$$

Kronecker coefficients are the multiplicities of the irreducibles in a restriction (for both partition and multiset partition algebra)

$$\mathrm{Res}^{P_{k+\ell}(n)}_{P_k(n)\otimes P_\ell(n)} W^{\lambda}_{P_{k+\ell}(n)} \simeq \bigoplus_{\nu\vdash n} \bigoplus_{\gamma\vdash n} (W^{\nu}_{P_k(n)} \otimes W^{\gamma}_{P_\ell(n)})^{g_{\lambda\mu\nu}}$$

"The partition algebra and the Kronecker coefficients"
Bowman, de Visscher, Orellana 2012

$$\mathrm{Res}^{MP_{r,k+\ell}(n)}_{MP_{d,k}(n)\otimes MP_{r-d,\ell}(n)} W^{\lambda}_{MP_{r,k}(n)} \simeq \bigoplus_{\nu\vdash n} \bigoplus_{\gamma\vdash n} (W^{\nu}_{MP_{d,k}(n)} \otimes W^{\gamma}_{MP_{r-d,\ell}(n)})^{g_{\lambda\mu\nu}}$$

Part II - How to get at the combinatorics of (reduced) Kronecker

Characters of irreducible Gl_n modules are Schur functions.

$$s_\lambda(x_1, x_2, x_3, \ldots, x_n)$$

$$W^\lambda_{Gl_n} \otimes W^\mu_{Gl_n} \simeq \bigoplus_\gamma (W^\gamma_{Gl_n})^{\oplus c^\gamma_{\lambda\mu}}$$

$$s_\lambda s_\mu = \sum_\gamma c^\gamma_{\lambda\mu} s_\gamma$$

$$S_n \subseteq Gl_n$$

Littlewood-Richardson

reduced Kronecker

Characters of irreducible S_n modules are "irreducible character basis"

$$\tilde{s}_\lambda(x_1, x_2, x_3, \ldots, x_n)$$

$$W^{(n-|\lambda|,\lambda)}_{S_n} \otimes W^{(n-|\mu|,\mu)}_{S_n} \simeq \bigoplus_\gamma (W^{(n-|\gamma|,\gamma)}_{S_n})^{\overline{g}_{\lambda\mu\gamma}}$$

$$\tilde{s}_\lambda \tilde{s}_\mu = \sum_\gamma \overline{g}_{\lambda\mu\gamma} \tilde{s}_\gamma$$

n big

Combinatorial interpretations of

$$\tilde{s}_\lambda \tilde{s}_{a_1} \tilde{s}_{a_2} \cdots \tilde{s}_{a_\ell}$$

coefficients are $\bar{g}_{\nu\lambda(a_1)(a_2)\cdots(a_\ell)}$

= number of multiset tableaux satisfying

- shape $(n - |\nu|, \nu)$
- content λ barred $(a_1, a_2, \ldots, a_\ell)$ unbarred
- lattice condition
- no singletons first row
- no repeated entries

see Rosa's talk tomorrow

Example:

$$\begin{array}{|c|c|c|c|}\hline \bar{1} & \bar{1}1 & \bar{2}1 & \bar{2}2 \\ \hline & & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c}\cline{1-4} \bar{1} & \bar{1} & \bar{2}1 & 2 & \multicolumn{1}{c}{} \\ \hline & & & & \multicolumn{1}{c|}{\bar{2}1} \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c}\cline{1-4} \bar{1} & \bar{1} & 1 & \bar{2}1 & \multicolumn{1}{c}{} \\ \hline & & & & \multicolumn{1}{c|}{\bar{2}2} \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c}\cline{1-4} \bar{1} & \bar{1} & 1 & \bar{2}2 & \multicolumn{1}{c}{} \\ \hline & & & & \multicolumn{1}{c|}{\bar{2}1} \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|}\hline \bar{1} & \bar{1} & \bar{2}1 & \bar{2}12 \\ \hline & & & \\ \hline \end{array}$$

$$\lambda = (2,2) \qquad a_1 = 2 \qquad a_2 = 1$$

coefficient of $\tilde{s}_4$ in $\tilde{s}_{(2,2)}\tilde{s}_2\tilde{s}_1 = 5$

approach #1 - rectification (Littlewood-Richardson version)

$$s_\lambda \underbrace{s_1 s_1 \cdots s_1}_{\ell \text{ times}} = \sum_{\gamma \vdash \ell} (\dim W_{S_\ell}^{\gamma}) s_\lambda s_\gamma$$

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$$s_\lambda \underbrace{s_1 s_1 \cdots s_1}_{\ell \text{ times}} = \sum_{\gamma \vdash \ell} (\dim W^{\gamma}_{S_\ell}) s_\lambda s_\gamma = \sum_{\gamma \vdash \ell} \sum_{\nu \vdash |\lambda| + \ell} (\dim W^{\gamma}_{S_\ell}) c^{\nu}_{\lambda\gamma} s_\nu$$

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# of skew standard tableaux shape ν/λ = $\sum_{\gamma \vdash \ell} (\dim\, W^\gamma_{S_\ell}) c^\nu_{\lambda\gamma}$

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$\nu = (3,2,2) \qquad \lambda = (2,1)$

$$\langle s_{21} s_1 s_1 s_1 s_1, s_{322} \rangle \quad \begin{array}{ccc} 1 & 3 & \\ & 2 & \\ & & 4 \end{array} \quad \begin{array}{ccc} 1 & 4 & \\ & 2 & \\ & & 3 \end{array} \quad \begin{array}{ccc} 2 & 3 & \\ & 1 & \\ & & 4 \end{array} \quad \begin{array}{ccc} 3 & 4 & \\ & 1 & \\ & & 2 \end{array} \quad \begin{array}{ccc} 2 & 4 & \\ & 1 & \\ & & 3 \end{array} \quad \begin{array}{ccc} 1 & 4 & \\ & 3 & \\ & & 2 \end{array} \quad \begin{array}{ccc} 3 & 4 & \\ & 2 & \\ & & 1 \end{array} \quad \begin{array}{ccc} 2 & 4 & \\ & 3 & \\ & & 1 \end{array}$$

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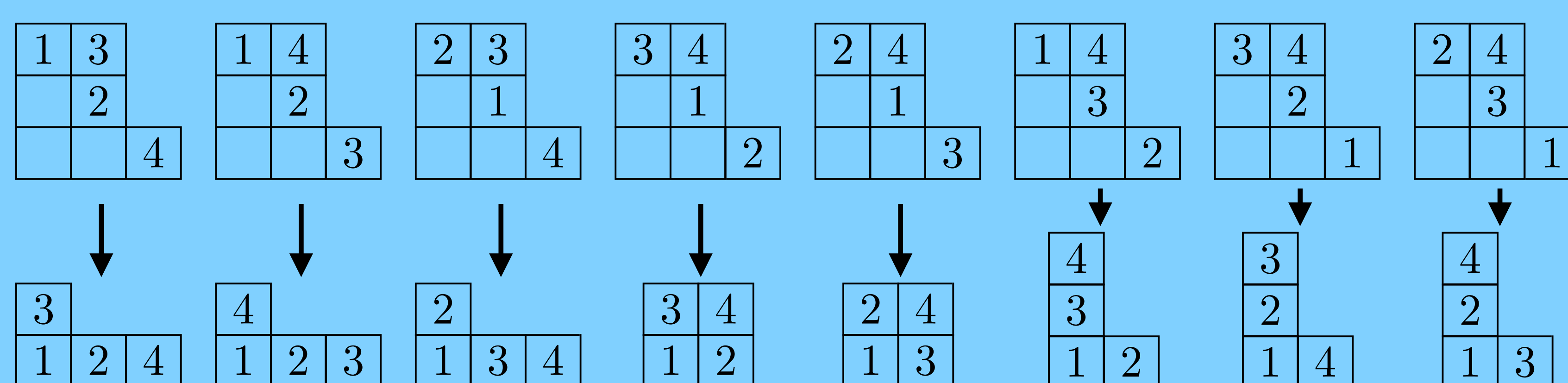

$c^{322}_{21,31} = c^{322}_{21,22} = c^{322}_{21,211} = 1$

jeu de Taquin

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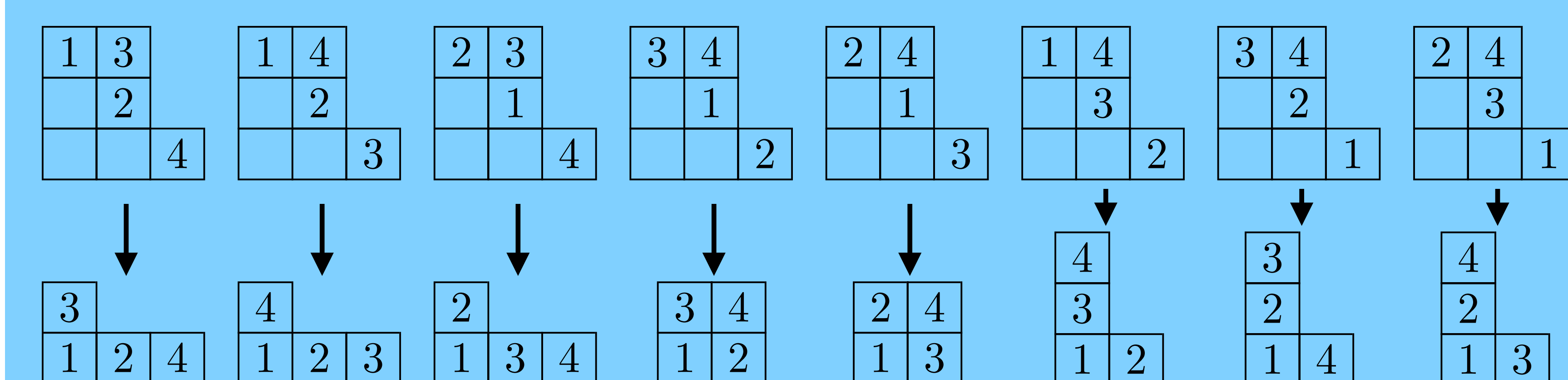

jeu de Taquin

$c^{322}_{21,31} = c^{322}_{21,22} = c^{322}_{21,211} = 1$

$c^\nu_{\lambda\gamma}$ = # of skew standard tableaux of shape ν/λ which rectify to super standard shape γ

approach #1 - “rectification” (reduced Kronecker version)

$$\tilde{s}_\lambda \underbrace{\tilde{s}_1 \tilde{s}_1 \cdots \tilde{s}_1}_{\ell \text{ times}} = \sum_{|\gamma| \leq \ell} (\dim W_{QP_\ell(n)}^{(n-|\gamma|,\gamma)}) \tilde{s}_\lambda \tilde{s}_\gamma$$

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$$\tilde{s}_\lambda \underbrace{\tilde{s}_1 \tilde{s}_1 \cdots \tilde{s}_1}_{\ell \text{ times}} = \sum_T \tilde{s}_{sh(T)}$$

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# of set valued tableaux of “inner shape” λ
and “outer” shape $(n - |\nu|, \nu)$
+ other conditions

= $\displaystyle\sum_{|\gamma|\le\ell} (\dim W^{(n-|\gamma|,\gamma)}_{QP_\ell(n)}) \overline{g}_{\lambda\gamma\nu}$

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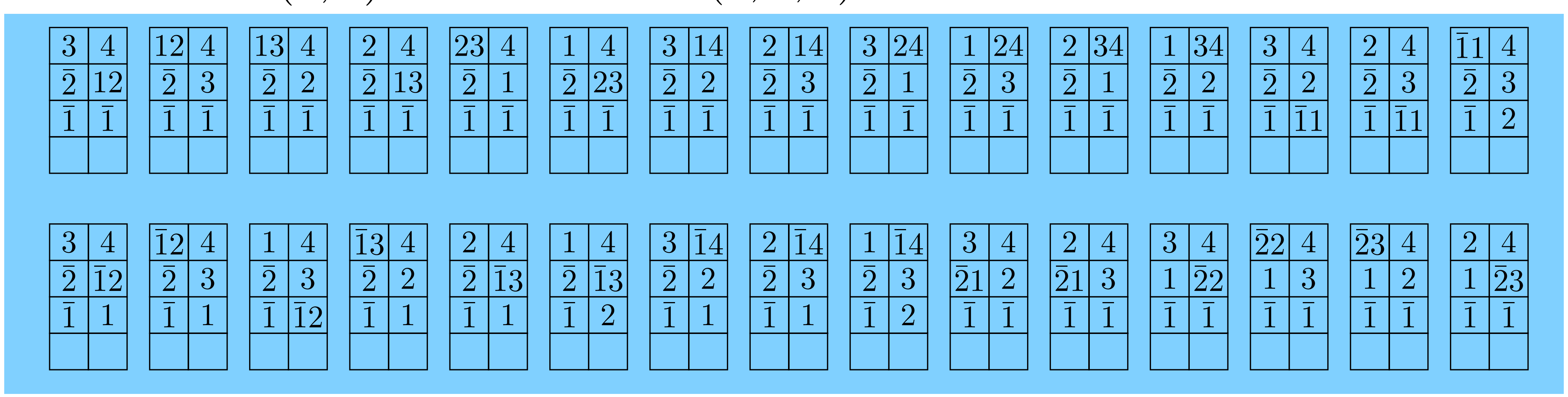
$$\tilde{s}_\lambda \underbrace{\tilde{s}_1 \tilde{s}_1 \cdots \tilde{s}_1}_{\ell \text{ times}} = \sum_{T} \tilde{s}_{sh(T)}$$

# of set valued tableaux of “inner shape” λ and “outer” shape $(n-|\nu|,\nu)$ + other conditions $= \sum_{|\gamma|\leq \ell} (\dim W^{(n-|\gamma|,\gamma)}_{QP_\ell(n)}) \overline{g}_{\lambda\gamma\nu}$

$\lambda = (2,1)$ $\nu = (2,2,2)$



λ	$\overline{g}_{(21)(222)\lambda}$	dim
(31)	2	3
(22)	2	2
(211)	3	3
(1111)	1	1
(21)	1	12

approach #2 - crystal bases or lattice condition (Littlewood-Richardson version)

$$s_\lambda s_{\mu_1} s_{\mu_2} \cdots s_{\mu_r} = s_\lambda s_\mu + \text{ other stuff}$$

combinatorial interpretation of LHS = skew
column strict tableaux of shape γ/λ
content μ

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combinatorial interpretation of LHS = skew
column strict tableaux of shape γ/λ
content μ

2	3	3			
	1	2	2		
			1	1	1

“lattice”

2	3	3			
	1	1	2		
			1	1	2

not “lattice”

a column strict tableau is “lattice” if the last r letters of the reading word contains at least as many i’s as i+1’s

approach #2 - crystal bases or lattice condition (Littlewood-Richardson version)

$$s_\lambda s_{\mu_1} s_{\mu_2} \cdots s_{\mu_r} = s_\lambda s_\mu + \text{ other stuff}$$

“lattice” **not “lattice”**

combinatorial interpretation of LHS = skew column strict tableaux of shape γ/λ content μ

$$s_\lambda s_\mu = \sum_T s_{sh(T)}$$

2	3	3			
	1	2	2		
			1	1	1

“lattice”

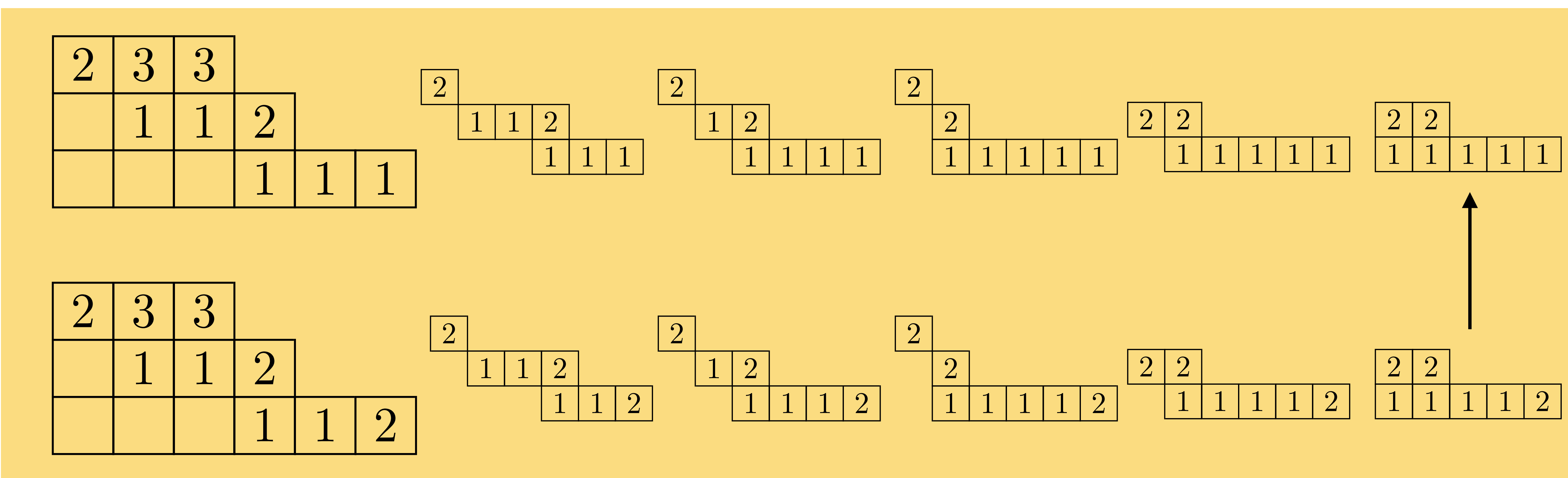
2	3	3			
	1	1	2		
			1	1	2

not “lattice”

a column strict tableau is “lattice” if the last r letters of the reading word contains at least as many i’s as i+1’s

The notion of “lattice” comes from the highest weights from a crystal structure on column strict tableaux

- **crystal operators**
- **jeu de Taquin**
- **reading word**
- **Bender-Knuth involution**
- **standardization**



approach #2 - crystal bases or lattice condition (reduced Kronecker version)

$$\tilde{s}_\lambda \tilde{s}_{\mu_1} \tilde{s}_{\mu_2} \cdots \tilde{s}_{\mu_r} = \tilde{s}_\lambda \tilde{s}_\mu + \text{ other stuff}$$

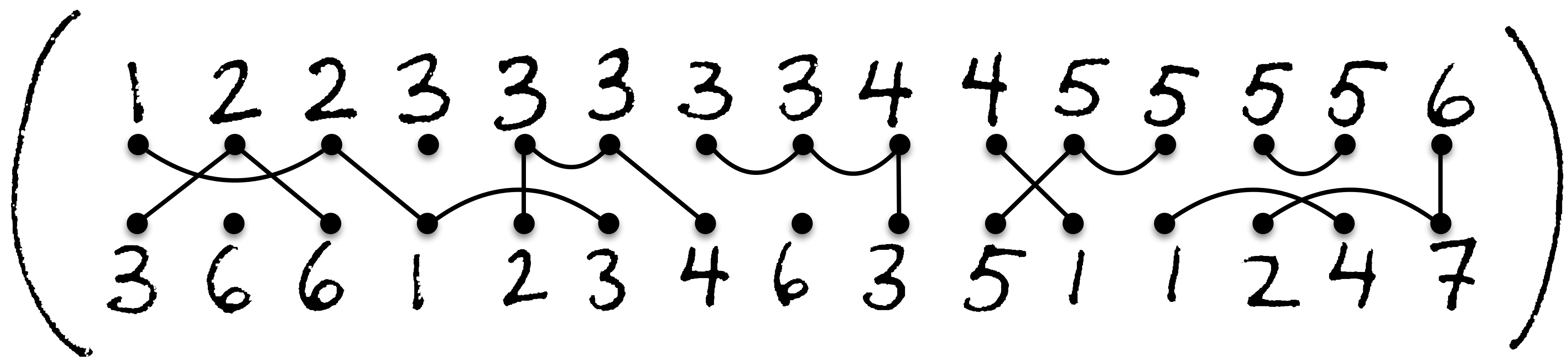
combinatorial interpretation of LHS =
multiset tableaux of content λ in barred entries and μ in unbarred entries w/certain lattice conditions

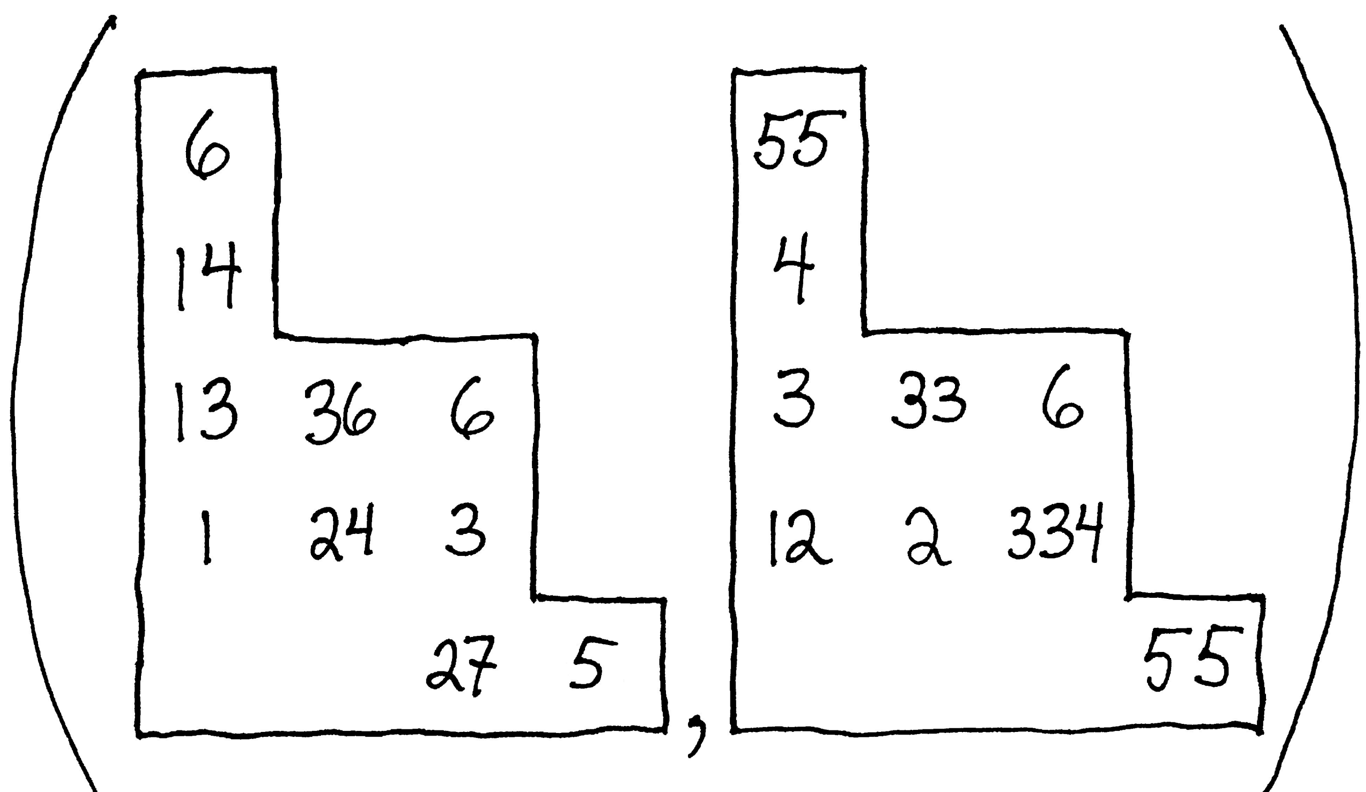
$$\nu = (2,2,2)$$

$$\lambda = (2,1) \quad \mu = (4,3,2)$$

$$\begin{array}{|c|c|cc}
\hline
12 & 13 & \multicolumn{1}{c}{} & \\
\cline{1-2}
\overline{2} & 3 & \multicolumn{1}{c}{} & \\
\cline{1-2}
\overline{1} & 2 & \multicolumn{1}{c}{} & \\
\hline
 & & \multicolumn{1}{c|}{\overline{1}1} & \multicolumn{1}{c|}{12} \\
\hline
\end{array}$$

Thank you!


1 2 2 3 3 3 3 3 4 4 5 5 5 5 6
3 6 6 1 2 3 4 6 3 5 1 1 2 4 7


6
14
13 36 6
1 24 3
27 5
55
4
3 33 6
12 2 334
55

Motivation:

Pieri rules

$S_n \to S_{n-1}$	remove a box
$Gl_n \to Gl_{n-1}$	remove a row
$P_k(n) \to P_{k-1}(n)$	remove a box add a box
????	remove a some cells add some cells

RSK and symmetric functions

$$n^k = \sum_{\lambda \vdash k} \#SemiStdTab_n^\lambda \times StdTab^\lambda = \sum_{\lambda \vdash n} \#StdTab^\lambda \times SetTab_k^\lambda$$

$$\binom{nk + r - 1}{r} = \sum_{\lambda \vdash r} \#SemiStdTab_n^\lambda \times SemiStdTab_k^\lambda = \sum_{\lambda \vdash n} \#StdTab^\lambda \times MultiSetTab_{r,k}^\lambda$$

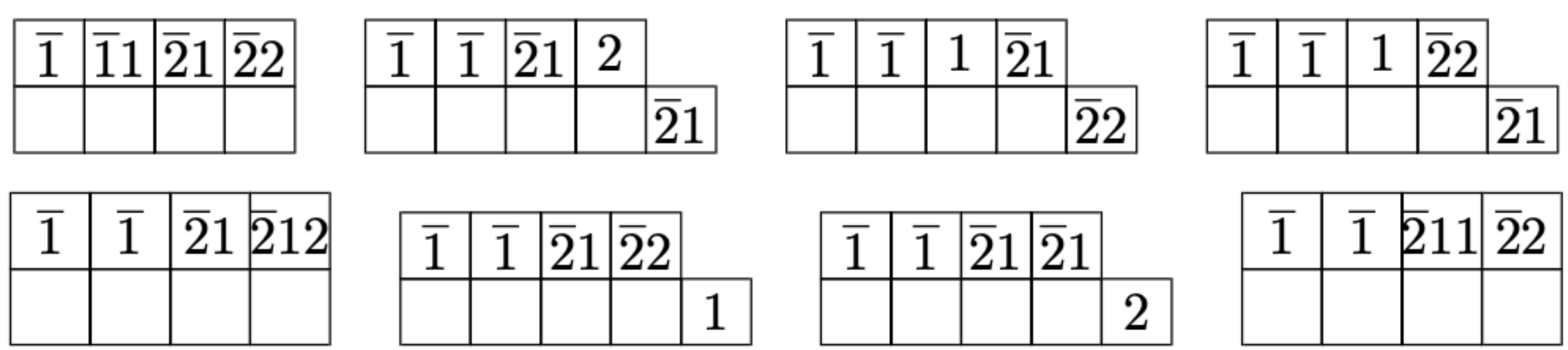
Combinatorial interpretations of

$\tilde{s}_\lambda \tilde{s}_{a_1} \tilde{s}_{a_2} \cdots \tilde{s}_{a_\ell}$ and $\tilde{s}_\lambda s_{a_1} s_{a_2} \cdots s_{a_\ell}$

= number of multiset tableaux satisfying
- shape $(n - |\nu|, \nu)$
- content λ barred $(a_1, a_2, \ldots, a_\ell)$ unbarred
- lattice condition
- no singletons first row
- no repeated entries

see Rosa's talk tomorrow

Example:



$\lambda = (2,2)$ $\quad a_1 = 2$ $\quad a_2 = 1$

coefficient of $\tilde{s}_4$ in $\tilde{s}_{(2,2)} \tilde{s}_2 \tilde{s}_1 = 5$

coefficient of $\tilde{s}_4$ in $\tilde{s}_{(2,2)} s_2 s_1 = 8$