Symmetric functions

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1 Preliminaries

Let R be a commutative ring and $x_1, \dots, x_n$ a set of indeterminants. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a vector of non-negative integer coefficients. The set of all α form a monoid under addition which is isomorphic to the monoid of all monomials x^α under multiplication:

$$x^\alpha x^\beta = x^{\alpha+\beta} \quad \text{with} \quad x^\alpha = x_1^{\alpha} \cdots x_n^{\alpha_n}\,.$$

The corresponding algebra is the **ring of polynomials**, denoted $R[x_1, \dots, x_n]$, consisting of all polynomials in n variables with coefficients in R.

We write $|\alpha| = deg\, x^\alpha = \alpha_1 + \cdots + \alpha_n$. An element of $R[x_1, \dots, x_n]$ of the form

$$f(x_1, \dots, x_n) = \sum_{|\alpha|=d} c_\alpha x^\alpha$$

is called a **homogeneous polynomial** of degree d. With set of all such polynomials denoted by $R[x_1, \dots, x_n]^d$, $R[x_1, \dots, x_n]$ is a graded ring:

$$R[x_1, \dots, x_n] = \bigoplus_{d \geq 0} R[x_1, \dots, x_n]^d$$

There is a natural degree-preserving S_n **action** on polynomials, where elements of the symmetric group act by permuting the variables. That is, given $P(\mathbf{x}) \in R[x_1, \ldots, x_n]$ and $\sigma \in S_n$:

$$\sigma P(x_1, x_2, \ldots, x_n) = P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n})\,.$$

For example, $\sigma = (1\,3\,2) \implies \sigma\left(x_1^2 + 2x_1x_3 + x_2\right) = x_1^2 + 2x_1x_2 + x_3\,.$

A polynomial is called **symmetric** if it is invariant under the action of S_n. The set of symmetric polynomials of degree d in n-variables is denoted Λ_n^d and we set,

$$\Lambda_n = \bigoplus_d \Lambda_n^d\,.$$

Λ_n is a subring of $R[x_1, \ldots, x_n]$, called the **ring of symmetric polynomials**. Examples include:

$$x_1 + 2x_1x_2x_3 + x_2 + x_3 \in \Lambda_3$$

$$x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4 + x_1x_3x_4 \in \Lambda_4^3$$

2 Monomial symmetric functions

The most obvious symmetric polynomials are constructed by symmetrizing the monomial term x^λ, for a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of weakly decreasing non-negative integers.

The monomial symmetric functions: m_λ

$$m_\lambda = \sum_{\beta : \beta^* = \lambda} x^\beta \quad \text{where} \quad x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$$

over all distinct β where β^* is the partition rearrangement of β.

For example, with $\lambda = (2,1,1)$ and $n = 4$:

$$(2,2,1,0)\,(2,2,0,1)\,(1,2,2,0)\,(1,2,0,2)\,(2,1,2,0)\,(2,1,0,2)\,(0,2,2,1)\,(0,2,1,2)\,(0,1,2,2)$$

$$m_{2,1,1} = x_1^2x_2^2x_3 + x_1^2x_2^2x_4 + x_1x_2^2x_3^2 + x_1x_2^2x_4^2 + x_1^2x_2x_3^2 + x_1^2x_2x_4^2 + x_2^2x_3^2x_4 + x_2^2x_3x_4^2 + x_2x_3^2x_4^2$$

Furthermore, the m_λ are clearly independent and if the homogeneous polynomial

$$f = \sum_{|\alpha| = d} c_\alpha x^\alpha$$

is symmetric, then the c_α must remain constant as α ranges over the S_n orbit. Thus, the set of all m_λ with $|\lambda| = d$ spans Λ_n^d. Note if $\ell(\lambda) > n$, we don't have enough variables and $m_\lambda = 0$. To avoid this issue, we instead work in the vector space spanned by all the m_λ called the **ring of symmetric functions:**

$$\Lambda = R[m_\lambda]\,.$$

Λ is closed under product, and is a graded ring. If Λ^d is the space spanned by all m_λ of degree d,

$$\Lambda = \bigoplus_{d \geq 0} \Lambda^d\,.$$

Theorem 1. *$\{m_\lambda : |\lambda| = d\}$ is a basis of Λ^d*

Corollary 2. *The dimension of Λ^d is the number of partitions of d.*

2.1 Multiplying Monomials

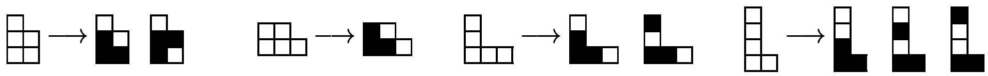
The study of the product of certain monomial symmetric functions will later help us understand other bases for Λ. More precisely, we are interested in determining the possible terms in the right hand side of

$$m_{1^k}\, m_\lambda = \sum_{|\nu|=|\lambda|+k} c^{\nu}_{\lambda 1^k}\, m_\nu \quad \text{where} \quad 1^k = (1,\ldots,1)\,.$$

Note that since the product of symmetric functions is symmetric,

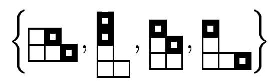
$$c^{\nu}_{\lambda 1^k} = \text{the coefficient of } m_\nu \text{ in } m_\lambda m_{1^k} = \text{the coefficient of } x^\nu \text{ in } m_\lambda m_{1^k}$$

Since the monomial terms in m_λ are x^α where α rearranges to λ, $c^{\nu}_{\lambda 1^k}$ is the number of ways to fill rows of ν with at most one row of λ and of $(1,\ldots,1)$. For example, filling shapes with (2,1) and (1,1) gives



$$m_{1,1} m_{2,1} = 2m_{2,2,1} + m_{3,2} + 2m_{3,1,1} + 3m_{2,1,1,1}$$

It turns out that the only shapes with at least one filling of λ and 1^k are those obtained by adding a vertical k-strip to the diagram of λ.

 + a vertical 2-strip = 

To convince yourself of this fact, note first that since ν is obtained from the parts of λ and 1^k, each row of ν must be λ_i or $\lambda_i + 1$ for some i. Thus, the length of the longest row will be at most $\lambda_1 + 1$ and at least λ_1 implying ν_1 contains λ_1. Similarly, the length of the second longest row will be $\lambda_2 + 1$ or λ_2 implying ν_2 contains λ_2, etc.

2.2 Dominance order

We put a partial order $\lhd$ on the set of all partitions (and consequently on the set of monomial symmetric functions $m_\mu \lhd m_\lambda$ when $\mu \lhd \lambda$) called **dominance order** by declaring that

$$\lambda \rhd \mu \iff \lambda_1 + \cdots + \lambda_j \geq \mu_1 + \cdots + \mu_j \quad \text{for all } j$$

Many bases are triangularly related with respect to $\rhd$. Intuitively, λ is greater than μ in dominance order if the Ferrers diagram of λ is wider than μ.

Remark 3. $\lambda \lhd \mu \iff \lambda' \rhd \mu'$

We can use dominance order to refine our result on the product of monomials:

$$m_{1^k} m_\lambda = \sum_{\nu=\lambda+\text{vertical } k\text{-strip}} c^{\nu}_{\lambda,1^k}\, m_\nu = m_{\lambda+1^k} + \sum_{\nu \,\lhd\, \lambda+1^k} c^{\nu}_{\lambda,1^k}\, m_\nu \tag{1}$$

where $\lambda + 1^k = (\lambda_1 + 1, \ldots, \lambda_k + 1, \lambda_{k+1}, \ldots)$.

3 Elementary symmetric functions

The **elementary symmetric functions** are $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_\ell}$ where

$$e_k(x_1,\ldots,x_n) = \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} x_{i_1} x_{i_2} \cdots x_{i_k} \quad \text{and} \quad e_0 = 1\,.$$

For example, $e_1 = x_1 + x_2 + \cdots$ and $e_2 = x_1x_2 + x_1x_3 + \cdots$. Clearly $m_{1^k} = e_k$. Note the degree$(e_k) = k$ and degree$(e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}) = |\lambda|$. Further, when $k > n$, $e_k = 0$.

Letting $\lambda = 1^{k_1}$ and $m_{1^{k_2}} = e_{k_2}$ in our result on multiplying monomials Eq. (1), implies

$$e_{k_2}\, m_{1^{k_1}} \;=\; m_{1^{k_1}+1^{k_2}} \;+\; \sum_{\nu \lhd 1^{k_1}+1^{k_2}} c_\nu\, m_\nu$$

Repeating this process implies

$$e_{k_n} e_{k_{n-1}} \cdots e_{k_2} m_{1^{k_1}} \;=\; m_{1^{k_1}+\cdots+1^{k_n}} \;+\; \sum_{\nu < 1^{k_1}+\cdots+1^{k_n}} c_\nu\, m_\nu$$

Then noting $\mu = (k_1, k_2, \cdots, k_n)$ implies $\mu' = 1^{k_1} + \cdots + 1^{k_n}$, we have

$$e_\mu = m_{\mu'} + \sum_{\nu \lhd \mu'} c_\nu m_\nu \quad \text{where} \quad c_\nu \in \mathbb{N} \tag{2}$$

The unitriangularity of this expansion of the elementary symmetric function in terms of the basis of monomial symmetric functions implies

Theorem 4. *$\{e_\mu\}$ forms a basis for Λ.*

Since the set $\{e_\lambda\}$ consists of all monomials $e_1^{a_1} e_2^{a_2} \cdots$ where $a_i \in \mathbb{N}^\infty$, and any element of Λ can be expressed as a linear combination of e_λ, every element of Λ is uniquely expressible as a polynomial in the e_r and the e_r are algebraically independent.

Theorem 5. $\Lambda \cong \mathbb{Q}[e_1, e_2, \ldots]$

4 Homogeneous symmetric functions: h_λ

Definition: Let $h_0 = 1$ and $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$ where

$$h_k(x_1, \ldots, x_n) \quad = \quad \sum_{|\lambda|=k} m_\lambda \quad = \quad \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

Note that $h_1 = e_1$. Sometimes h_k is called the complete symmetric function since it is the sum over *all* monomials: $h_1 = \sum x_i$ and $h_2 = \sum x_i^2 + \sum x_i x_j = x_1^2 + x_2^2 + x_1 x_2 + \cdots$.

The homogeneous functions are not triagulary related to the monomials. We shall thus appeal to the use of generating functions to show that the homogeneous symmetric functions provide a basis for Λ.

4.1 Generating Functions

Define

$$E(t) = \sum_{k \ge 0} e_k\, t^k \qquad H(t) = \sum_{k \ge 0} h_k\, t^k$$

Theorem 6.

$$E(t) \;=\; \prod_{i \ge 1} (1 + x_i t) \qquad H(t) \;=\; \prod_{i \ge 1} \frac{1}{1 - x_i t}$$

Proof.

$$\prod_{i \ge 1} (1 + x_i t) = (1 + t x_1)(1 + t x_2) \cdots = 1 + \sum_{i \ge 1} t x_i + \sum_{i<j} t^2 x_i x_j + \cdots$$

The identity for $H(t)$ follows similarly, starting with the geometric series expansion. □

Theorem 7. *$\{h_\lambda : \lambda \vdash d\}$ is a basis for Λ^d.*

Proof. Since the number elements h_λ is the number of partitions of d, it suffices to show that they generate the e_μ. Further, since both h_λ and e_λ are multiplicative we shall simply show that $e_k = f(h_1, \ldots, h_k)$ for some polynomial f. Our identies reveal that $H(t) = \frac{1}{E(-t)}$, or $H(t)E(-t) = 1$. Substituting h_k and e_ℓ in the summations gives

$$\left(\sum_k t^k h_k\right)\left(\sum_l (-t)^l e_l\right) = 1\,,$$

and by comparing coefficients of t^n in both sides we see that

$$\sum_{k+l=n} (-1)^\ell h_k e_\ell = 0 \qquad n \neq 0 \tag{3}$$

Thus, $e_n = h_1 e_{n-1} - h_2 e_{n-2} + \cdots \pm h_{n-1} e_1$ which is a polynomial in the h's by induction on n. □

5 Power symmetric functions: p_λ

$$p_k = \sum_i x_i^k \quad \text{and} \quad p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$$

$$p_1 = x_1 + x_2 + \cdots = e_1 = h_1 \qquad p_r = x_1^r + x_2^r + \cdots = m_{(r)}$$

Theorem 8. *$\{p_\lambda : \lambda \vdash d\}$ is a basis of Λ^d*

Proof. Let $p_\lambda = \sum_\mu c_{\lambda\mu} m_\mu$. If we can show that $c_{\lambda\mu} = 0$ for all $\mu \geq \lambda$ and $c_{\lambda\lambda} \neq 0$ then the transition matrix is invertible and then p_λ must be a basis. If $x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$ appears in $p_\lambda = (x_1^{\lambda_1} + x_2^{\lambda_1} + \cdots)(x_1^{\lambda_2} + x_2^{\lambda_2} + \cdots)\cdots$, then each μ_i is a sum of λ_j's. Since adding together parts of a partition makes it become larger in dominance order, m_λ must be the smallest term that occurs. □

5.1 Generating Function:

Studying the generating function for the power sums reveals:

Proposition 9.

$$P(t) \stackrel{def}{=} \sum_{n\geq 1} p_n \frac{t^n}{n} = \ln H(t)$$

Proof. Using the Taylor expansion of $\ln \frac{1}{1-x}$,

$$\ln \prod_{i\geq 1} \frac{1}{(1-x_i t)} = \sum_{i\geq 1} \ln \frac{1}{(1-x_i t)} = \sum_{i\geq 1}\sum_{n\geq 1} \frac{(x_i t)^n}{n} = \sum_{n\geq 1} \frac{t^n}{n} \sum_{i\geq 1} x_i^n$$

□

Proposition 10. *Let $z_\lambda = \prod_i i^{r_i} r!$*

$$h_n = \sum_{\lambda\vdash n} \frac{1}{z_\lambda} p_\lambda \quad \textit{and} \quad e_n = \sum_{\lambda\vdash n} \frac{(-1)^{|\lambda|-\ell(\lambda)}}{z_\lambda} p_\lambda$$

Proof. It suffices to show

$$H(t) = \sum_{n\geq 0} h_n t^n = \sum_\lambda \frac{1}{z_\lambda} p_\lambda t^{|\lambda|}$$

Using the proposition,

$$\begin{aligned} H(t) &= exp\left(\sum_{n\geq 1} p_n \frac{t^n}{n}\right) = \sum_{k\geq 0} \frac{1}{k!}\left(\sum_{n\geq 1} p_n \frac{t^n}{n}\right)^k = \sum_{\substack{k\geq 0 \\ r_1+r_2+\cdots=k}} \frac{1}{k!}\binom{k}{r_1, r_2, \ldots}\binom{p_1 t}{1}^{r_1}\binom{p_2 t^2}{2}^{r_2}\cdots \\ &= \sum_{r_1, r_2, \ldots} \frac{1}{r_1! r_2! \cdots} \frac{p_1^{r_1} p_2^{r_2} \cdots}{1^{r_1} r^{r_2} \cdots} t^{r_1 + 2r_2 + \cdots} = \sum_\lambda \frac{1}{z_\lambda} p_\lambda t^{|\lambda|} \end{aligned}$$

where $\lambda = (\cdots, 3^{r_3}, 2^{r_2}, 1^{r_1})$. □

6 Schur functions

6.1 Skew Schur functions

Semi-standard **skew tableaux** are defined on any skew shape will the condition that the fillings are weakly increasing in rows and strictly increasing in columns. The **evaluation**(content) of any such tableau T is $(m_1, m_2, \ldots)$ where m_i is the number of time i occurs in T.

The **skew Schur function** is defined

$$s_{\lambda/\mu} = \sum_{T \in \mathcal{T}(\lambda/\mu)} x^{ev(T)},$$

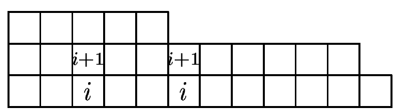
where $\mathcal{T}(\lambda/\mu)$ is the set of all semi-standard skew tableaux of shape λ/μ.

The first job is to prove these functions are symmetric. As such, let τ_i denote the simple transposition of letters i and $i+1$. Since $\tau_i \in S_n$ where $\tau_i(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots) = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots)$, it suffices to show that there is an involution σ_i on $\mathcal{T}(\lambda/\mu)$ that permutes letters in the tableau so that

$$ev(\sigma_i(T)) = \tau_i(ev(T)). \tag{4}$$

6.2 Symmetric group action

Any two entries i and $i+1$ of a tableau T are **married** if they occur in the same column. The conditions on rows and columns of skew tableaux imply that if i is married, then its spouse $i+1$ occurs in the cell directly above it. And, if any two entries i on the same row are both married, then all entries occuring between these cells are part of a married couple.



Furthermore, since columns are strictly increasing, any unmarried entries i or $i+1$ in a row must lie to the right of all married i, and similarly any unmarried entries i or $i+1$ in a row must lie to the left of all married $i+1$. Thus, the set of all unmarried i and $i+1$ in any row lie in a contiguous sequence such any cell above these contains on letters strictly larger than $i+1$ and those below this sequence have letters strictly smaller than i. Thus, σ_i is defined to act on a tableaux by replacing every such sequence of $r \geq 0$ unmarried i's and $s \geq 0$ unmarried $i+1$'s with s i's followed by r $i+1$'s. Since the number of married i's and $i+1$'s is the same, this involution meets the condition (4).

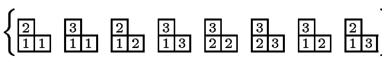
6.3 Schur functions

An important special case of skew Schur functions is the **Schur function**, $s_\lambda = s_{\lambda/\emptyset}$. More precisely,

$$s_\lambda = \sum_{T \in \mathcal{T}(\lambda)} x^{ev(T)},$$

where $T(\lambda)$ is the set of all semi-standard tableaux of shape $\lambda = \lambda/\emptyset$.

In three variables, we consider tableaux with fillings on the set $\{1, 2, 3\}$ and

$\mathcal{T}(2,1) = \{$  $\}$

$\xrightarrow{ev}$ $(2,1,0)$ $(2,0,1)$ $(1,2,0)$ $(1,0,2)$ $(0,2,1)$ $(0,1,2)$ $(1,1,1)$ $(1,1,1)$

$$\Longrightarrow s_{2,1}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + x_1 x_2 x_3 + x_1 x_2 x_3 = m_{2,1} + 2m_{1,1,1}$$

In the case that $\lambda = (k)$ or (1^k), the Schur functions reduce to the kth complete or elementary symmetric function. That is,

$$s_{1^k} = e_k = \sum_{T \in \mathcal{T}(1^k)} x^T \quad \text{where } x^T = x_1^{m_1} x_2^{m_2} \cdots$$

Since the set $\mathcal{T}(1^k)$ contains only vertical k-strips, the multiplicity of any entry in T must be at most 1. Thus, any term x^T in the sum must be a monomial with variables to the first power only. On the other hand,

$$s_k = h_k = \sum_{T \in \mathcal{T}(k)} x^T$$

where we sum over all horizontal k-strips with entries from $\{1, \ldots, n\}$. Since a horizontal k-strip may contain any k subset of n, i.e. anything from k 1's to $\{1, 2, \ldots, k\}$, we see a term x^T is a monomial in variables to any power.

Since the action σ_i on tableaux implies the Schur functions are symmetric, we can expand any Schur function in terms of the monomial basis. For any partition λ,

$$s_\lambda = \sum_{\mu \vdash |\lambda|} K_{\lambda\mu} m_\mu$$

where the **Kostka number** $K_{\lambda\mu}$ is the number of tableau of shape λ and evaluation μ.

Note that the coefficient of $\mu = 1^n$ will simply be the number of standard tableaux of shape λ since evaluation $(1, \ldots, 1)$ implies each entry occurs only once.

Theorem 11. *$\{s_\lambda\}$ is an integral basis for Λ*

Proof. It suffices to show $K_{\lambda\mu} = 0$ for all $\mu > \lambda$ and $K_{\lambda\lambda} = 1$. If $K_{\lambda\mu}$ is non-zero then there must exist a tableau of shape λ with μ_1 ones, μ_2 twos, ... Assume $\mu > \lambda$. This implies $\lambda_1 + \cdots + \lambda_{i-1} = \mu_1 + \cdots + \mu_{i-1}$ and $\lambda_i < \mu_i$ for some i. The column strict condition implies the first row must have all μ_1 ones, the second row the μ_2 twos, ... In row i of size λ_i, since $\mu_i > \lambda_i$, the μ_i i's will not all fit. However, a leftover i cannot go above this row without violating the increasing column condition and all rows below have already been filled. Therefore, for a λ shaped tableau to exist with evaluation μ, $\mu \leq \lambda$. Moreover, our argument shows that if $\lambda = \mu$, our only option is to fill each row λ_j with μ_j $j's$. Thus $K_{\lambda\lambda} = 1$. □

6.4 The Pieri Rule

Recall we showed that the formal sum of tableaux

7 Littlewood-Richardson coefficients

More generally is the question of the Schur function expansion of a product of two Schur functions.

7.1 The bi-alternant formula

The definition for Schur functions was originally given by a ratio determinants called **alternants**:

$$a_\beta = det(x_i^{\beta_j}) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, x^{\sigma\beta}\,.$$

Note that for any $\sigma \in S_n$ we have $a_{\sigma\beta} = \mathrm{sgn}(\sigma) a_\beta$. Letting $\rho = (n-1, n-2, \ldots, 0)$, the Schur functions were then defined by the "bi-alternant" formula:

$$\tilde{s}_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}$$

As the ratio of determinants, the functions are clearly symmetric. However, the proof that $\tilde{s}_\lambda = s_\lambda$ is equivalent to the functions defined by tableux follows from an interesting identity.

Our point of departure is to consider subtableaux of $T \in \mathcal{T}(\lambda/\mu)$. let $T_{\geq j}$ denote the tableau formed by the columns $j, j+1, \ldots$ of T. Similarly defined are $T_{>j}, T_j$, and $T_{<j}$. For any partition λ, T is **λ-yamanouchi** if $\lambda + ev(T_{\geq j})$ is a partition.

Proposition 12.

$$a_{\lambda+\rho}\, s_{\mu/\nu} = \sum_{\substack{T\in\mathcal{T}(\mu/\nu)\\ T=\lambda-yamanouchi}} a_{\lambda+ev(T)+\rho} \tag{5}$$

Note that there is only one tableau of shape μ (a partition) that is $\emptyset$-yamanouchi, the tableau of content μ. Thus, when $\lambda=\nu=\emptyset$, this identity reduces to $a_\rho s_\mu = a_{\mu+\rho}$ implying

Corollary 13. *For any partition μ,*

$$\tilde{s}_\mu = s_\mu$$

Proof. The symmetric group action on $\mathcal{T}(\mu/\nu)$ implies that the number of tableaux of evaluation e equals the number with evaluation σe for any $\sigma\in S_n$. Thus $\sigma(\lambda+\rho)+ev(T)$ and $\sigma(\lambda+\rho+ev(T))$ occur the same number of times as T ranges over $\mathcal{T}(\mu/\nu)$. Therefore

$$a_{\lambda+\rho}\, s_{\mu/\nu} = \sum_{\sigma\in S_n}\sum_{T\in\mathcal{T}(\mu/\nu)} \mathrm{sgn}(\sigma)x^{\sigma(\lambda+\rho)}x^{ev(T)} = \sum_{\sigma\in S_n}\sum_{T\in\mathcal{T}(\mu/\nu)} \mathrm{sgn}(\sigma)x^{\sigma(\lambda+\rho+ev(T))} = \sum_{T\in\mathcal{T}(\mu/\nu)} a_{\lambda+ev(T)+\rho}$$

We thus are left to prove that the alternants die when T is not λ-yamanouchi. The idea is as follows: suppose $\lambda_i=\lambda_{i+1}$ and T has no i's and exactly one $i+1$. T is not λ-yamanouchi since $\lambda+ev(T)$ has $\lambda_{i+1}+ev_{i+1}(T)=\lambda_i+ev_i(T)+1$ – illegal. In this case, the corresponding alternant $a_{\lambda+ev(T)+\rho}$ is zero since $\rho_{i+1}=\rho_i-1$ and $\lambda_{i+1}+ev_{i+1}(T)+\rho_{i+1}=\lambda_i+ev_i(T)+\rho_i$ imply that two columns in the determinant are equal.

In the general case, we shall prove that there is a unique pairing of all tableaux that are not λ-yamanouchi such that the corresponding alternants cancel out. To this end, suppose T is not λ-yamanouci. Then $\lambda+ev(T_{\geq j})$ is not a partition for some j, we choose the largest such j. Let k denote the first row that is larger then its successor. This given,

$$\lambda_k+ev_k(T_{\geq j}) < \lambda_{k+1}+ev_{k+1}(T_{\geq j}) \quad\text{and}\quad \lambda_k+ev_k(T_{>j}) \geq \lambda_{k+1}+ev_{k+1}(T_{>j})\,.$$

Since columns of T are strictly increasing, there is at most one k and at most one $k+1$ in any column. Thus, the inequality on the right is forced to be an equality and the jth column of T contains $k+1$ but no k:

$$\lambda_{k+1}+ev_{k+1}(T) = \lambda_k+ev_k(T)+1.$$

The entry (in column $j+1$) southwest of the $k+1$ in column j is strictly smaller than k implying all entries to the west of this are also smaller than k. Thus the tableau $\hat{T}$ obtained by acting with the symmetric group on $T_{<j}$ is such $\hat{T}_{\geq j}=T_{\geq j}$, and is therefore not λ-yamanouchi. Further $\tau_k ev(T_{<j}) = ev(\hat{T}_{<j})$ while $\lambda_{k+1}+ev_{k+1}(T)=\lambda_k+ev_k(T)+1$, implying $\tau_k(\lambda+ev(T)+\rho)=\lambda+ev(\hat{T})+\rho$. Therefore, $a_{\lambda+ev(T)+\rho} = -a_{\lambda+ev(\hat{T})+\rho}$ as claimed. □

7.2 Littlewood-Richardson coefficients

Now, dividing the identity (5) by a_ρ and using the bi-alternant formula gives

$$s_\lambda\, s_{\mu/\nu} = \sum_{\substack{T\in\mathcal{T}(\mu/\nu)\\ T=\lambda-yamanouchi}} s_{\lambda+ev(T)}\,.$$

When $\lambda=\emptyset$, the yamanouchi tableaux describe the Schur function expansion of a skew Schur function.

$$s_{\mu/\nu} = \sum_{\substack{T\in\mathcal{T}(\mu/\nu)\\ T=yamanouchi}} s_{ev(T)}\,,$$

and taking $\nu=\emptyset$ gives the formula for the Schur function expansion of a product of two Schur functions:

$$s_\lambda\, s_\mu = \sum_{\substack{T\in\mathcal{T}(\mu)\\ T=\lambda-yamanouchi}} s_{\lambda+ev(T)}\,.$$

Note that the Pieri formulas are the special case when $\mu=r$ or 1^r.

7.3 RSK-Correspondence and the Cauchy identity

We have seen that there is a bijection between standard tableaux and permutations. The RS-correspondence can be generalized to pinpoint a precise relation between pairs of tableaux and **generalized permutations**. Such a permutation is a two-line array of positive integers whose columns are in lexicographic order with the top entry taking precedence:

$$w = \begin{pmatrix} u_1 & u_2 & \cdots & u_m \\ v_1 & v_2 & \cdots & v_m \end{pmatrix}$$

Lexicographic order [1] implies $u_1 \leq u_2 \leq \cdots \leq u_m$ and $v_{k-1} \leq v_k$ if $u_{k-1} = u_k$.

The Robinson-Schensted-Knuth correspondence takes a generalized permutation π to a pair of tableaux by the following algorithm:

$$w = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 \\ 3 & 4 & 1 & 1 & 2 & 1 \end{pmatrix}$$

$$\emptyset \leftarrow 3 = \begin{array}{|c|}\hline 4 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline\end{array}$$

$$\begin{array}{|c|}\hline 4 \\ \hline\end{array} \leftarrow 4 = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array} \leftarrow 1 = \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{4} \\ \hline\end{array} \qquad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline\end{array}$$

$$\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{4} \\ \hline\end{array} \leftarrow 1 = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 1 \\ \hline\end{array} \leftarrow 2 = \begin{array}{|c|c|c}\hline 3 & 4 & \\ \hline 1 & 1 & \multicolumn{1}{|c|}{2} \\ \hline\end{array} \qquad \begin{array}{|c|c|c}\hline 2 & 2 & \\ \hline 1 & 1 & \multicolumn{1}{|c|}{2} \\ \hline\end{array}$$

$$\begin{array}{|c|c|c}\hline 3 & 4 & \\ \hline 1 & 1 & \multicolumn{1}{|c|}{2} \\ \hline\end{array} \leftarrow 1 = \begin{array}{|c|c|c}\hline 3 & & \\ \hline 2 & \multicolumn{1}{|c|}{4} & \\ \hline 1 & 1 & \multicolumn{1}{|c|}{1} \\ \hline\end{array} \qquad \begin{array}{|c|c|c}\hline 3 & & \\ \hline 2 & \multicolumn{1}{|c|}{2} & \\ \hline 1 & 1 & \multicolumn{1}{|c|}{2} \\ \hline\end{array}$$

Note that when the top row is $(1, \ldots, n)$ and the bottom row is a permutation of these letters, a two-rowed array is a permutation in two line notation.

As with the RS-correspondence, we refer to the resulting tableaux as the insertion and recording tableau respectively.

Theorem 14. *The RSK-correspondence is a bijection between two-rowed arrays in lexicographic order and pairs of same-shaped tableaux.*

Proof. Let P and Q denote the insertion and recording tableau respectively. The insertion algorithm ensures that P is a tableau. Now Q has weakly increasing rows since the cells are added to the periphery at each step and entries are taken from the top of the two-rowed array – a non-decreasing sequence. When two entries of the top row of this array are equal, the corresponding entries in the bottom row are non-decreasing by lexicographic order. Thus, the insertion process of these consecutive entries $x \geq x'$ gives rise to cells that do not lie in the same column by the row bumping lemma and therefore the columns of Q are increasing.

On the other hand, given a corner and a tableau we have the deletion algorithm that is the precise inverse of Schensted insertion. Choosing corners from largest to smallest, where the rightmost of repeated entries is the largest, we can construct a two-rowed array with lexicographic ordering. □

[1] Note: the ordering is lexicographic as an ordering on pairs

$$\binom{u}{v} \leq \binom{u'}{v'} \quad \text{if } u < u' \text{ or if } u = u' \text{ and } v \leq v'$$

Theorem 15. *Cauchy Identity*

$$\prod_{i,j} \frac{1}{(1 - x_i y_j)} = \sum_{\lambda} s_\lambda(x) s_\lambda(y) \tag{6}$$

Proof. The coefficient of $x^\alpha y^\beta$ in the right hand side of () is $K_{\lambda\alpha} K_{\lambda\beta}$, or the number of possible pairs of λ-shaped tableaux (P, Q) such that the evaluation of P is α and $\text{ev}(Q) = \beta$. On the other hand, the left hand side can be expanded using a product of geometric series,

$$\prod_{i,j} \frac{1}{(1 - x_i y_j)} = \prod_{i,j} \sum_{a_{i,j} \geq 0} (x_i y_j)^{a_{i,j}} \tag{7}$$

$$= \left(\sum_{a_{1,1} \geq 0} (x_1 y_1)^{a_{1,1}} \right) \left(\sum_{a_{1,2} \geq 0} (x_1 y_2)^{a_{1,2}} \right) \cdots \tag{8}$$

$$\left(\sum_{a_{2,1} \geq 0} (x_2 y_1)^{a_{2,1}} \right) \left(\sum_{a_{2,2} \geq 0} (x_2 y_2)^{a_{2,2}} \right) \cdots \tag{9}$$

By our recent bijection, we have proved the coefficients of $x^\alpha y^\beta$ in both sides of the identity are equivalent.