Immaculate basis of Quasisymmetric functions

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(with **Chris Berg, Franco Saliola, Luis Serrano and Mike Zabrocki**)

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Outline

- NSym, Sym, QSym Combinatorial Hopf Algebras.
- Immaculate Basis Created from emptyness.
- Nice properties and posets is this basis interesting?
- Hall-Littlewood if time.

Sym, NSym, QSym

$$\begin{array}{ccc} \mathrm{NSym} & & \\ \downarrow\!\!\downarrow & & \\ \mathrm{Sym} & \hookrightarrow & \mathrm{QSym} \end{array}$$

Sym, NSym, QSym

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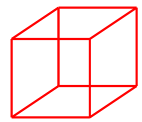
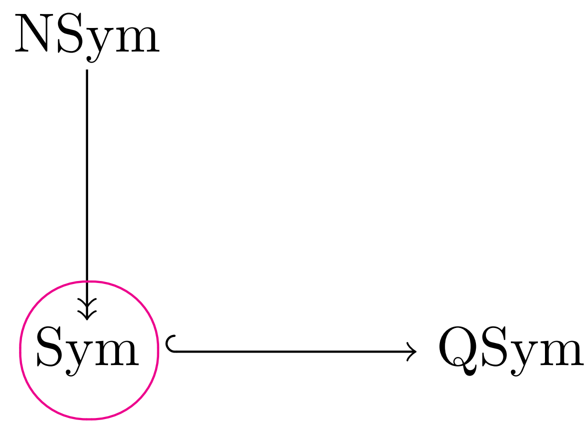


Sym, NSym, QSym



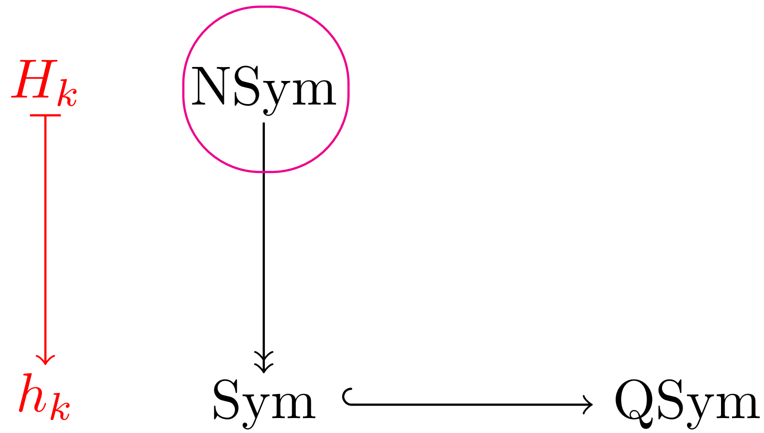
Sym: Symmetric functions

Basis: e_λ (Elementary); m_λ (monomials); p_λ (Power sums);

$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$ (Homogeneous) $\sum_{k\geq 0} h_k t^k = \prod_{i\geq 1} \frac{1}{1-x_i t}$

$s_\lambda = \det(h_{\lambda_i+j-i})$ (Schur)

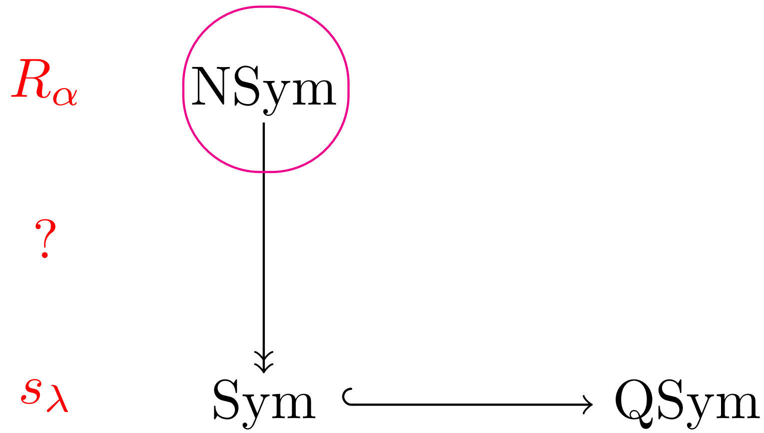
Sym, NSym, QSym



Sym: Symmetric functions $= \mathbb{Z}[h_1, h_2, \ldots]$

NSym: noncommutative symmetric functions $= \mathbb{Z}\langle H_1, H_2, \ldots \rangle$

Sym, NSym, QSym



Sym: Symmetric functions $= K_0\left(\bigoplus S_n\right)$: Irreducible $\leftrightarrow$ Schur

NSym: noncommutative symmetric functions $= \mathbb{Z}\langle H_1, H_2, \ldots\rangle$

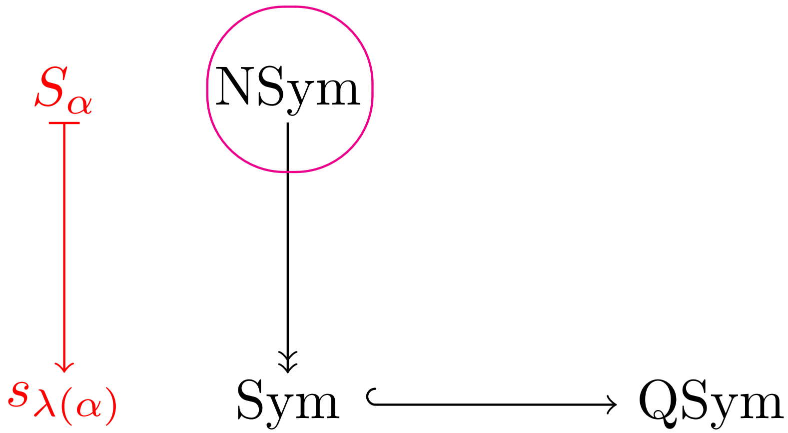
a quest for Schur function in NSym

$$\text{NSym} = K_0\left(\bigoplus H_n(0)\right)$$

[Grothendick group of representation of Hecke algebra at q=0]

Irreducible $\leftrightarrow$ noncommutative Ribon

Sym, NSym, QSym



Sym: Symmetric functions

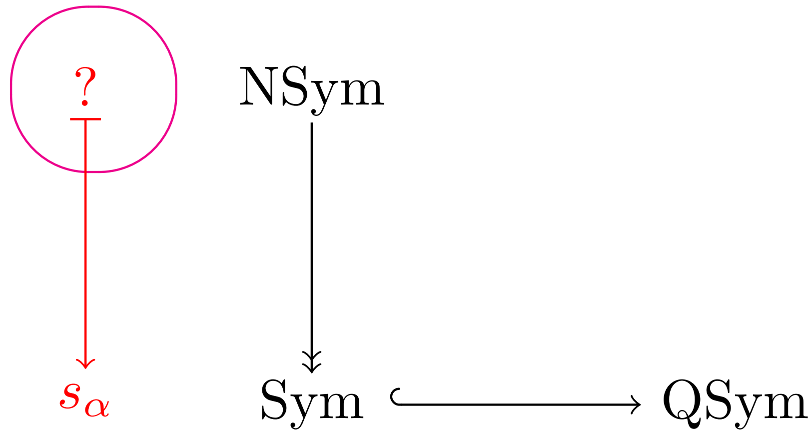
NSym: noncommutative symmetric functions $= \mathbb{Z}\langle H_1, H_2, \ldots \rangle$

a quest for Schur function in NSym

Schur noncommutative symmetric functions

[Haglund-Luoto-Mason-van Willigenburg]

Sym, NSym, QSym



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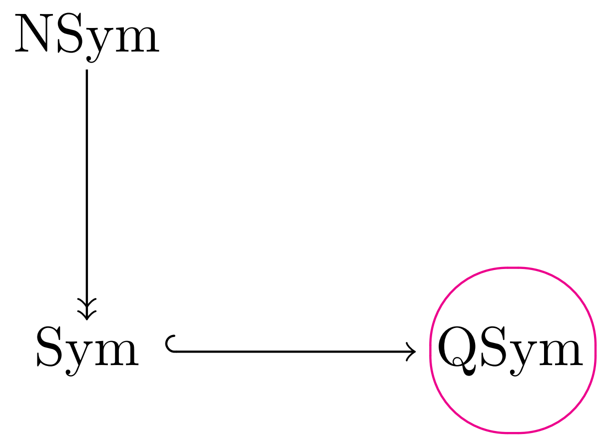
Immaculate noncommutative symmetric functions

[B-B-S-S-Z]

Sym, NSym, QSym



Sym: Symmetric functions

NSym: noncommutative symmetric functions $= \mathbb{Z}\langle H_1, H_2, \ldots \rangle$

QSym: Quasisymmetric functions $= NSym^*$

polytopes and posets

Immaculate noncommutative symmetric functions

Schur functions revisited: For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$

$$s_\lambda := \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+\ell-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_\ell-\ell+1} & h_{\lambda_\ell-\ell+2} & \cdots & h_{\lambda_\ell} \end{bmatrix}$$

where we use the convention that $h_0 = 1$ and $h_{-m} = 0$ for $m > 0$.

Immaculate noncommutative symmetric functions

Schur functions revisited: For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ where $\alpha_i > 0$

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where we use the convention that $h_0 = 1$ and $h_{-m} = 0$ for $m > 0$.

$s_{13} = -s_{22}$; $s_{12} = 0$; $s_\alpha = \pm s_\mu$ for a unique μ, or 0.

$$s_{13} = \det \begin{bmatrix} h_1 & h_2 \\ h_2 & h_3 \end{bmatrix} = -\det \begin{bmatrix} h_2 & h_3 \\ h_1 & h_2 \end{bmatrix} = s_{22}$$

Immaculate noncommutative symmetric functions

Immaculate noncommutative symmetric functions: For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ where $\alpha_i > 0$

$$\boxed{\mathfrak{S}_\alpha := \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1+\sigma_1-1} H_{\alpha_2+\sigma_2-2} \cdots H_{\alpha_m+\sigma_m-m}}$$

where we use the convention that $H_0 = 1$ and $H_{-m} = 0$ for $m > 0$.

$$\mathfrak{S}_{13} = \widetilde{\det} \begin{bmatrix} H_1 & H_2 \\ H_2 & H_3 \end{bmatrix} = H_1 H_3 - H_2 H_2$$

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$$\mathfrak{S}_{22} = \widetilde{\det} \begin{bmatrix} H_2 & H_3 \\ H_1 & H_2 \end{bmatrix} = H_2 H_2 - H_3 H_1 \neq \pm \mathfrak{S}_{13}$$

Immaculate noncommutative symmetric functions

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$$\mathfrak{S}_{12} = \widetilde{\det} \begin{bmatrix} H_1 & H_2 \\ H_1 & H_2 \end{bmatrix} = H_1 H_2 - H_2 H_1 \neq 0$$

Immaculate noncommutative symmetric functions

Immaculate noncommutative symmetric functions: For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ where $\alpha_i > 0$

$$\mathfrak{S}_\alpha = \widetilde{\det} \begin{bmatrix} H_{\alpha_1} & H_{\alpha_1+1} & \cdots & H_{\alpha_1+\ell-1} \\ H_{\alpha_2-1} & H_{\alpha_2} & \cdots & H_{\alpha_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{\alpha_\ell-\ell+1} & H_{\alpha_\ell-\ell+2} & \cdots & H_{\alpha_\ell} \end{bmatrix}$$

We have $\mathfrak{S}_\alpha = H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_\ell}$ + higher lex term

Hence

$\{\mathfrak{S}_\alpha\}$ is a basis of NSym

Why Immaculate? Is it interesting?

Creation operator $\mathbf{B}_m : \mathrm{Sym}_n \to \mathrm{Sym}_{m+n}$

$$\mathbf{B}_m := \sum_{i \geq 0} (-1)^i h_{m+i} e_i^{\perp},$$

where

$$e_i^{\perp} f = \sum \langle e_i^{\perp} f, s_\lambda \rangle s_\lambda = \sum \langle f, e_i s_\lambda \rangle s_\lambda$$

$$\boxed{\mathbf{B}_m s_\alpha = s_{(m,\alpha)}}$$

In particular

$$s_\lambda = \mathbf{B}_{\lambda_1} \mathbf{B}_{\lambda_2} \cdots \mathbf{B}_{\lambda_\ell} 1$$

Why Immaculate? Is it interesting?

Creation operator $\mathbb{B}_m : \mathrm{NSym}_n \to \mathrm{NSym}_{m+n}$

$$\mathbb{B}_m := \sum_{i\geq 0} (-1)^i H_{m+i} F_{1^i}^{\perp},$$

where

$$F_{1^i}^{\perp}\varphi = \sum \langle F_{1^i}^{\perp}\varphi, F_\alpha \rangle R_\alpha = \sum \langle \varphi, F_{1^i} F_\alpha \rangle R_\alpha$$

Here $F_\alpha \in \mathrm{QSym} = \mathrm{NSym}^*$ where $\langle R_\beta, F_\alpha \rangle = \delta_{\beta=\alpha}$

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THEOREM

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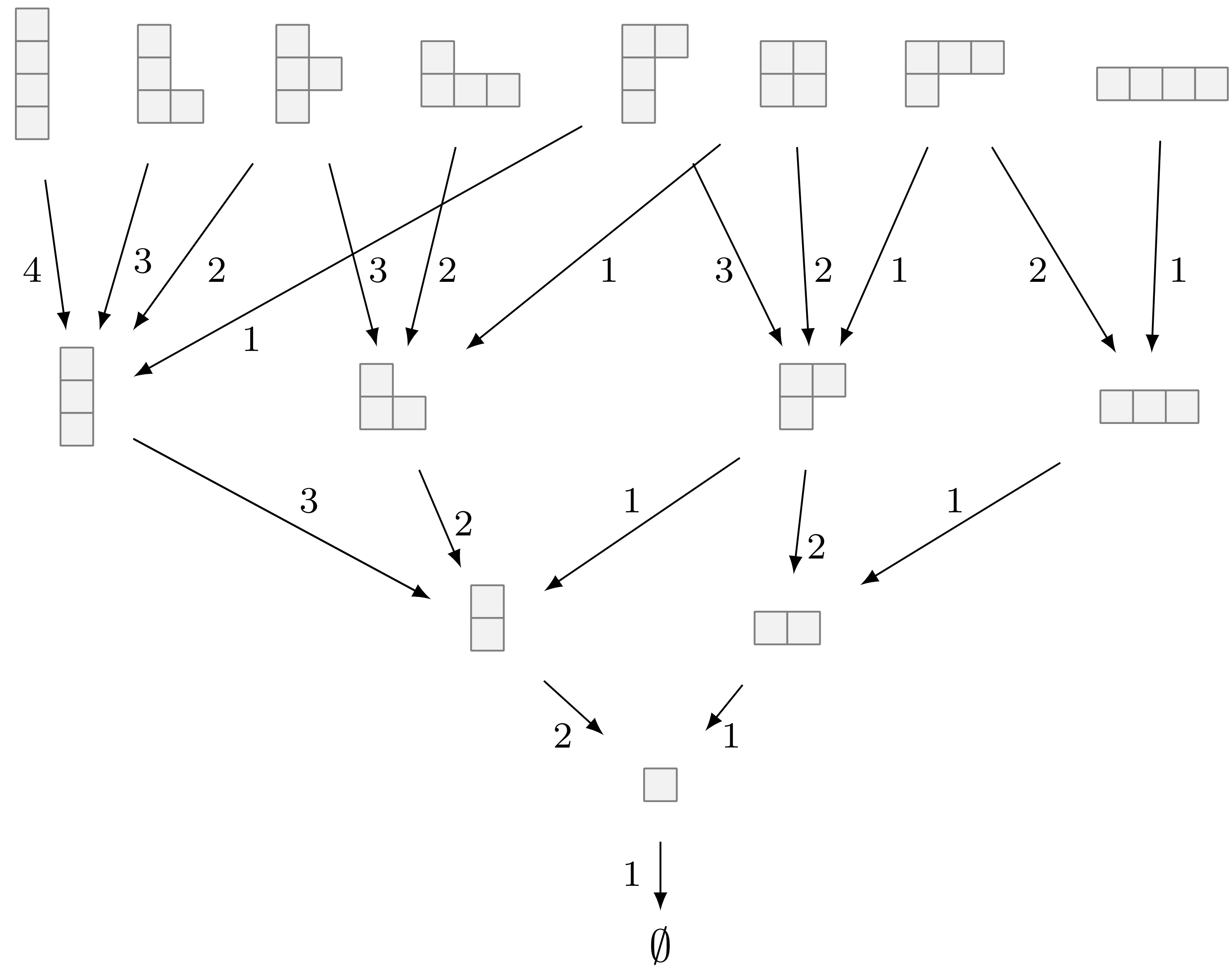
immaculately concieved?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Why Immaculate? Is it interesting?

$$\mathfrak{S}_\alpha = \mathbb{B}_{\alpha_1}\mathbb{B}_{\alpha_2}\cdots\mathbb{B}_{\alpha_\ell}1$$

THEOREM Pieri Rule

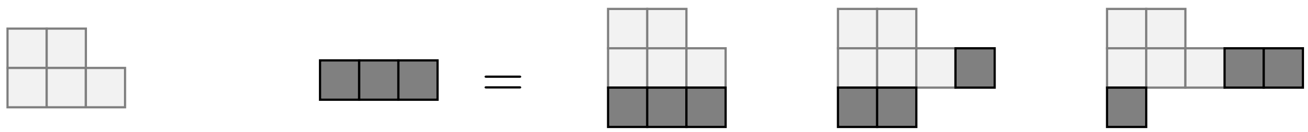
$$\mathfrak{S}_\alpha H_s = \sum_{\alpha \subset_s \beta} \mathfrak{S}_\beta$$

where $\alpha \subset_s \beta$ if:

1. $|\beta| = |\alpha| + s$,
2. $\alpha_j \leq \beta_j$ for all $1 \leq j \leq \ell(\alpha)$,
3. $\ell(\beta) \leq \ell(\alpha) + 1$.

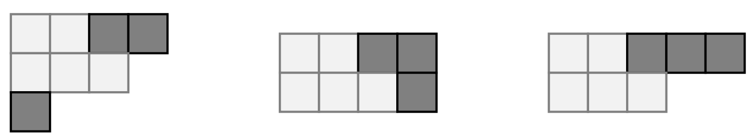
Why Immaculate? Is it interesting?

$$\mathfrak{S}_\alpha H_s = \sum_{\alpha \subset_s \beta} \mathfrak{S}_\beta$$



$$\mathfrak{S}_{23} \quad * \quad H_3 = \quad \mathfrak{S}_{233} \quad + \mathfrak{S}_{242} \quad + \mathfrak{S}_{251}$$

$$+ \mathfrak{S}_{26} \quad + \mathfrak{S}_{332} \quad + \mathfrak{S}_{341} \quad + \mathfrak{S}_{35}$$



$$+ \mathfrak{S}_{431} \quad + \mathfrak{S}_{44} \quad + \mathfrak{S}_{53}$$

Why Immaculate? Is it interesting?

THEOREM Pieri Rule

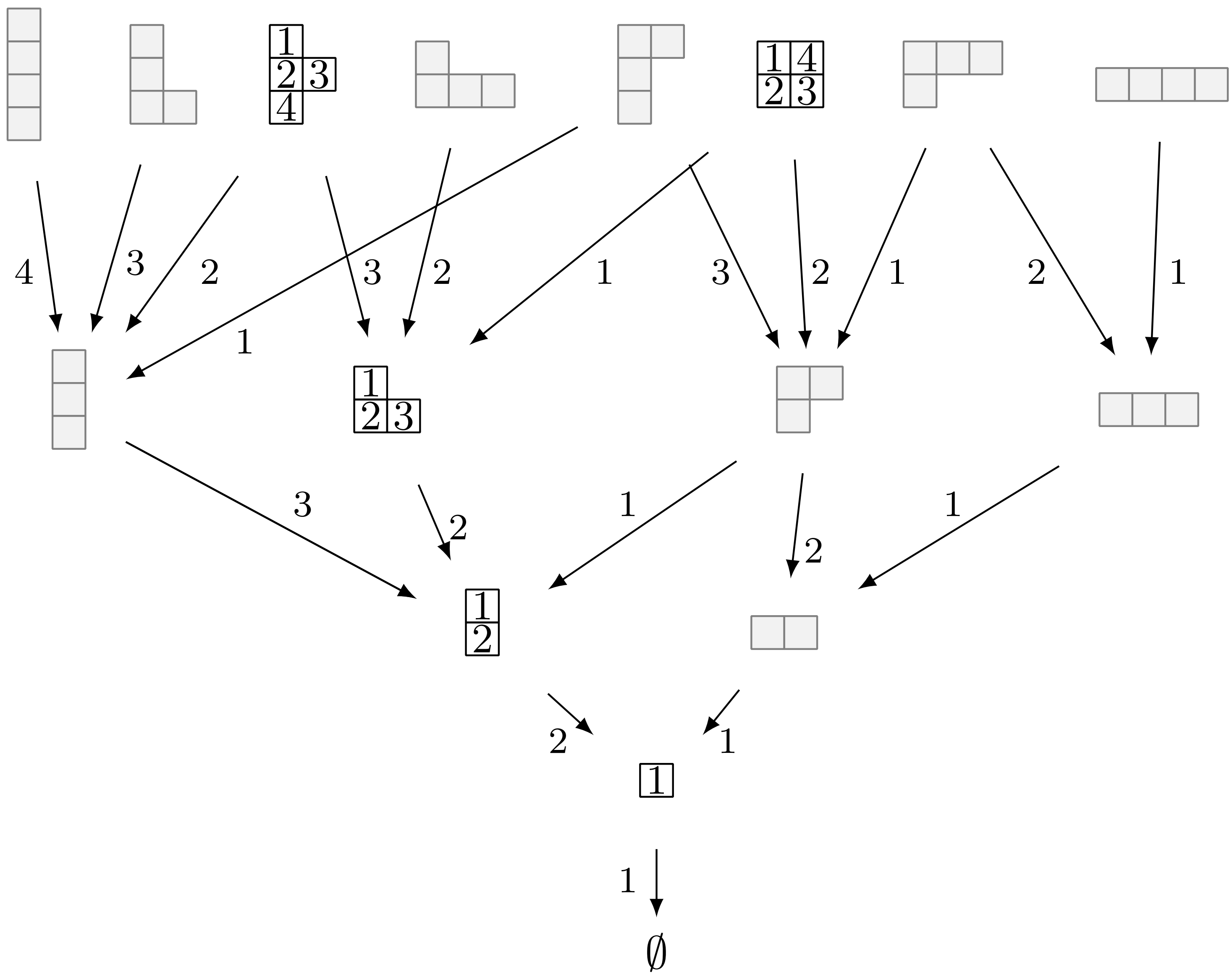
$$\mathfrak{S}_\alpha H_s = \sum_{\alpha \subset_s \beta} \mathfrak{S}_\beta$$

THEOREM Noncommutative Kotska

$$H_\beta = \sum_{\alpha \geq_\ell \beta} K_{\alpha,\beta} \mathfrak{S}_\alpha$$

Conjecture/THEOREM Noncommutative Littlewood-Richardson ($c^\beta_{\alpha,\lambda} \geq 0$)

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_\beta c^\beta_{\alpha,\lambda} \mathfrak{S}_\beta$$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Hall-Littlewood

Define $\widetilde{\mathbf{B}}_m : Sym[q]_n \to Sym[q]_{n+m}$ by:

$$\widetilde{\mathbf{B}}_m := \sum_{i \geq 0} q^i \mathbf{B}_{m+i} h_i^{\perp}.$$

then let

$$Q'_\alpha := \widetilde{\mathbf{B}}_{\alpha_1} \cdots \widetilde{\mathbf{B}}_{\alpha_m}(1) \ .$$

The Q'_λ are the Hall-Littlewood symmetric functions dual to "$P_\lambda(q)$" Hall-littlewood.

Noncommutative Hall-Littlewood

Define $\widetilde{\mathbb{B}}_m : NSym[q]_n \to NSym[q]_{n+m}$ by:

$$\widetilde{\mathbb{B}}_m := \sum_{i \geq 0} q^i \mathbb{B}_{m+i} H_i^{\perp}.$$

then let

$$\mathcal{Q}_\alpha := \widetilde{\mathbb{B}}_{\alpha_1} \cdots \widetilde{\mathbb{B}}_{\alpha_m}(1)\ .$$

These are lifting to NSym of Hall-Littlewood symmetric

THEOREM

Under $NSym \twoheadrightarrow Sym$, we have $\mathcal{Q}_\alpha \mapsto Q'_\alpha$

This is the first lift of Hall-Littlewood to NSym. All many previous lift where Hall-Littlewood *like*!

Are Noncommutative Hall-Littlewood interesting?

$$\mathcal{Q}'_{1111} = \mathfrak{S}_{1111} + q\mathfrak{S}_{112} + (q+q^2)\mathfrak{S}_{121} + q^3\mathfrak{S}_{13} + (q+q^2+q^3)\mathfrak{S}_{211} + (q^2+q^3+q^4)\mathfrak{S}_{22} + (q^3+q^4+q^5)\mathfrak{S}_{31} + q^6\mathfrak{S}_4\ .$$

Under $NSym \twoheadrightarrow Sym$,

$$Q'_{1111} = s_{1111} + (q+q^2+q^3)s_{211} + (q^2+q^4)s_{22} + (q^3+q^4+q^5)s_{31} + q^6 s_4\ .$$

For any partition λ, $\mathfrak{S}_\lambda \mapsto s_\lambda$.

$\mathfrak{S}_{112} \mapsto 0$, $\quad \mathfrak{S}_{121} \mapsto 0 \quad$ and $\quad \mathfrak{S}_{13} \mapsto -s_{22}$.

Are Noncommutative Hall-Littlewood interesting?

THEOREM

$$\mathcal{Q}'_\alpha H_s = \sum_{\alpha \subset_s \beta} (1-q)^{n(\alpha,\beta)} \mathcal{Q}'_\beta$$

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$$\mathcal{Q}'_\alpha H_s = \sum_{\alpha \subset_s \beta} (1-q)^{n(\alpha,\beta)} \mathcal{Q}'_\beta$$

$$\mathfrak{S}_{1^n} = \sum_{\alpha \models n} (-q)^{n-\ell(\alpha)} \mathcal{Q}'_\alpha .$$

Are Noncommutative Hall-Littlewood interesting?

THEOREM

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CONJECTURE

$$\mathcal{Q}'_\lambda = \sum_T q^{st(T)} \mathfrak{S}_{shape(T)}$$

for some statistic st, over all immaculate tableau of content λ.

Thank You

$$\begin{array}{ccc} \mathrm{NSym} & & \\ \downarrow\!\!\downarrow & & \\ \mathrm{Sym} & \hookrightarrow & \mathrm{QSym} \end{array}$$