Hypergraphical polytope and antipode for hypergraphs

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[with C. Benedetti and J. Machacek]

Outline

- Warm up with Graphs.
- Theorems of Humpert-Martin and Aguiar-Ardila.
- The antipode of any Linearized co-com Hopf Monoid reduced to Hypergraphs.
- Orientation of hypergraphs.
- An Antipode formula for hypergraphs.
- Hypergraphical Polytope.

Hopf Algebra of Graphs

$H = \bigoplus_{n \geq 0} H_n$ graded Hopf algebra

$H_n = \mathbb{Q}\big[G : G$ iso class of graphs on V and $|V| = n\big]$

$G_1 \cdot G_2 = G_1 \cup G_2$

$$\Delta(G) = \sum_{S \subseteq [n]} G|_S \otimes G|_{S^c}$$

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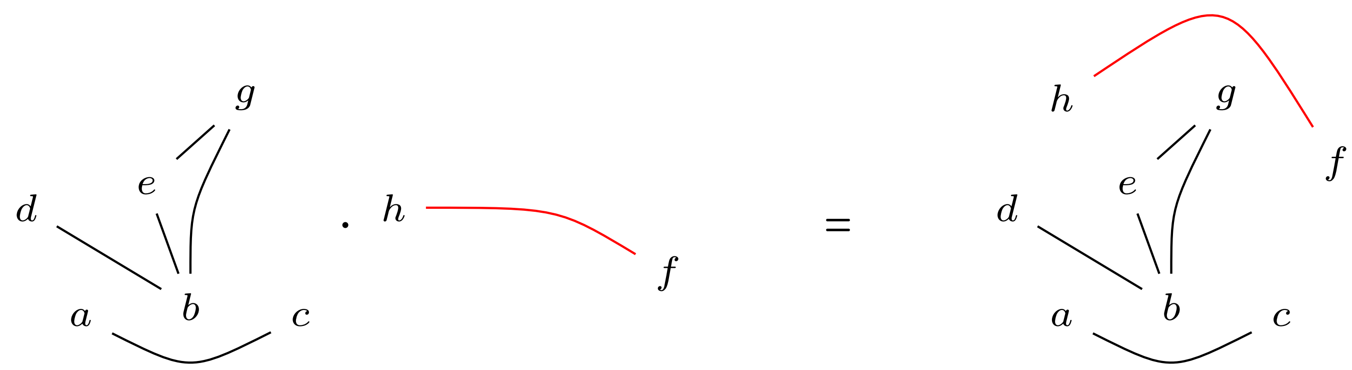
EXAMPLE



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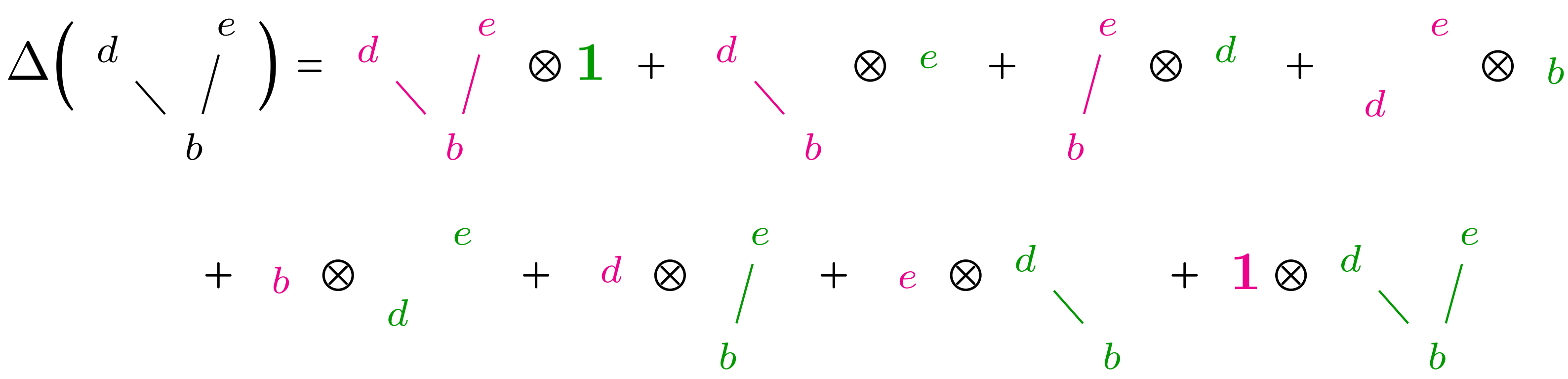

Antipode for graded (connected) Hopf algebras

Graded vector space: $H = \bigoplus_{n \geq 0} H_n$

multiplication: $m\colon H \otimes H \to H$

graded: $m = \sum m_{a,b}$ where

$$m_{a,b}\colon H_a \otimes H_b \to H_{a+b}$$

comultiplication: $\Delta\colon H \to H \otimes H$

graded: $\Delta = \sum \Delta_{a,b}$ where

$$\Delta_{a,b}\colon H_{a+b} \to H_a \otimes H_b$$

Antipode for graded (connected) Hopf algebras

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multiplication: $m_\alpha : H_{\alpha_1} \otimes \cdots \otimes H_{\alpha_\ell} \to H_n$

comultiplication: $\Delta_\alpha : H_n \to H_{\alpha_1} \otimes \cdots \otimes H_{\alpha_\ell}$

ANTIPODE: $S : H \to H$ [Takeuchi] For $x \in H_n$

$$S(x) = \sum_{\alpha \vDash n} (-1)^{\ell(\alpha)} m_\alpha \Delta_\alpha(x)$$

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MANY CANCELATIONS

Theorem [Humpert-Martin]

$$S(G) = \sum_{F\ flats} (-1)^{\ell(F)} a(G/_F) G|_F$$

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$$S(G) = \sum_{F \ flats} (-1)^{\ell(F)} a(G/_F) G|_F$$

FLAT:

For a set partition $A = \{A_1, A_2, \dots A_\ell\} \vdash V$:

$$G|_A = G|_{A_1} G|_{A_2} \cdots G_{A_\ell}$$

F is a flat if for some H

$$F = \text{FINNEST}\{A \vdash V : G|_A = H\}$$

Theorem [Humpert-Martin]

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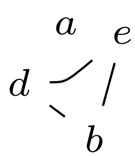
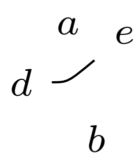
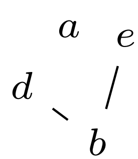
FLAT:

For $G =$ 

and $H =$ 

$\{\{e,d\},\{a\},\{b\}\}$ is a flat but $\{\{e,d\},\{a,b\}\}$ is NOT

REMARK $H =$ 

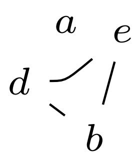
defines NOT FLAT since $\forall A \vdash V$

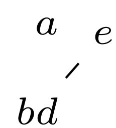
$$G|_A \neq H$$

Theorem [Humpert-Martin]

$$S(G) = \sum_{F \ flats} (-1)^{\ell(F)} a(G/_F) G|_F$$

$a(G/_F)$: Number of acyclic orientation of $G/_F$

For $G =$ 

$a\left(G/_{\{\{b,d\},\{a\},\{e\}\}}\right) = a\Big($ 

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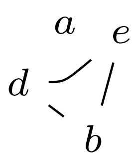
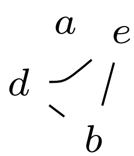
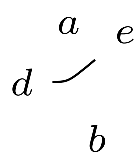
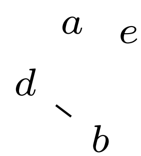
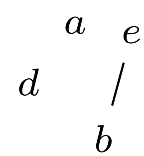
EXAMPLE

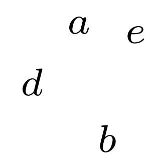
$S\Big($ 

$\Big) = 1$ 

$+\ 2$ 

$+\ 2$ 

$+\ 2$ 

$+\ 6$ 

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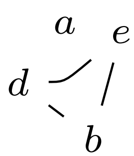
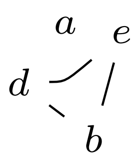
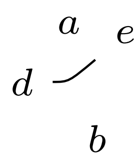
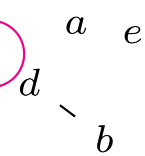
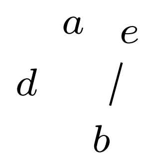
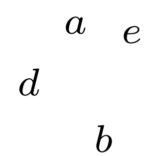
$S\Big($ 

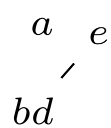
$\Big) = 1$ 

$+ \ 2$ 

$+ \ 2$ 

$+ \ 2$ 

$+ \ 6$ 

$a\Big($ 

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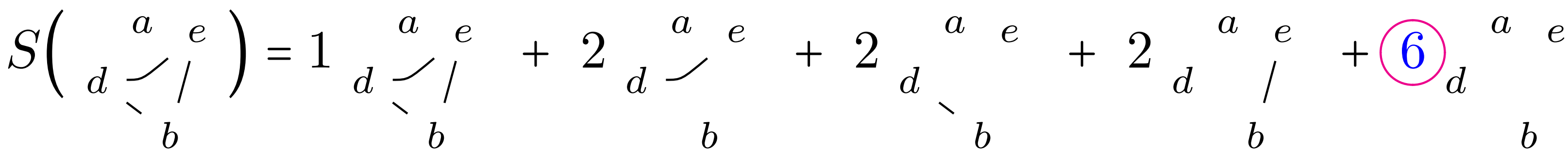

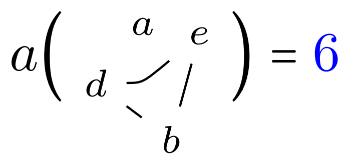

Theorem [Aguiar-Ardila]

Related to $S(G) = \sum_{F\ flats} (-1)^{\ell(F)} a(G/_F) G|_F$

THEOREM The ℓ-faces of the graphical zonotope Z_G are in bijection with the acyclic orientations of the ℓ-flats of G

ℓ-flats: Flats F of G such that $\ell(F) = \ell$.

Z_G: graphical zonotope

$$Z_G = \sum_{\{i,j\}\ edge\ of\ G} \Delta_{ij}$$

where Δ_{ij} is the segment from e_i to e_j in $\mathbb{R}^n$.

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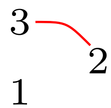
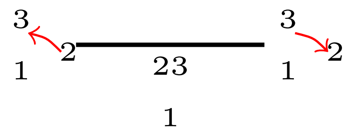
EXAMPLE: $G =$ 

$\Delta_{23} =$ 

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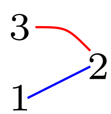
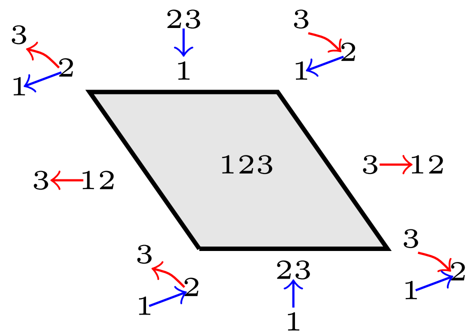
EXAMPLE: $G =$ 

$\Delta_{23} + \Delta_{12} =$ 

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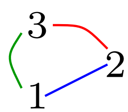
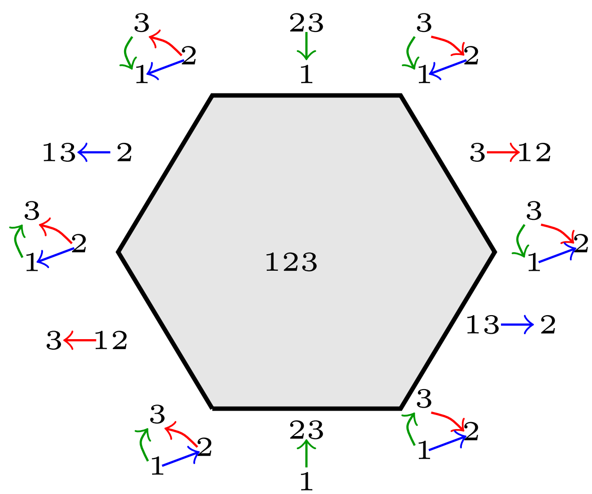
EXAMPLE: $G =$ 

$\mathbf{\Delta}_{23} + \mathbf{\Delta}_{12} + \mathbf{\Delta}_{13} =$ 

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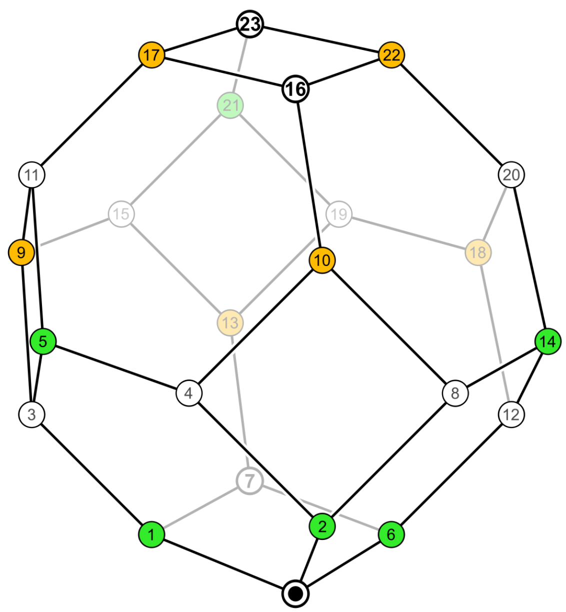
EXAMPLE: $G = K_4$

$Z_G =$ 

THEOREM [Benedetti-B]

For any linearilized (co and coco)mutative Hopf monoid H with basis $\mathbf{h}$.

Let x and y in $\mathbf{h}[I]$

$$\boxed{\text{Coeff of } y \text{ in } S(x) = \text{Coeff of } \epsilon \text{ in } S(G_x^y)}$$

where G_x^y is a hypergraph.

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MORAL: we better understand the antipode for hypergraph.

Hopf Algebra of HyperGraphs

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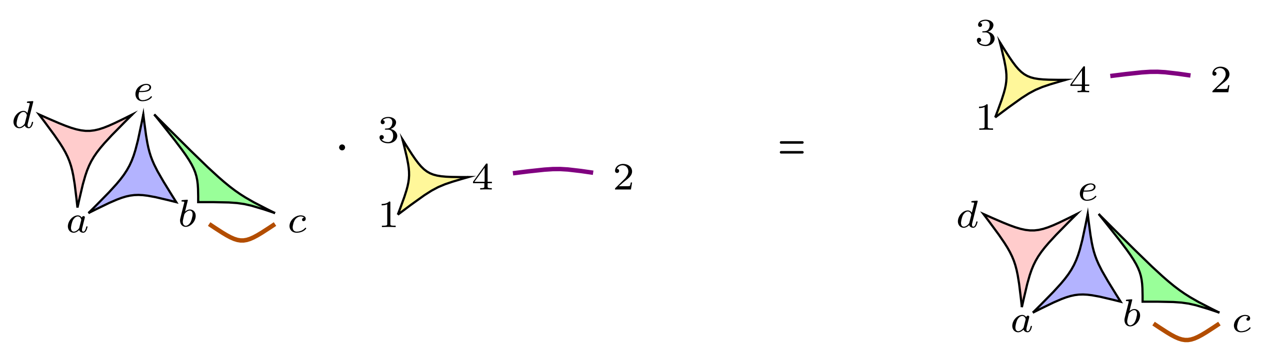

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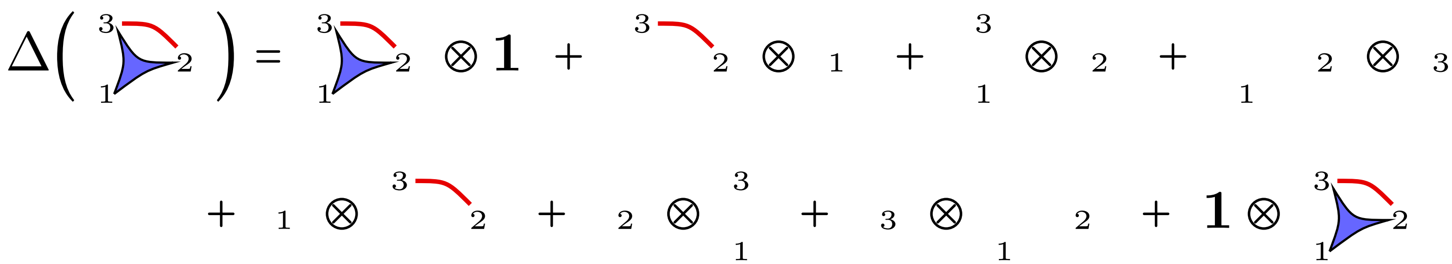

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FLAT: Same as before

$a(G/_F)$: Sum of signed **acyclic orientation** of $G/_F$???

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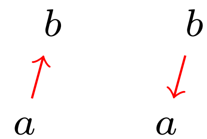
$H =$ 

acyclic orientations: 

$$a(H) = 1 + 1 = 2$$

Theorem [Benedetti-B]

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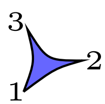
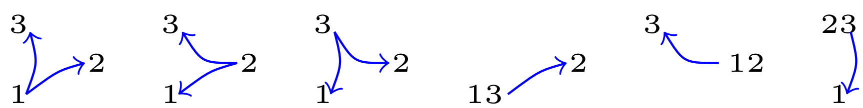
Orientation of a hyperedge is a flow from a unique source

$H =$ 

acyclic orientations:



$$a(H) = -1 + -1 + -1 + 1 + 1 + 1 = 0$$

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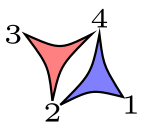
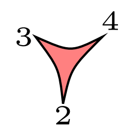
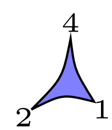
$H =$ 

has 36 orientations:

$(2^3 - 2) = 6$ orientations of 

$(2^3 - 2) = 6$ orientations of 

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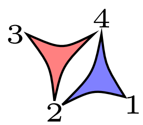
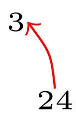
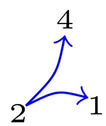
$H =$ 

some orientations are not acyclic:

For 

and 

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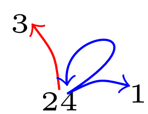

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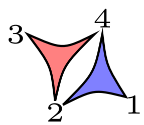
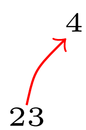
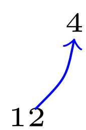
$H =$ 

some orientations are acyclic:

For 

and 

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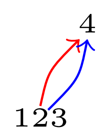

the sign is $(-1)^2 = 1$

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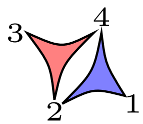
$a(G/_F)$: Sum of signed **acyclic orientation** of $G/_F$???

$H =$  has **20** acyclic orientations:

(4,3,2,1); (3,4,2,1); (34,2,1); (3,2,4,1);

(2,4,3,1); (23,4,1); (1,4,3,2); (3,1,4,2);

(1,2,4,3); (1,23,4); (1,24,3); (1,34,2);

(3,12,4); (12,4,3); (123,4); (14,3,2);

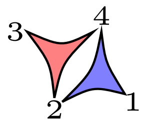
(3,14,2); (134,2); (3,24,1); (24,3,1).

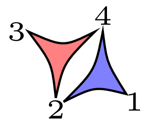
$$a(H) = 9 - 11 = -2$$

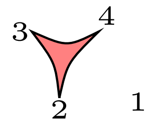
Theorem [Benedetti-B]

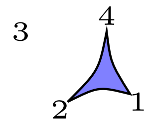
$$S(G) = \sum_{F \ flats} a(G/_F)G|_F$$

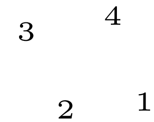
EXAMPLE

$S($ 

$) = -$ 

$+ 2$ 

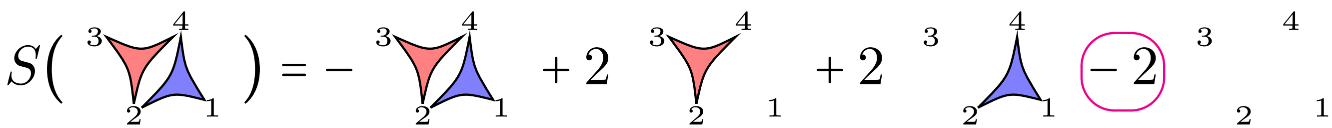
$+ 2$ 

$- 2$ 

Theorem [Benedetti-B]

$$S(G) = \sum_{F \ flats} a(G/_F)G|_F$$

EXAMPLE



$$a(H) = 9 - 11 = -2$$

Theorem [Benedetti-B-Machacek]

Related to $S(G) = \sum_{F \ flats} a(G/_F) G|_F$

Hypergraphical polytope P_G :

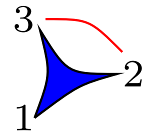
$$P_G = \sum_{U \ hyperedge \ of \ G} \Delta_U$$

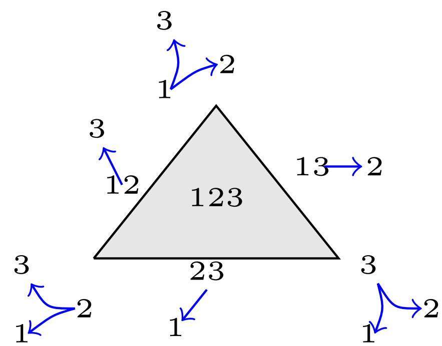
where Δ_U is the simplex spanned by e_i in $\mathbb{R}^n$ for $i \in U$.

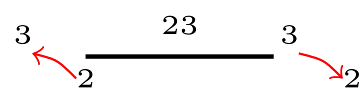
THEOREM:

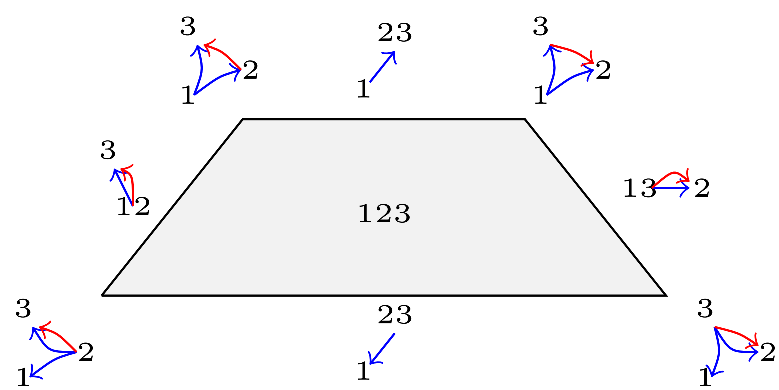
$\pm a(G/_F)$ is the homology of a (sub) complex in P_G.

Theorem [Benedetti-B-Machacek]

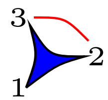
Hypergraphical polytope P_G : G =  .

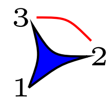
$\mathbf{\Delta}_{123}$ = 

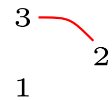
$\mathbf{\Delta}_{23}$ = 

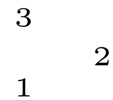
$P_G = \mathbf{\Delta}_{123} + \mathbf{\Delta}_{23}$ = 

Theorem [Benedetti-B-Machacek]

$S($ 

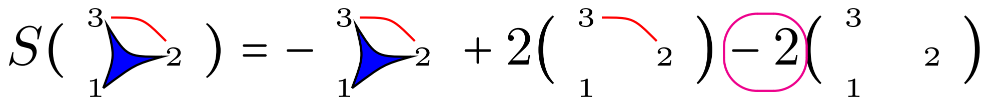
$) = -$ 

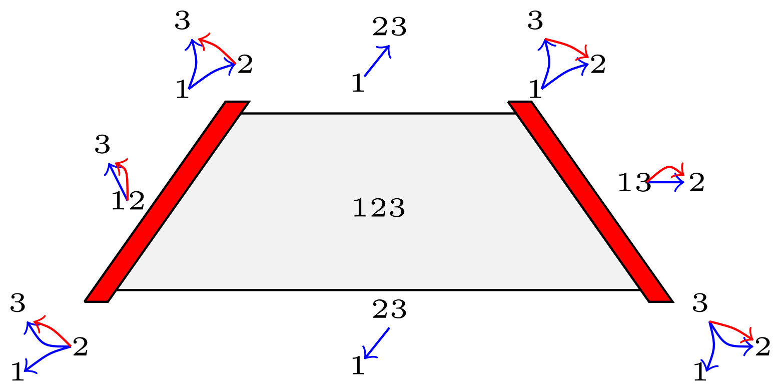
$+ 2($ 

$) - 2($ 

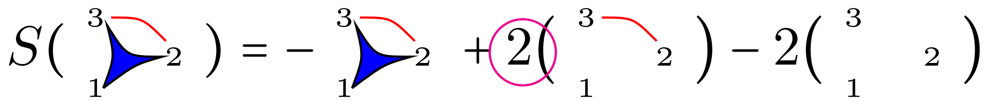
$)$

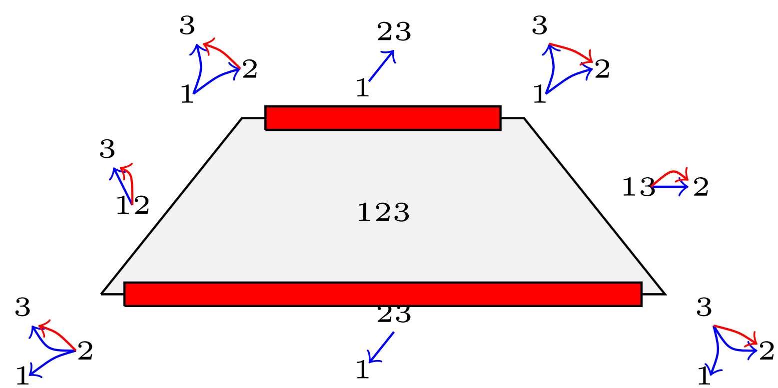
Theorem [Benedetti-B-Machacek]



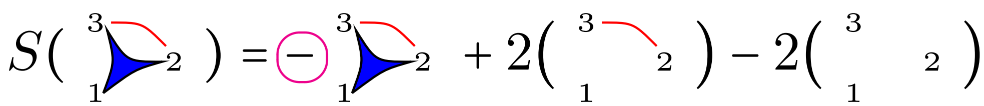


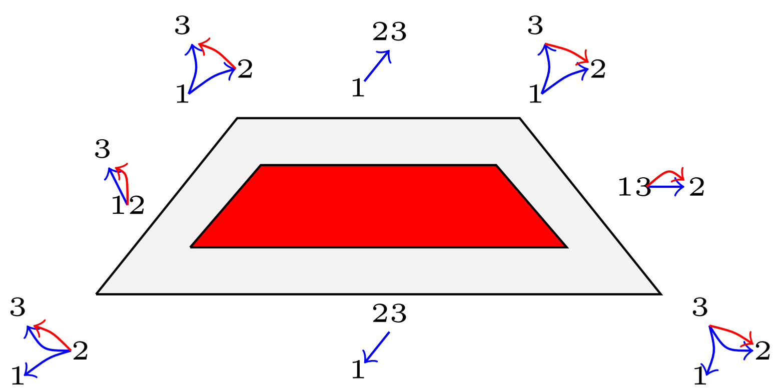
Theorem [Benedetti-B-Machacek]



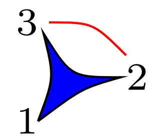


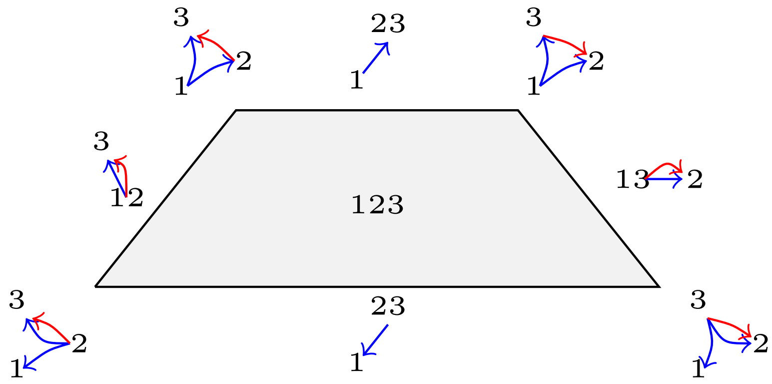
Theorem [Benedetti-B-Machacek]





M E R C I





T H A N K S

G R A C I A S