QuasiSymmetric functions
Part II

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Outline

(part I) QSym

– definition

– basis

– P-partitions

– structure (Hopf + internal product)

– Relation to others Hopf algebras

– ABS Theory

Outline

(part II) Positivity

– F-positivity

– Schur positivity

– A Question of Billera

Antipode

– formula

– Multiplicity free formula?

– A proof of Stanley's acyclic theorem

Quasisymmetric functions (Review)

$QSym$ graded subspace of bounded degree series

Bases indexed by compositions $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell) \vDash n \geq 0$.

Monomial basis $M_\alpha = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$

Fundamental basis $F_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n \\ k \in D(\alpha) \implies i_k < i_{k+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}$

Relation with P-partitions

$$\overline{F}_{P,\gamma} = \sum_{f \in \mathcal{F}_{P,\gamma}} \prod_{i \in P} x_{f(i)} = \sum_{(L,\gamma) \in \mathcal{L}(P,\gamma)} \overline{F}_{L,\gamma}$$

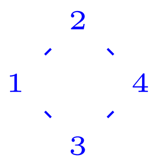
$$F_\alpha = \overline{F}_{|,\gamma} \qquad \textit{and} \qquad S_\lambda = \overline{F}_{\lambda,\gamma}$$

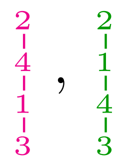
$QSym$ Hopf Algebra

Quasisymmetric functions (POSITIVITY)

$$S_\lambda = \sum_{(L,\gamma)\in\mathcal{L}(P,\gamma)} \overline{F}_{L,\gamma}$$

EXAMPLE

$(P,\gamma) =$ 

$\mathcal{L}(P,\gamma) = \{$  $\}$

$$S_{(2,2)} = F_{(1,2,1)} + F_{(2,2)}$$

$Sym \hookrightarrow QSym$ Useful to show positivity...

Quasisymmetric functions (POSITIVITY)

$$S_\lambda = \sum_{(L,\gamma)\in\mathcal{L}(P,\gamma)} \overline{F}_{L,\gamma}$$

Sym $\hookrightarrow$ *QSym*

THEOREM | Schur-positivity $\Longrightarrow$ F-positivity. |

Given a Schur-positivity problem [For example $\tilde{H}_\lambda(X;q,t)$]

-1- Show it is F-positive [much easier]

-2- "try" to resolve/group/inverse term to get Schur coefficient [still hard, see Egge, Loehr and Warrington and others]

Many Other Approach Schur quasisymetric, Dual Imaculate quasisymmetric

Quasisymmetric functions (A Question)

$$S_\lambda = \sum_{(L,\gamma)\in\mathcal{L}(P,\gamma)} \overline{F}_{L,\gamma}$$

$Sym \hookrightarrow QSym$

THEOREM $\boxed{\text{Schur-positivity} \implies F\text{-positivity.}}$

QUESTION [BBvW] (discussion at Fields Institute ~2000)

What are the maximal ray of the F-positive cone in $Sym \cap QSym$

Answer

Quasisymmetric functions (A Question)

$$S_\lambda = \sum_{(L,\gamma)\in\mathcal{L}(P,\gamma)} \overline{F}_{L,\gamma}$$

$Sym \hookrightarrow QSym$

THEOREM | Schur-positivity $\Longrightarrow$ F-positivity. |

QUESTION [BBvW] (discussion at Fields Institute ~2000)

What are the maximal ray of the F-positive cone in $Sym \cap QSym$

Answer | Schur functions (No!!! but what is the answer) |

Quasisymmetric functions (Graded Hopf Algebra)

Graded vector space: $QSym = \bigoplus_{n\geq 0} QSym_n$

Associative multiplication: $m: QSym \otimes QSym \to QSym$

Unity: $u: \mathbb{Q} \to QSym$

Associative comultiplication: $\Delta: QSym \to QSym \otimes QSym$

Counity: $\epsilon: QSym \to \mathbb{Q}$

Compatibility: Δ and ϵ are algebra morphism

ANTIPODE: $S: QSym \to QSym$

Graded Hopf Algebra

Graded vector space: $H = \bigoplus_{n \geq 0} H_n$

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Given $f, g \colon H \to H$ we define

$$f * g \colon H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{m} H$$

This operation is associative with unity $u \circ \epsilon$

What is (if it exists) the convolution inverse of $\mathrm{Id} \colon H \to H$

Graded Hopf Algebra

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ANTIPODE: $S\colon H \to H$

$S = \mathrm{Id}^{<-1>_*}$ is recursively constructed degree by degree

[Takeuchi] For $x \in H_n$

$$S(x) = \sum_{\alpha \vDash n} (-1)^{\ell(\alpha)} m_\alpha \Delta_\alpha(x)$$

where

$m_\alpha\colon H_{\alpha_1} \otimes \cdots \otimes H_{\alpha_\ell} \to H_n$ and $\Delta_\alpha\colon H_n \to H_{\alpha_1} \otimes \cdots \otimes H_{\alpha_\ell}$

Graded Hopf Algebra

some other examples

$$\begin{array}{ccc} \mathrm{NSym} & \hookrightarrow & \mathrm{SSym} \\ \downarrow & & \downarrow \\ \mathrm{Sym} & \hookrightarrow & \mathrm{QSym} \end{array}$$

Combinatorial Hopf Algebra

$H = \bigoplus_{n\geq 0} H_n$ graded Hopf algebra with character $\zeta\colon H \to \mathbb{Q}$

$H_n = \mathbb{Q}\big[G : G \text{ iso class of graphs on } [n]\big]$

$G_1 \cdot G_2 = G_1 \cup G_2$

$\Delta(G) = \sum_{S\subseteq[n]} G|_S \otimes G|_{S^c}$

$$\zeta(G) = \begin{cases} 1 & \text{if } G \text{ has no edges} \\ o & \text{otherwise} \end{cases}$$

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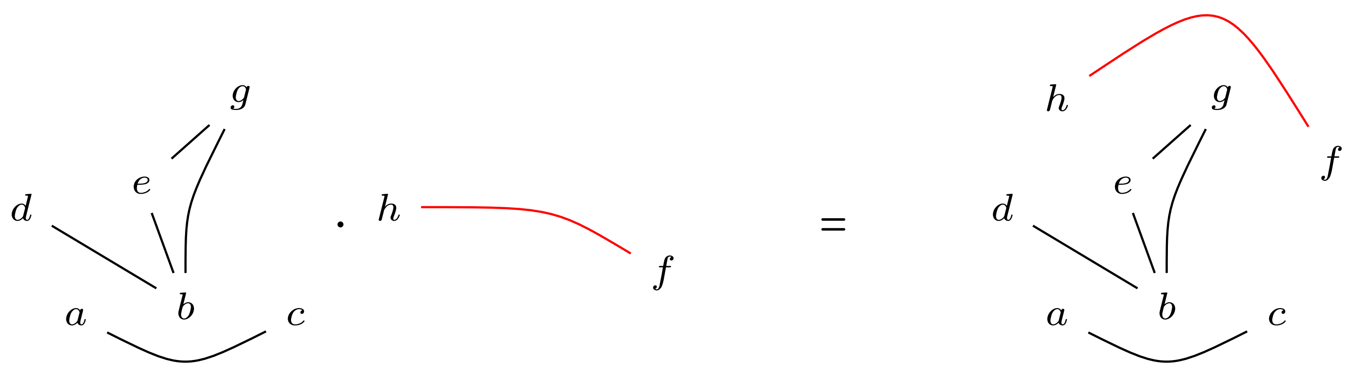

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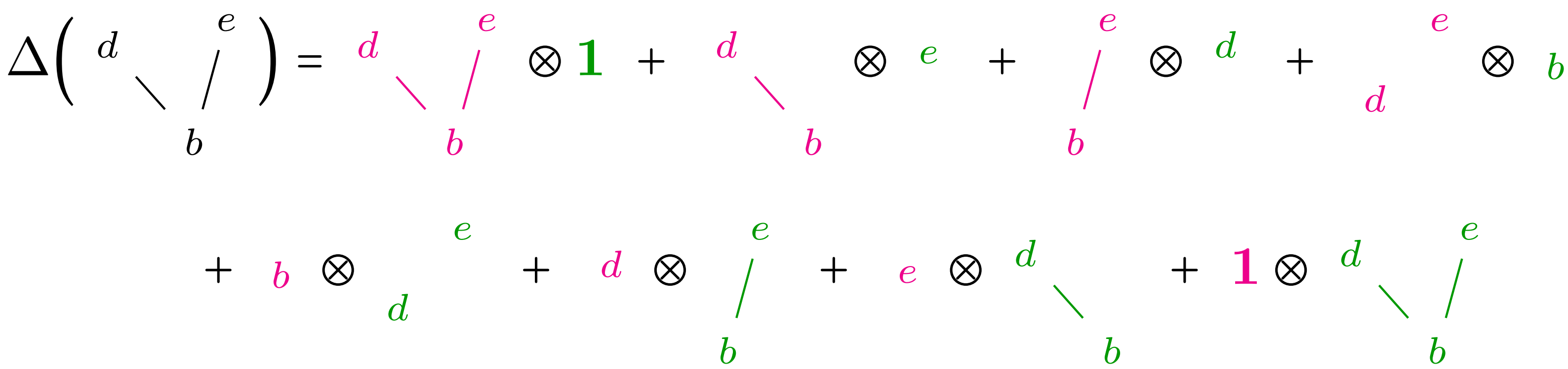

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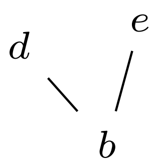
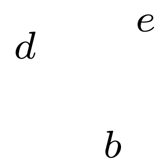
$\zeta\Big($ 

$\Big) = \mathbf{0}$ and $\zeta\Big($ 

$\Big) = \mathbf{1}$

Combinatorial Hopf Algebra

[Aguiar-Bergeron-Sottile]

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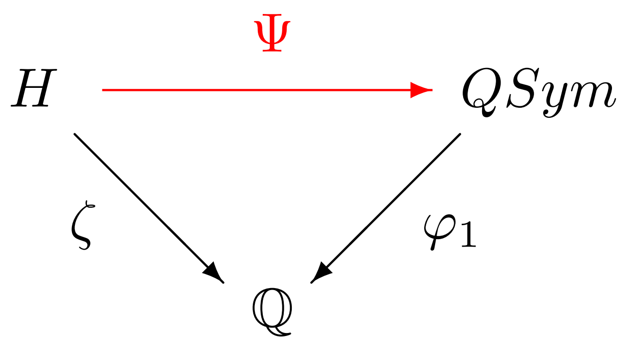

where $\varphi_1(f) = f(1, 0, 0, \cdots)$.

Combinatorial Hopf Algebra

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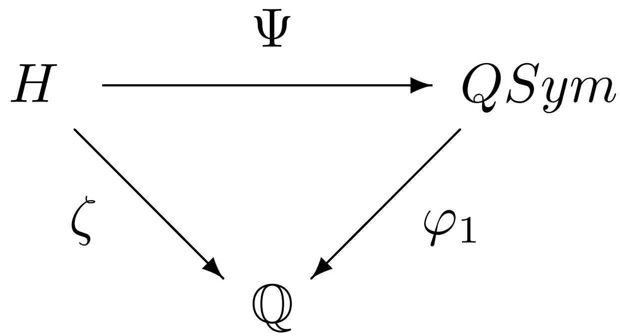
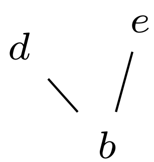


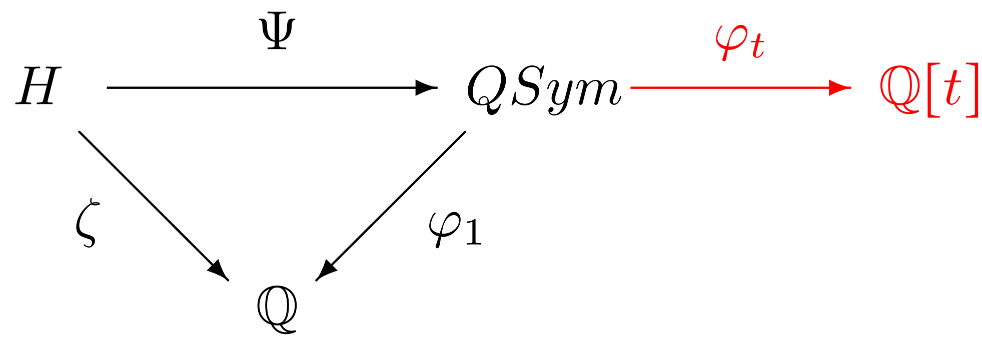
EXAMPLE

$\Psi\Big($  $\Big) = 0M_{(3)} + 1M_{(2,1)} + 1M_{(1,2)} + 6M_{(1,1,1)}$

$= 6m_{(1,1,1)} + m_{(2,1)}$

Combinatorial Hopf Algebra (ABS)

$H = \bigoplus_{n\geq 0} H_n$ graded Hopf algebra with character $\zeta\colon H \to \mathbb{Q}$



where $\varphi_1(f) = f(1,0,0,\cdots)$.

$\varphi_t(M_\alpha) = \binom{t}{\ell(\alpha)}$ is a Hopf morphism

$$\varphi_t\big(f(x_1,x_2,\ldots)\big)\Big|_{t=n} = f(\underbrace{1,\ldots,1}_{n},0,0,\ldots)\,.$$

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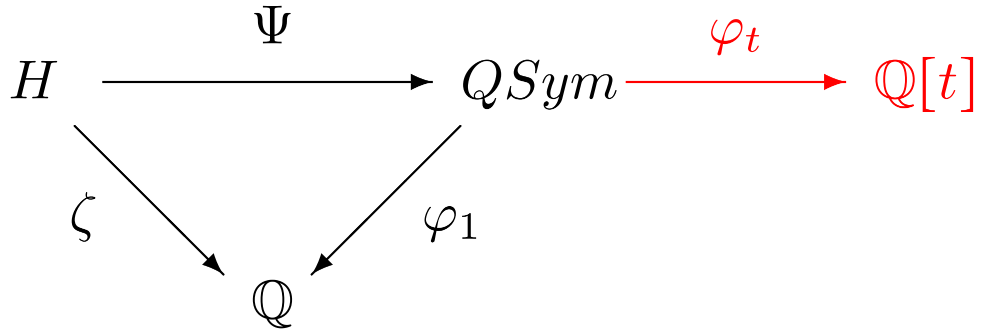

$\varphi_t(M_\alpha) = \binom{t}{\ell(\alpha)}$ is a Hopf morphism

EXAMPLE

$$\Psi\left(\begin{smallmatrix} d & & e \\ & b & \end{smallmatrix}\right) = 0M_{(3)} + 1M_{(2,1)} + 1M_{(1,2)} + 6M_{(1,1,1)}$$

$$\varphi_t \circ \Psi\left(\begin{smallmatrix} d & & e \\ & b & \end{smallmatrix}\right) = \binom{t}{2} + \binom{t}{2} + 6\binom{t}{3} = t(t-1)^2$$

Generalized Chromatic Polynomial

[Gringberg-Reiner]

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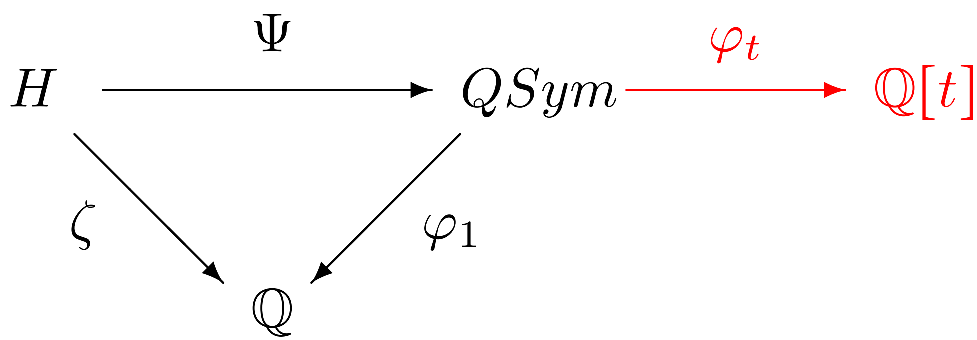

Generalized Chromatic Polynomial

$$\chi_x(t) = \varphi_t \circ \Psi(x)$$

$$\varphi_t \circ \Psi\Big|_{t=1} = \left(\varphi_t\big|_{t=1}\right) \circ \Psi = \varphi_1 \circ \Psi = \zeta.$$

Generalized Chromatic Polynomial

[Grinberg-Reiner]

$H = \bigoplus_{n \geq 0} H_n$ graded Hopf algebra with character $\zeta : H \to \mathbb{Q}$

$$H \xrightarrow{\quad \Psi \quad} QSym \xrightarrow{\varphi_t} \mathbb{Q}[t]$$

When $H_n = \mathbb{Q}\big[G : G \text{ iso class of graphs on } [n]\big]$

$$\chi_G(t) = \varphi_t \circ \Psi(G)$$

is the usual Chromatic polynomial.

Theorem [Stanley]

$$\boxed{\chi_G(-1) = \pm a(G)}$$

For a graph G; $a(G)$ is number of acyclic orientation.

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For a graph G; $a(G)$ is number of acyclic orientation.

Antipode: $S\colon \mathbb{Q}[t] \to \mathbb{Q}[t]$

$$f(t) \mapsto f(-t)$$

$$\chi_G(-1) = S \circ \varphi_t \circ \Psi(G)\Big|_{t=1} = \varphi_t \circ \Psi \circ S(G)\Big|_{t=1} = \zeta \circ S(G)$$

$$\boxed{\chi_G(-1) = \zeta\big(S(G)\big)}$$

Theorem [Humpert-Martin]

Jeremy: *What can I do with a Hopf algebra I have just rediscovered*

Nantel: *Compute its antipode*

THEOREM

$$S(G) = \sum_{F \ flats} (-1)^{c(F)} a(G/_F) G|_F$$

Corollary $\chi_G(-1) = \zeta\big(S(G)\big) = (-1)^{c(G)} a(G)$

Theorem [Humpert-Martin]

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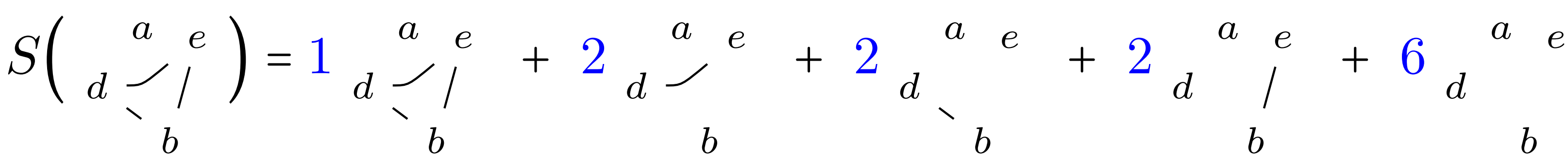

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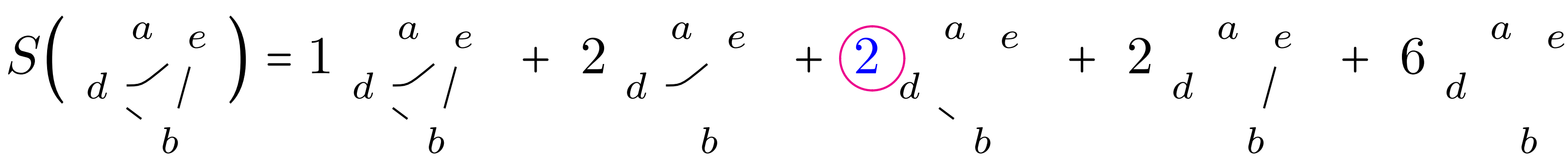

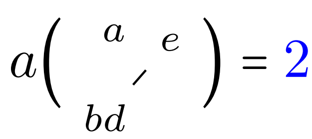

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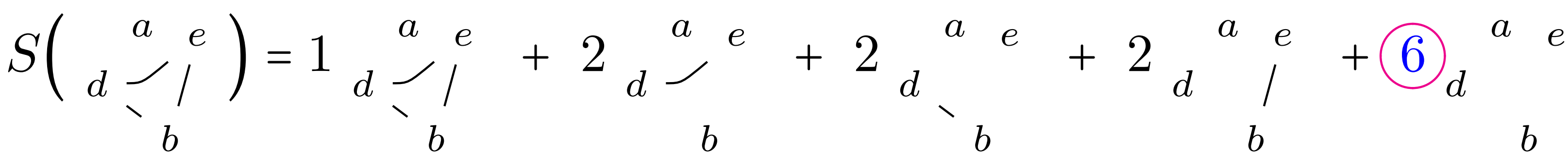

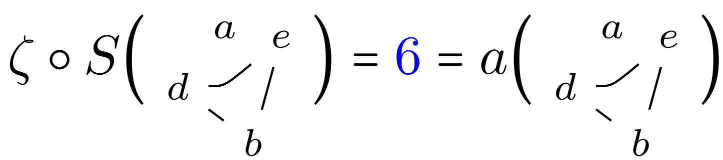

So now I want better formulas for antipode

For $H = \bigoplus_{n\geq 0} H_n$ and $x \in H_n$ [Takeuchi]

$$S(x) = \sum_{\alpha \vDash n} (-1)^{\ell(\alpha)} m_\alpha \Delta_\alpha(x)$$

where

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$$S(x) = \sum_{\alpha \vDash n} (-1)^{\ell(\alpha)} m_\alpha \Delta_\alpha(x)$$

MANY CANCELATIONS

ANSWER is FINER THAN GRADING

Cancelation FREE (Positivity ha ha ha)

감사해요 KAMSAHAEYO

MERCI

THANKS

GRACIAS