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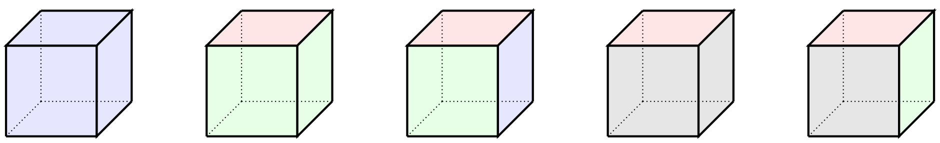
GRUPOS

NANTEL BERGERON

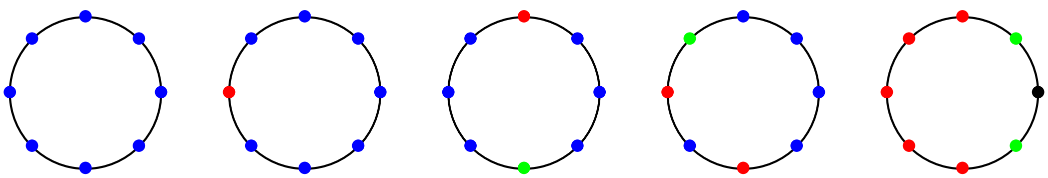
Abstract.

1. Quick Introduction

In this mini course we will see how to count several attribute related to symmetries of an object. For example, how many different dices with four colors can we construct?

 ......???

There are many possibilities and it is hard to answer without the tools we will build. Another typical example is how many different necklaces can we build using 3 kind of beads?

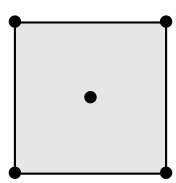
 ...??

To achieve this we need to learn about groups of symmetries, action of groups, cosets, etc We will see a much more interesting enumerations and applications.

2. Simetrias y groupos

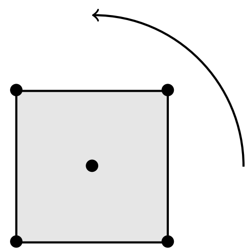
We are interested in understanding all the symmetries of an object we study and the algebraic structures on those symmetries.

2.1. **Conjuntos de Simetrias.** Suppose you have a solid square attached in the middle:

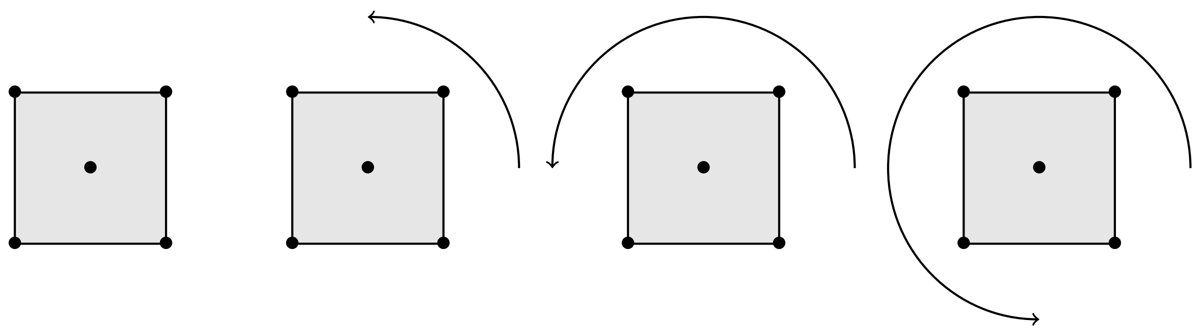


We can turn the square and get back the same thing. This is a symmetry.

Research of Bergeron supported in part by NSERC and York Research Chair.

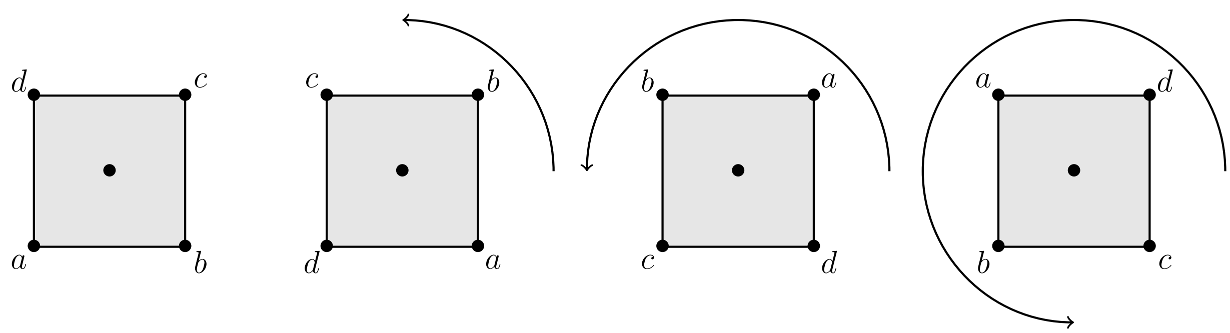


We want to list all the possibilities:



Is there more? **When are two symmetries considered the same??**

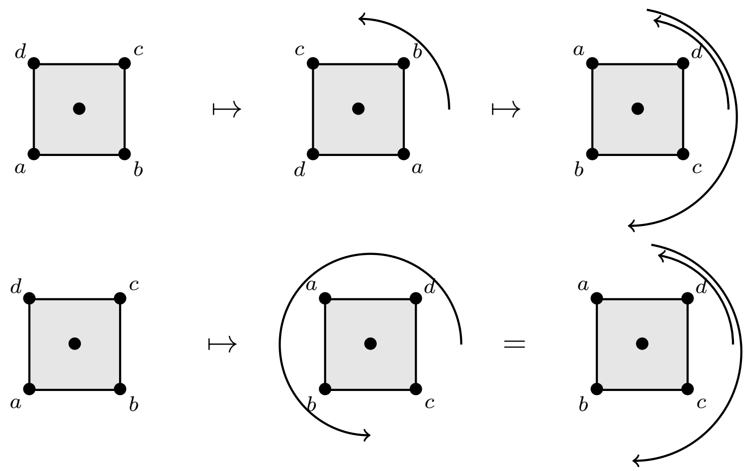
To keep track of what happen, We can name the vertices of the square to see what happen to them. This is a secret marking, it is not part of the object, it is only there to help us keep track of what happen:



We can then encode what happen to the square by only listing what happen to a, b, c, and d:

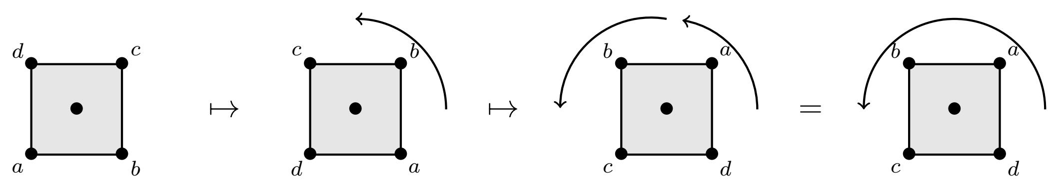
$$\sigma_0 = \begin{matrix} abcd \\ abcd \end{matrix}\,, \qquad \sigma_1 = \begin{matrix} abcd \\ dabc \end{matrix}\,, \qquad \sigma_2 = \begin{matrix} abcd \\ cdab \end{matrix}\,, \qquad \sigma_3 = \begin{matrix} abcd \\ bcda \end{matrix}\,.$$

This is a mathematical model of the symmetries of the square. Now we say that two symmetries are **equal** if starting with the same marking, they end up with the same marking after the symmetry.



This model is useful to understand the structures of those symmetries and focus on what interests us at the moment (the vertices at the corner of the square). This describe in some way all the symmetries of the square. I think of these as maps. This is not unique, we will consider soon other descriptions of the **same** symmetries and see that they can have different presentations.

2.2. **operations on symmetries.** We now want to describe operation we can do on symmetries. First,
composing symmetries: What happen if I do a rotation, and then another one from what we have done?



Now using our mathematical model:

$$\begin{array}{cccc} a & b & c & d \\ \downarrow & \downarrow & \downarrow & \downarrow \\ d & a & b & c \\ \downarrow & \downarrow & \downarrow & \downarrow \\ c & d & a & b \end{array} \quad = \quad \begin{array}{c} abcd \\ cdab \end{array}.$$

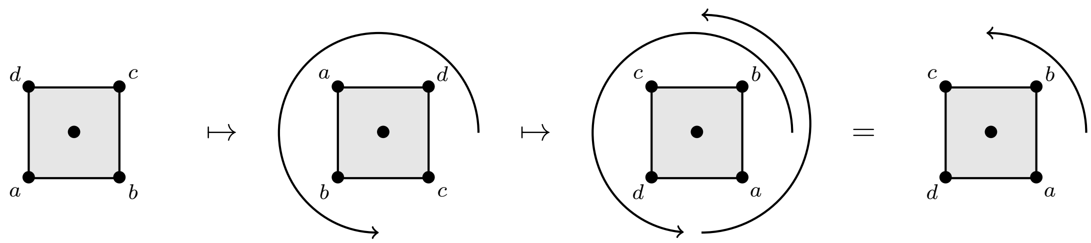
I think of these as composition of maps:

$$\begin{array}{rcl} \sigma_1\colon \{a,b,c,d\} & \to & \{a,b,c,d\} \\ a & \mapsto & d \\ b & \mapsto & a \\ c & \mapsto & b \\ d & \mapsto & c \end{array} \qquad \begin{array}{rcl} \sigma_2\colon \{a,b,c,d\} & \to & \{a,b,c,d\} \\ a & \mapsto & c \\ b & \mapsto & d \\ c & \mapsto & a \\ d & \mapsto & b \end{array}$$

What we have seen above is that

$$\sigma_1 \circ \sigma_1 = \sigma_2$$

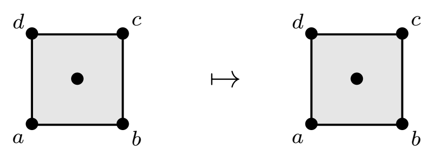
it is the same maps. Another example



In all case composing rotations gives us another rotation. Now using our mathematical model we can look at all cases

	$\begin{matrix}abcd\\abcd\end{matrix}$	$\begin{matrix}abcd\\dabc\end{matrix}$	$\begin{matrix}abcd\\cdab\end{matrix}$	$\begin{matrix}abcd\\bcda\end{matrix}$
$\begin{matrix}abcd\\abcd\end{matrix}$	$\begin{matrix}abcd\\abcd\end{matrix}$	$\begin{matrix}abcd\\dabc\end{matrix}$	$\begin{matrix}abcd\\cdab\end{matrix}$	$\begin{matrix}abcd\\bcda\end{matrix}$
$\begin{matrix}abcd\\dabc\end{matrix}$	$\begin{matrix}abcd\\dabc\end{matrix}$	$\begin{matrix}abcd\\cdab\end{matrix}$	$\begin{matrix}abcd\\bcda\end{matrix}$	$\begin{matrix}abcd\\abcd\end{matrix}$
$\begin{matrix}abcd\\cdab\end{matrix}$	$\begin{matrix}abcd\\cdab\end{matrix}$	$\begin{matrix}abcd\\bcda\end{matrix}$	$\begin{matrix}abcd\\abcd\end{matrix}$	$\begin{matrix}abcd\\dabc\end{matrix}$
$\begin{matrix}abcd\\bcda\end{matrix}$	$\begin{matrix}abcd\\bcda\end{matrix}$	$\begin{matrix}abcd\\abcd\end{matrix}$	$\begin{matrix}abcd\\dabc\end{matrix}$	$\begin{matrix}abcd\\cdab\end{matrix}$

The identity symmetry: You remarked that there is a special symmetry that "does nothing":

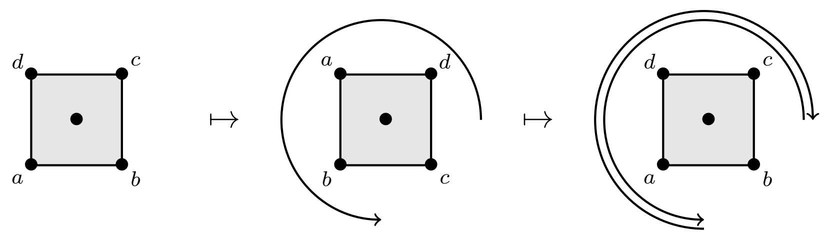


encoded by $Id = \begin{matrix}abcd\\abcd\end{matrix}$

This symmetry is special in the sense that if we compose by Id on the right or on the left we do not change anything:

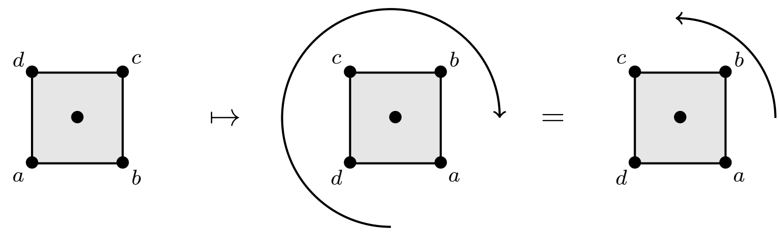
$$Id \circ \sigma = \sigma \circ Id = \sigma$$

Inverting symmetries: Finally, you see that any symmetry can be undone:



$= \quad Id$

Here we have that



We say that it is the **inverse**. If σ is a symmetry, we denote its inverse by σ^{-1}. Using notation as above, we have

$$\sigma_0^{-1} = \sigma_0 \qquad \sigma_1^{-1} = \sigma_3 \qquad \sigma_2^{-1} = \sigma_2 \qquad \sigma_3^{-1} = \sigma_1$$

2.3. **Abstract groups.** When we talk about the symmetries of an object we have seen four important features

(1) We have a **set** of symmetries
(2) We can **compose** symmetries and it gives back a symmetry in our set
(3) There is a special symmetry called the **identity**.
(4) Every symmetry has an **inverse**

In mathematical terms, we have a **group**. More formally, a group is:

(1) A **set** G
(2) An associative operation $m \colon G \times G \to G$
[we often write $m(a, b) = ab$ or $m(a, b) = a + b$]

$$\text{associative: } a(bc) = (ab)c$$

(3) A unique element $1 \in G$: for all $g \in G$

$$1g = g1 = g$$

(4) Every $g \in G$ has an inverse g^{-1}:

$$gg^{-1} = g^{-1}g = 1$$

Example 1. $C_4 = \{1, a, a^2, a^3\}$ where we assume $a^4 = 1$ so

m	1	a	a^2	a^3
1	1	a	a^2	a^3
a	a	a^2	a^3	1
a^2	a^2	a^3	1	a
a^3	a^3	1	a	a^2

It is associative. We have the special element 1 and every element has an inverse

$$a^{-1} = a^3, \qquad a^{-2} = a^2, \qquad a^{-3} = a.$$

Example 2. $C_2 \times C_2 = \{1, a, b, ab\}$ where we assume $a^2 = 1$, $b^2 = 1$ and $ab = ba$ so

m	1	a	b	ab
1	1	a	b	ab
a	a	1	ab	b
b	b	ab	1	a
ab	ab	b	a	1

It is associative. We have the special element 1 and every element has an inverse

$$a^{-1} = a, \qquad b^{-1} = b, \qquad (ab)^{-1} = ab.$$

Example 3. $\mathbb{S}_3 = \{1, s, t, st, ts, sts\}$ where we assume $s^2 = 1$, $t^2 = 1$ and $sts = tst$ (is it all?), so

m	1	s	t	st	ts	sts
1	1	s	t	st	ts	sts
s	s	1	st	t	sts	ts
t	t	ts	1	tst	s	st
st	st	sts	s	ts	1	t
ts	ts	t	sts	1	st	s
sts	sts	st	ts	s	t	1

It is associative. We have the special element 1 and every element has an inverse

$$a^{-1} = a, \qquad b^{-1} = b, \qquad (ab)^{-1} = ab.$$

2.4. **Symmetric group.** We have seen above that symmetries of an object can be encoded by bijective maps on a finite set. This raises two questions

– Is it possible to realize any (abstract) group with bijective group (equivalently, is it true that abstract groups are the symmetries of something?)

– Is the set of all bijections of a finite set a group?

Let us first answer the second question. Let $[n] = \{1, 2, \ldots, n\}$ and consider the set

$$S_n = \{\sigma | \sigma \colon [n] \to [n] \textbf{ bijection}\}$$

We say that the element of S_n are **permutations**. For example

$$S_3 = \{123, 132, 213, 231, 312, 321\}$$

where we encoded the permutations by its list of values. That is using the natural order $1 < 2 < \ldots < n$, we list the values $\sigma(1), \sigma(2), \ldots, \sigma(n)$. For example the permutation 231 is the map

$$\begin{array}{rcl} \{1,2,3\} & \to & \{1,2,3\} \\ 1 & \mapsto & 2 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 1 \end{array}$$

We have the **identity** permutation $Id = 123 \cdots n \in S_n$ and given two permutations $\sigma, \pi \in S_n$ we can compose the two maps and we get a permutation $\sigma \circ \pi \in S_n$. Moreover, for every permutation $\sigma \in S_n$ we can find σ^{-1} such that

$$\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = Id.$$

This gives us a nice group and we call it **the symmetric group**.

2.5. **Representation by permutation.** Let us now consider the question of seeing an abstract group as a group of symmetry. Look at Example 1. You can check that the map

$$\mathbf{1} \mapsto 1234; \qquad a \mapsto 2341; \qquad a^2 \mapsto 3412; \qquad a^3 \mapsto 4123.$$

is a realization of the group C_4. Now compare this with our starting example of rotating the square and see that up to changing the names, we have the same group of symmetries. The abstract group C_4 is realized using permutation (a subset of S_4).

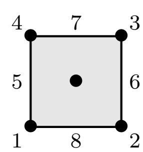
$$\{1234, 2341, 3412, 4123\} \subset S_4.$$

We need to make some observations. When we have $\{Id\} \subseteq H \subseteq S_n$ and H is a group by itself we say that H is a **permutation subgroup** of S_n. For a group G, if we have a map $\varphi\colon G \to S_n$ such that $\varphi(1) = Id$ and $\varphi(ab) = \varphi(a) \circ \varphi(b)$, then we say that φ is a **homomorphism** of G. You can check that in this case $\{Id\} \subseteq \varphi(G) \subseteq S_n$ is then a permutation subgroup of S_n and we say that $\varphi(G)$ is a **permutation representation** of G. If moreover $|G| = |\varphi(G)|$ then we say that $\varphi(G)$ is a **permutation realization** of G.

Exercises.

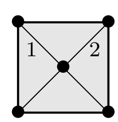
Ex.2.1 For C_4 in Example 1, find a homomorphism $\varphi\colon C_4 \to S_5$. Find 5 points on the square that are maps to themselves after rotations and visualize your construction of φ.

Ex.2.2 Put numbers on the vertices and the sides of the square



use this to define a homomorphism $\phi\colon C_4 \to S_8$. Now you see that the same group may have many different permutation realizations.

Ex.2.3 Draw the two diagonals of the square and call them 1 and 2:



use this to define a homomorphism $\phi\colon C_4 \to S_2$. Is this a permutation representation but is it permutation realizations? Why?

Ex.2.4 Consider the group $C_2 \times C_2$ in Example 2. Find a way to see it as the symmetry of an object. Use that to give a homomorphism $\varphi\colon C_2 \times C_2 \to S_4$ and give a permutation realizations of this group.

Ex.2.5 Show that $\mathbb{S}_3$ in Example 3 is the same as S_3 in Section 2.4. That is find a homomorphism $\varphi\colon \mathbb{S}_3 \to S_3$ What are the permutations $\varphi(s)$ and $\varphi(t)$? Can you find a geometric object such that S_3 describe the symmetries of that object?

Ex.2.6 Some enumeration. What is the cardinality of S_n. That is how many permutations of $[n]$ is there?

Ex.2.7 Take a geometrical object in $\mathbb{R}^3$ (prism, cube, tetrahedra, ...). Describe its symmetry. Then, put label on your object to get a permutations realization of that group of symmetries.

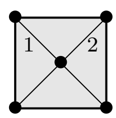
3. Group actions and enumeration

In the exercises of Section 2, we have seen that to get a permutation realization of a group we consider an object to visualize the symmetries and then name some portions of the object to get the permutation. Now I will discuss how to do this systematically.

3.1. **Action.** An **action** of a group G on a finite set X is a map $G \times X \to X$ such that

- 1.x=x
- (gh).x=g.(h.x)

Remark that the notation here put emphasis on the fact that G do something to element of X. Look at exercise 1.3 where we used $C_4 = \{1, a, a^2, a^3\}$ as the symmetries of the square



This gives us an action of C_2 on $X = \{1, 2\}$.

$$\begin{array}{lll} a.1 = 2 & a^2.1 = 1 & a^3.1 = 2 \\ a.2 = 1 & a^2.2 = 2 & a^3.2 = 1 \end{array}$$

We see that for a fix $g \in G$, the map $X \to X$ given by $x \mapsto g.x$ is a permutation of X. It is easy to check that this map is invertible since

$$g^{-1}.(g.x) = (g^{-1}g).x = 1.x = x\,.$$

So any time we have an action, we will have a permutation representation. When do we know if it is a realization or not? In the example above we see that $\{1, a^2\}$ gives the same permutation (the identity) on X. As well, $\{a, a^3\}$ also gives the same permutation on X. We can partition de group C_4 into classes according to the permutation they give

$$\Big\{\{1, a^2\},\ \{a, a^3\}\Big\}$$

This gives us a lot of clue on what happen. Before we really do this let me first give some general propositions.

Proposition 4. *Given a finite group G and H a subgroup of G. Let*

$$gH = \{gh \mid h \in H\}.$$

(1) $\Big\{gH \;\Big|\; g \in G\Big\}$ *is a partition of* G
(2) $|gH| = |H|$ *for all* $g \in G$
(3) $|G|$ *is divisible by* $|H|$ *and* $\Big|\Big\{gH \;\Big|\; g \in G\Big\}\Big| = \frac{|G|}{|H|}$

A **subgroup** here, as before, is a subset $\{1\} \subseteq H \subseteq G$ such that H is itself a group inside G. You see that (3) follows from (1) and (2). For any $g \in G$ we see that a one to one correspondence

$$\begin{array}{rcl} H & \longleftrightarrow & gH \\ h & \mapsto & gh \\ g^{-1}gh = h & \leftarrow & gh \end{array}$$

This gives us that (2) is always true. Now (1) need to be well understood. What are wee really saying? Lets look at C_4 above and remark that $H = \{1, a^2\}$ is indeed a subgroup. Now we for the set of sets

$$1H = \{1, a^2\}; \quad aH = \{a, a^3\}; \quad a^2H = \{1, a^2\}; \quad a^3 = \{a, a^3\}.$$

So when we write

$$\Big\{gH \mid g \in C_4\Big\} = \Big\{\{1, a^2\},\ \{a, a^3\}\Big\}$$

we mean **do not repeat** element that are the same. So what (1) is saying is that

$$g_1H \cap g_2H \neq \emptyset \quad \implies \quad g_1H = g_2H.$$

Proposition 5. *Given a finite group G acting on a finite set X. Let*

$$H = \{g \in G \mid g.x = x \text{ for all } x \in X\}.$$

1. *H is a subgroup of G.*
2. *The action gives a permutation realization if and only if $|H| = 1$*
3. *We can take $X = G$ and the action is left multiplication, in this case we get a permutation realization.*

I will leave as an exercise to show (1). If you have never done that it is good to do it. If you have seen that before it is very easy.

The item (2) is subtle. When we have an action $G \times X \to X$ you have seen in exercise that we can build a homoporphism $\varphi\colon G \to S_{|X|}$. For a fix g, the permutation $\varphi(g)$ is given by how g permutes X under the map $x \mapsto g.x$. Now we have

$$\varphi(g_1) = \varphi(g_2) \quad \iff \quad g_1H = g_2H.$$

indeed

$$g_1.x = g_2.x \quad \iff \quad g_2^{-1}g_1.x = g_2^{-1}g_2.x = x$$

This is true for all x if and only if $g_2^{-1}g_1 \in H$. That is $g_2^{-1}g_1H = H$ (since H is a subgroup) and thus $g_1H = g_2H$. You see that we have as many permutations in $\varphi(G)$ as we have element in $\Big|\Big\{gH \mid g \in G\Big\}\Big| = \frac{|G|}{|H|}$ so

$$|\varphi(G)| = \Big|\Big\{gH \mid g \in G\Big\}\Big| = \frac{|G|}{|H|} = |G| \quad \iff \quad |H| = 1.$$

Now (3) is much easier. Remark that the action here is $G \times \mathbf{G} \to \mathbf{G}$ the usual multiplication but we think of $\mathbf{G}$ as being the set we act on. The fact that it is an action is clear. If we have $gh = h$ then it is clear that $g = ghh^{-1} = hh^{-1} = 1$. so $H = \{1\}$ only.

Remark 6. The Proposition 5 (3) tell us that any abstract group can be realize (in at least one way) as a permutation group. This is known as Cayley's theorem.

Lets do some examples. Take $C_2 \times C_2 = \{1, a, b, ab\}$ (see Example 2) and $X = \{1, 2, 3, 4\}$. We can define

$$\begin{array}{rcl} \varphi(a)\colon X & \to & X \\ 1 & \mapsto & a.1 = 2 \\ 2 & \mapsto & a.2 = 1 \\ 3 & \mapsto & a.3 = 3 \\ 4 & \mapsto & a.4 = 4 \end{array} \qquad \begin{array}{rcl} \varphi(b)\colon X & \to & X \\ 1 & \mapsto & b.1 = 1 \\ 2 & \mapsto & b.2 = 2 \\ 3 & \mapsto & b.3 = 4 \\ 4 & \mapsto & b.4 = 3 \end{array} \qquad \begin{array}{rcl} \varphi(ab)\colon X & \to & X \\ 1 & \mapsto & ab.1 = 2 \\ 2 & \mapsto & ab.2 = 1 \\ 3 & \mapsto & ab.3 = 4 \\ 4 & \mapsto & ab.4 = 3 \end{array}$$

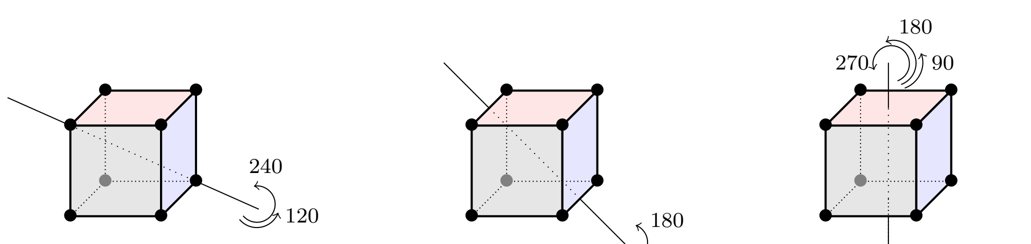
You see that $\big(\varphi(a)\big)^2 = Id$, $\big(\varphi(b)\big)^2 = Id$ and $\varphi(ab) = \varphi(a)\varphi(b) = \varphi(b)\varphi(a)$. Hence this is a well defined action and we do have a homomorphism. Moreover only $\varphi(1) = Id$ and we have a permutation realization of $C_2 \times C_2$.

If instead we take

$$\begin{array}{rcl} \phi(a)\colon X & \to & X \\ 1 & \mapsto & a.1 = 2 \\ 2 & \mapsto & a.2 = 1 \\ 3 & \mapsto & a.3 = 4 \\ 4 & \mapsto & a.4 = 3 \end{array} \qquad \begin{array}{rcl} \phi(b)\colon X & \to & X \\ 1 & \mapsto & b.1 = 2 \\ 2 & \mapsto & b.2 = 1 \\ 3 & \mapsto & b.3 = 4 \\ 4 & \mapsto & b.4 = 3 \end{array} \qquad \begin{array}{rcl} \phi(ab)\colon X & \to & X \\ 1 & \mapsto & ab.1 = 1 \\ 2 & \mapsto & ab.2 = 2 \\ 3 & \mapsto & ab.3 = 3 \\ 4 & \mapsto & ab.4 = 4 \end{array}$$

we see again that $\big(\phi(a)\big)^2 = Id$, $\big(\phi(b)\big)^2 = Id$ and $\phi(ab) = \phi(a)\phi(b) = \phi(b)\phi(a)$ and this is a different action of the same group on X. But this time $\phi(1) = \phi(ab) = Id$ and here we do not have a permutation realization since $H = \{1, ab\}$.

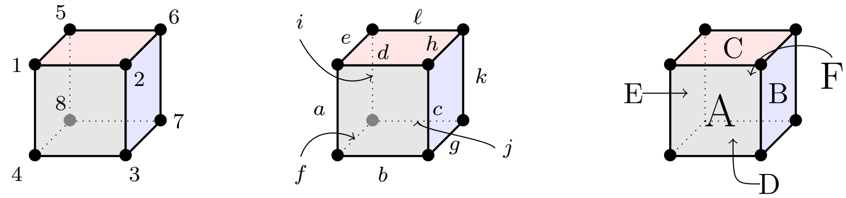
3.2. **The cube.** We would like to have a much bigger example to play with. Let us consider the group B_3 of symmetry of a cube



pair of opposite vertices (4 such pairs) — pair of opposite edges (6 such pairs) — pair of opposite faces (3 such pairs)

This group has 1 identity map, $4 * 2$ rotation of $120°$ and $240°$ for each pair of opposite vertices, 6 rotation of $180°$ for each pair of opposite edges and $3 * 3$ rotation of $90°$, $180°$ and $270°$ for each pair of opposite faces. That is $24 = 1 + 8 + 6 + 9$. This group is strating to be complicated and it would take some time to write down the full table of multiplication. We will soon name all its element.

To help us, let us first name all the components of the cube



8 vertices 12 edges 6 faces

The group B_3 acts on the set

$$X = \{1,2,3,4,5,6,7,8,a,b,c,d,e,f,g,h,i,j,k,\ell,A,B,C,D,E,F\}$$

For example the permutation corresponding to the rotation of $120°$ around the line through the vertices 3 and 5 is

$$\begin{array}{cccccccccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & a & b & c & d & e & f & g & h & i & j & k & \ell & A & B & C & D & E & F \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
6 & 7 & 3 & 2 & 5 & 8 & 4 & 1 & h & c & g & k & \ell & d & b & j & e & a & f & i & B & D & F & A & C & E
\end{array}$$

It is sometime better to write that into disjoint cycle notation. For this we simply follow what happen to the element as we iterate the same symmetry: $1 \to 6 \to 8 \to 1$ and we have a cycle. We simply write $(1\,6\,8)$ and we assume the cycle close up. So the permutation above is

$$(1\,6\,8)(2\,7\,4)(3)(5)(a\,h\,j)(b\,c\,g)(d\,k\,f)(e\,\ell\,i)(A\,B\,D)(C\,F\,E)$$

The action we describe gives us a permutation representation $\varphi\colon B_3 \to S_{26}$. This is a permutation realization of B_3 (you can see why?). In this permutation representation the element above has 8 cycles of length 3 and 2 cycle of length 1. We call that the **cycle structure** of the element. Cycle structure will play a very important role in Polya Theory. Of course if we change the permutation representation, we get different cycle structures.

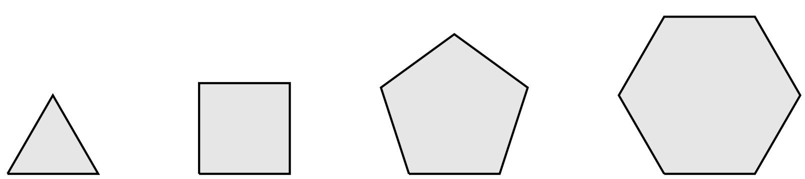
Exercises.

Ex.3.1 Show that H_X in Proposition 5 is a subgroup.

Ex.3.2 Let $G = S_3$. Find all different actions of S_3 on $X = \{a,b,c\}$. In each case describe the corresponding homomorphism $\varphi\colon S_3 \to S_3$. Describe the cycle structure each elements of S_3, for each permutation representations you constructed.

Ex.3.3 Look again at Ex.2.2. Describe the cycle structure of each element of C_4 for this permutation representation.

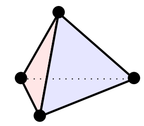
Ex.3.4 D_3, D_4, D_5, D_6: Study the dihedral groups. These are the group of symmetries



$D_3 = S_3$ D_4 D_5 D_6

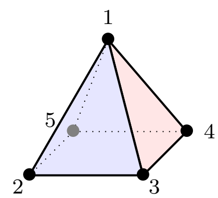
where here we can rotate the figure **and** we can also take it off the table, flip it, and put it back (reflection).

Ex.3.5 A_4: study the group of rotation symmetries of a tetrahedra.



3.3. **orbits and counting points.** When we studied the actions of B_3 on the set X of vertices, edges, faces of the cube you notices that a symmetry sends a vertex to another vertex, an edge to another edge and a face to a face. A symmetry preserve the types. This lead us to define the notion of orbit of a point $x \in X$.

Before we start let us consider the smaller example. Lets look at the symmetries of



We see that any symmetry need to fix the point 1. One the other hand, we can always find a symmetry that will send any point of $\{2,3,4,5\}$ to another point of $\{2,3,4,5\}$. We say that the group of symmetry acting on the points $\{1,2,3,4,5\}$ has two orbits: $\{1\}$ and $\{2,3,4,5\}$. We see that the orbits encode the nature of the points on our objects. So the orbits gives us interesting information about the symmetries of the objects. We will now define orbits and see how to count points within an orbit.

Given a group G acting on a set X, we say that the **orbit** $G.x$ of a point $x \in X$ is the set

$$G.x = \{g.x \mid g \in G\} \subseteq X.$$

This is the set of points of X we can reach from x via the action of G. We saw above example of orbits.

How to count the number of points inside the orbits of x? That is can we find a nice formula for $|G.x|$. We have already encountered this principal. Let

$$Stab(x) = \{g \in G \mid g.x = x\} \subseteq G$$

As you see $G.x \subseteq X$ and $Stab(x) \subseteq G$. The set $Stab(x)$ is in fact a subgroup of G. This gives us a nice way to compute $G.x$:

Proposition 7 (Lagrange's theorem)**.** *For $x \in X$, we have*

$$|G.x| = \frac{|G|}{|Stab(x)|}$$

The idea is again to construct an explicit bijection between the two sets

$$G.x \quad \leftrightarrow \quad \{gS \mid g \in G\}$$

where $S = Stag_x(G)$. For any $y \in G.x$, there are many possible $g \in G$ such that $y = g.x$.

$$\begin{array}{rcl} \alpha\colon G.x & \to & \{gS \mid g \in G\} \\ y & \mapsto & gS \text{ where g is any such that } g.x = y \end{array}$$

Of course we have to make sure that it is well define. That is if

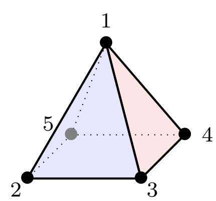
$$y = g.x = h.x \quad \Longleftrightarrow \quad h^{-1}g.x = x \quad \Longleftrightarrow \quad gS = hS$$

The map in the other direction is given by

$$\begin{array}{rcl} \beta\colon \{gS \mid g \in G\} & \to & G.x \\ gS & \mapsto & g.x \end{array}$$

The two maps construct a correspondence between the two sets.

If we go back to our small example. The group of symmetries of



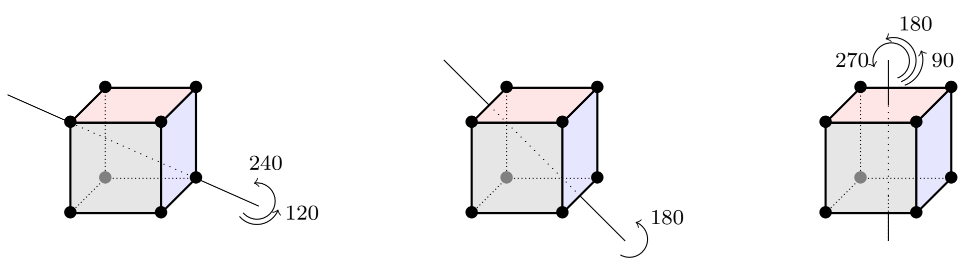
is C_4 (you see this?). We noticed 2 orbits:$\{1\}$ and $\{2, 3, 4, 5\}$. For 1, we see that every symmetry fix 1, Hence

$$\frac{|C_4|}{|Stab(1)|} = \frac{4}{4} = 1$$

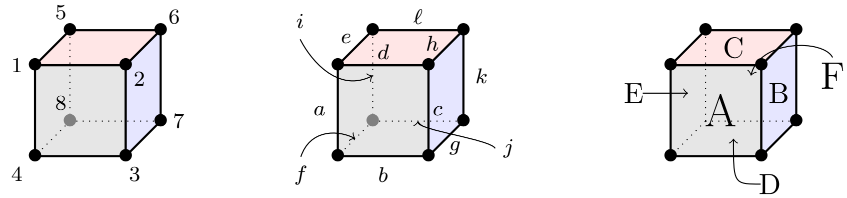
For any of the points $2, 3, 4$ or 5, only the identity symmetry fix such point. We get

$$\frac{|C_4|}{|Stab(2)|} = \frac{4}{1} = 4$$

If we look at Section 3.2 the group B_3



pair of opposite vertices (4 such pairs) | pair of opposite edges (6 such pairs) | pair of opposite faces (3 such pairs) acts on the set of vertices, edges and faces of the cube:



8 vertices 12 edges 6 faces

We see that exactly three symmetries fix the vertex 1. Hence

$$|B_3.1| = \frac{|B_3|}{3} = \frac{24}{3} = 8.$$

Similarly only two symmetries fix the edge a, hence

$$|B_3.a| = \frac{|B_3|}{2} = \frac{24}{2} = 12.$$

Finally four symmetries fix a face hence

$$|B_3.A| = \frac{|B_3|}{4} = \frac{24}{4} = 6.$$

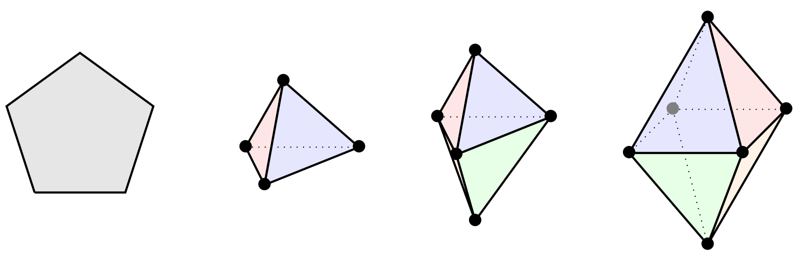
This is useful to count vertices, edges, faces, of object with a lot of symmetries. It is even more useful for things we cannot see (like in higher dimensions).

Exercises.

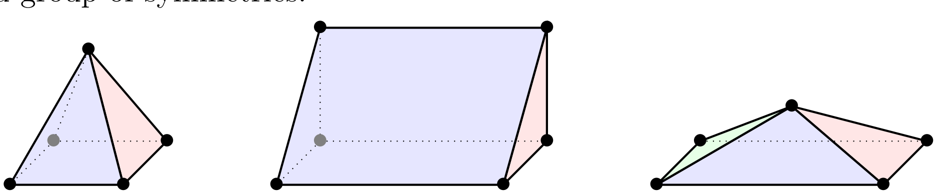
Ex.3.6 Show that $Stab(x)$ is a subgroup.

Ex.3.7 In Proposition 7, show that $\alpha \circ \beta = Id$ and $\beta \circ \alpha = Id$.

Ex.3.8 Count the number of points, edges and faces of the following objects using Proposition 7.



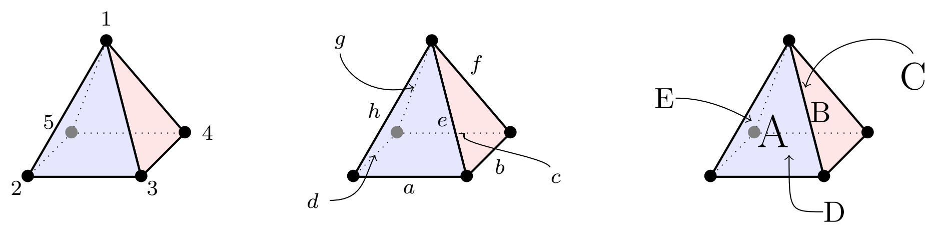
Ex.3.9 The object are not always fully symmetric and points, edges and faces are broken into smaller orbits (different kind of points, edges, faces). Understand the orbits to count the number of points, edges and faces of the following objects using Proposition 7 and group of symmetries.



Ex.3.10 Count the number of vertices, edges, 2-faces, 3-faces, ..., of a herpercube of dimension n.
[hint. find a group of symmetry of the hypercube that contains exactly one orbit for each type of faces]

3.4. **counting number of orbits.** At this point I hope you are convinced that using group theory is powerful to count things related to symmetry. Our next step is to count the number of orbits. As we have seen above, we can see this as counting the number of type of points we have.

Lets look again at C_4 acting on the square pyramid and consider its vertices, edges and faces.



The group C_4 is acting on the set $Y = \{1, 2, 3, 4, 5, a, b, c, d, e, f, g, h, A, B, C, D, E\}$. This set decompose into disjoint orbits. We denote this as

$$Y/C_4 = \Big\{\{1\}, \{2, 3, 4, 5\}, \{a, b, c, d\}, \{e, f, g, h\}, \{A, B, C, D\}, \{E\}\Big\}$$

We see that the number of orbits count the different types of things.

As for the number of element in an orbits, there is very beautiful formula to count the number of orbits of an action. Let G acts on a set X and define

$$Fix(g) = \{x \in X \mid g.x = x\}$$

.

Proposition 8 (Burnside's lemma)**.** *For G acting on a set X, we have*

$$\big|X/G\big| = \frac{1}{|G|} \sum_{g \in G} \big|Fix(g)\big|$$

This is not hard to see after a few manipulations of the concepts we have seen

$$\begin{aligned}
\sum_{g \in G} \big|Fix(g)\big| &= \sum_{g \in G} \sum_{x \in X} \delta_{x,g.x} = \sum_{x \in X} \sum_{g \in G} \delta_{x,g.x} \\
&= \sum_{x \in X} \big|Stab(x)\big| = \sum_{x \in X} \frac{|G|}{|G.x|} \\
&= |G| \textstyle\sum_{x \in X} \frac{1}{|G.x|} = |G| \big|X/G\big|
\end{aligned}$$

The last equality is a nice trick.

So if we want to count the number of orbit of an action, we simply need to find the cardinality of the set $Fix(g)$ for each $g \in G$. Lets look again at $C_4 = \{\mathbf{1}, a, a^2, a^3\}$ acting on Y above. We find that

$Fix(\mathbf{1}) = Y$
$Fix(a) = \{1, D\}$
$Fix(a^2) = \{1, D\}$
$Fix(a^3) = \{1, D\}$

Hence

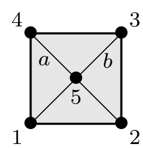
$$\big|Y/C_4\big| = \frac{1}{4}(18 + 2 + 2 + 2) = \frac{24}{4} = 6$$

I Alway marvel at the fact that this works so well.

Exercises.

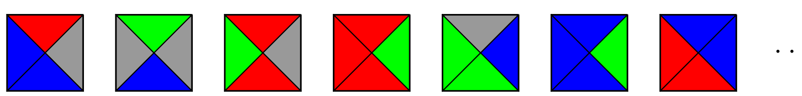
Ex.3.10 Look at the actions you defined in Ex.3.8 and Ex.3.9 and use Proposition 8 to count the number of orbits

Ex.3.11 Consider the actions of $C_4 = \{\mathbf{1}, a, a^2, a^3\}$ on the set $Z = \{1, 2, 3, 4, 5, a, b\}$ as the five vertices $\{1, 2, 3, 4, 5\}$ and two diagonal segment $\{a, b\}$ pictured bellow. Use Proposition 8 to count the number of orbits

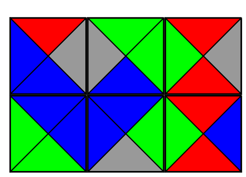


4. Coloring and Polya theory

Suppose your friend Laura wants to create a game with pieces that are tiles like



The game will be to tile a 2×3 board so that color matches where the tile touch. A good game will end with a tiling like this:



But she would like to know how many kind of tile we need, and how many different good game will be possible. Lets try to help....

4.1. **coloring of an action.** When we have a finite set X, a **coloring** of X with certain colors is a function $w\colon X \to \{colors\}$. Indeed if you have a set like $X = \{1, 2, 3\}$ and your have two colors *red* and *blue*, you can do 8 different colorings. Namely

123; 123; 123; 123; 123; 123; 123; 123;

This is the same as functions $\{1,2,3\} \to \{red, blue\}$. If C is a set of color, then we denote by C^X the set of all coloring of X by C. That is the set of all functions

$$C^X = \{w \colon X \to C \text{ functions}\}$$

For our example above with $X = \{1,2,3\}$ and $C = \{red, blue\}$ we get

$$C^X = \{123, 123, 123, 123, 123, 123, 123, 123\}$$

Now, when a group G acts on X, then G **also** acts on the larger set of all colorings C^X in a natural way. Indeed, if $w \in C^X$, let $(g.w)(x) = w(g^{-1}.x)$ and it works. Here $g.w$ is a new function builded from w. we need to use the inverse of g in this definition so that the "technicality" of the definition of action works for the action on C^X.

$$((hg).w)(x) = w((hg)^{-1}.x) = w((g^{-1}h^{-1}).x) = w(g^{-1}.(h^{-1}.x)) = (g.w)(h^{-1}.x)) = (h.(g.w))(x)$$

I put this here to show you the technicality in showing that $(hg).w = h.(g.w)$. Try to understand each equality? This technicality is important for things to work ok, but will not play a big role in the end.

Lets work with our example and the group of permutation S_3 acting on $X = \{1,2,3\}$ (it permutes the numbers). Recall that

$$S_3 = \{123, 132, 213, 231, 312, 321\}$$

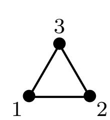
and each $\sigma \in S_3$ is a permutation $\sigma \colon X \to X$ and we have an action. Let $C = \{red, blue\}$ and consider $w \in C^X$. For example 123 is the function $w(1) = red$, $w(2) = blue$ and $w(3) = red$. Let $\sigma = 231$ and check that $\sigma^{-1} = 312$. So the new function $\sigma.w$ is defined as follow:

$$(\sigma.w)(1) = w(\sigma^{-1}(1)) = w(3) = red$$
$$(\sigma.w)(2) = w(\sigma^{-1}(2)) = w(1) = red$$
$$(\sigma.w)(3) = w(\sigma^{-1}(3)) = w(2) = blue$$

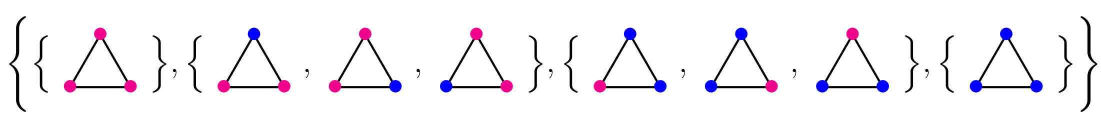
So $\sigma.w = 123$. If we look at the orbits of X under the action of S_3 we see that there is only one orbit (X is a single orbit). But if we look at the orbits of the action on C^X a very different picture emerge:

$$C^X/S_3 = \Big\{\{123\}, \{123, 123, 123\}, \{123, 123, 123\}, \{123\}\Big\}$$

We can represent this on a triangle $S_3 = D_3$ is the symmetry of the triangle allowing flips and rotation)

$X =$ 

The set C^X/S_3 can be pictured as

$C^X/S_3 =$ 

As you can see, each orbit represent **one** possible type of necklace with three beads and two different color. So we can use Proposition 8 to count the number of possibilities. First we need to figure out what is $Fix(g)$ for each symmetry on the set C^X. Let us first look at our example. For $Id = 123 \in S_3$ it is easy as 123 always fix all coloring

$$Fix(Id) = C^X \qquad \Longrightarrow \qquad \big|Fix(123)\big| = |C|^{|X|} = 2^3 = 8$$

Then we look at the other permutations:

$$Fix(132) = \{\{123, 123, 123, 123\} \qquad \Longrightarrow \qquad \big|Fix(132)\big| = 4 = 2^2$$
$$Fix(213) = \{\{123, 123, 123, 123\} \qquad \Longrightarrow \qquad \big|Fix(213)\big| = 4 = 2^2$$
$$Fix(231) = \{\{123, 123\} \qquad \Longrightarrow \qquad \big|Fix(231)\big| = 2 = 2^1$$
$$Fix(312) = \{\{123, 123\} \qquad \Longrightarrow \qquad \big|Fix(312)\big| = 2 = 2^1$$
$$Fix(321) = \{\{123, 123, 123, 123\} \qquad \Longrightarrow \qquad \big|Fix(321)\big| = 4 = 2^2$$

Using Proposition 8 we get the right answer:

$$\big|C^X/S_3\big| = \frac{1}{6}(8+4+4+2+2+4) = \frac{24}{6} = 4$$

You noticed that $\big|Fix(g)\big|$ is always a power of $|C| = 2$, this is not an accident.

Proposition 9. *Let G act on X. As above we get an action of G on C^X. For $g \in G$, we have that*

$$w \in Fix(g) \qquad \Longleftrightarrow \qquad w(x) = w(g.x) = w(g^2.x) = w(g^3.x) = \ldots \text{ for all } x \in X$$

That is, w is constant over the cycles of G for the action on X

Be careful here, we have **two** actions to consider. The action of G on X and the action of G on C^X. We are interested for the set $Fix(g)$ for the action of G on C^X, but to compute it, we use the action of G on X. We see here that we are interested in the disjoint cycles of g on X. Remember that the cycle of $g \in G$ are obtained by looking what happen when we apply successive power of g to the element of X. $x \to g.x \to g^2.x \to \cdots$ until we get bact x. If w is fixed by g, that is $(g.w)(x) = w(x)$, then applying g^{-1} both side we get

$$w(x) = (g^{-1}.w)(x) = w(g.x)$$

Applying g^-1 again to the equality above we get

$$w(g.x) = (g^{-1}.w)(x) = (g^{-2}.w)(x) = w(g^2.x)$$

and we can continue like that to show one of the implication of the Proposition 9.

In cycle notation, the permutation 213 is $(1\,2)(3)$. Look again at $Fix(213)$ and indeed all coloring in the set have the same color for the cycle $(1\,2)$ and for the cycle (3). On the other hand, 231 is a single cycle $(1\,2\,3)$ and only 2 coloring are constant on this cycle. I leave it to you to see the converse of the theorem above. What we have seen so far is that when G acts on C^X we have

$$\big|Fix(g)\big| = |C|^{|cyc_X(g)|}$$

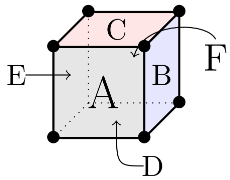
where $cyc_X(g)$ is the cycle decomposition of g for the action on X.

Theorem 10. *Given G acting on X. We have that G acts on C^X. Let $c = |C|$,*

$$\big|C^X/G\big| = \frac{1}{|G|}\sum_{g\in G} c^{|cyc_X(G)|}$$

You understand that this follow from Proposition 8 and Proposition 9.

Exercises. Our friend Laura that just learned about this theorem is pretty sure we can use that to solve her question. Lets do that together...first some warm up.

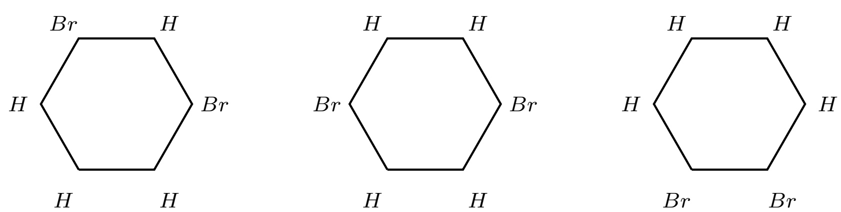
Ex.4.1 Let B_3 act on the cube 

and let $X = \{A, B, C, D, E, F\}$ be the set of faces. Consider $C = \{red, black, blue, green\}$ and B_3 acting on the coloring C^X. Now answer the first question we had: how many different dices with four colors can we construct.

Ex.4.2 Now answer the second questions we have seen: how many necklaces of 8 beads of 3 different kind can we make? [hint: use D_8 acting on an octogone]

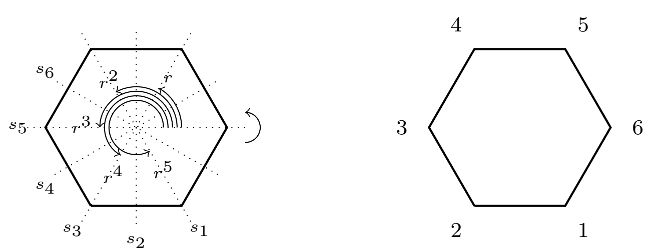
Ex.4.3 Solve the two questions of Laura. Keep in mind that the tiles are colored on one side only.

4.2. **Polya Theory.** Today, Laura said that it is a bad idea to have the same color on a single tile more than twice. It would make the game more interesting if we do not allow the same color more than twice on each tile. But now all our counting need to be redone? She is not sure we can solve her problem anymore. But she heard that a chemist name Polya had a similar problem and found a solution. The molecule $C_6H_4Br_2$ is 6 atoms of carbon (C) arrange on an hexagon and the atoms of hydrogen (H) and Bromine (Br) are attached in a circular way around it. But in nature there is more than one possibility and the molecule may have different behavior depending of the configuration



He wanted as well to understand the possible configuration of more complex molecule and for this he needed to refine the counting principle and be able to count coloring with a specified type of colors (how many of each color). For this we need to refine our counting recording which colors was used.

Lets follow an example. Recall that $D_6 = \{1, r, r^2, r^3, r^4, r^5, s_1, s_2, s_3, s_4, s_5, s_6\}$ act on the hexagon as depicted bellow and this define an action on the vertices $X = \{1, 2, 3, 4, 5, 6\}$:



We have seen in Teorem 10 that the number of cycles for each element of the group play an important role in counting colorings. If we want to refine our counting, we now have to study exactly the cycle type of each element of D_6 using the permutation representation with respect to X. For this we will use monomial to encode the cycle type of each element. For example, if an element has 2 cycles of length 1, 3 cycles of length 2 and 1 cycle of length 4, then we will encode this with a monomial

$$x_1^2 x_2^3 x_4$$

In general the exponent of x_i is the number of cycles of length i in our permutation representation of the element. For D_6 acting on $X = \{1, 2, 3, 4, 5, 6\}$ above we have

elements of D_6	Cycle type	cycle monomial
1	(1)(2)(3)(4)(5)(6)	x_1^6
r	(1 2 3 4 5 6)	x_6
r^2	(1 3 5)(2 4 6)	x_3^2
r^3	(1 4)(2 5)(3 6)	x_2^3
r^4	(1 5 3)(2 6 4)	x_3^2
r^5	(1 6 5 4 3 2)	x_6
s_1	(1)(4)(2 6)(3 5)	$x_1^2 x_2^2$
s_2	(1 2)(3 6)(4 5)	x_2^3
s_3	(2)(5)(1 3)(4 6)	$x_1^2 x_2^2$
s_4	(2 3)(1 4)(5 6)	x_2^3
s_5	(3)(6)(2 4)(1 5)	$x_1^2 x_2^2$
s_6	(3 4)(2 5)(1 6)	x_2^3

We now put all this information together in a polynomial (the sum of all the monomial we produce

$$P_{D_6,X}(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{12}\left(x_1^6 + 2x_6 + 2x_3^2 + 4x_2^3 + 3x_1^2 x_2^2\right)$$

We call this the **cycle index** of the permutation representation of D_6. The factor $\frac{1}{12}$ is there as it will play a role in the final result. This polynomial does not use the variable x_4 and x_5 in this case.

In general, consider a group G acting on a finite set X where $n = |X|$. The cycle index of the permutation representation of G on X is

$$P_{G,X}(x_1, x_2, \ldots, x_n) = \frac{1}{G} \sum_{g \in G} x_1^{|cyc_{X,1}(g)|} x_2^{|cyc_{X,2}(g)|} \cdots x_n^{|cyc_{X,n}(g)|}$$

where $cyc_{X,i}(g)$ is the number of cycle of length i in the permutation representation of g with X. We remark right away that if we color the action with $c = |C|$ colors, Theorem 10 can be written as follow

$$\left|C^X/G\right| = P_{G,X}(c, c, ..., c).$$

You see this?

For D_6

$$P_{D_6,X}(c,c,c,c,c,c) = \frac{1}{12}\big(c^6 + 2c + 2c^2 + 4c^3 + 3c^2c^2\big) = \frac{1}{12}\big(c^6 + 2c + 2c^2 + 4c^3 + 3c^{2+2}\big).$$

Now the exponent of c in the right hand side, count exactly the number of cycle as in Theorem 10.

If we want to refine our counting we can use the cycle index polynomial that knows about the size of each cycles. Remember that a coloring must respect the cycles of an element in order to be fixed by this element. The variable x_i in the cycle index polynomial contribute a cycle of length i. Instead of substituting $x_i \leftarrow c$ we do a can try to keep track of which color was used and how many time. The way to encode that with generating function is to do

$$x_i \leftarrow a_1^i + a_2^i + \cdots + a_c^i$$

We understand the right hand side as "choosing" (the "+") a color from $\{a_1, a_2, \ldots, a_c\}$ and using it i-times.

Lets look what happen when we do this with $P_{D_6,X}$ having two colors $\{a, b\}$:

$$P_{D_6,X}(a+b, a^2+b^2, a^3+b^3, a^4+b^4, a^5+b^5, a^6+b^6)$$

$$= \frac{1}{12}\big((a+b)^6 + 2(a^6+b^6) + 2(a^3+b^3)^2 + 4(a^2+b^2)^3 + 3(a+b)^2(a^2+b^2)^2\big)$$

Lets expand each monomial:

$$\begin{aligned}
&(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6\\
&a^6 + b^6 = a^6 + b^6\\
&(a^3+b^3)^2 = a^6 + 2a^3b^3 + b^6\\
&(a^2+b^2)^3 = a^6 + 3a^4b^2 + 3a^2b^2 + b^6\\
&(a+b)^2(a^2+b^2)^2 = (a^2+2ab+b^2)(a^4+2a^2b^2+b^4)\\
&\qquad = a^6 + 2a^5b + 3a^4b + 2a^3b + 3a^2b + 2ab^5 + b^6
\end{aligned}$$

Each monomial expansion "count" the number of coloring fixed by an element of D_6. For example, $s_5 \in D_6$ has cycles $(3)(6)(2\,4)(1\,5)$. We know that a coloring will be fixed once we choose a color for 3, 6, $\{2,4\}$ (same color) and $\{1,5\}$ (same color). The cycle monomial is $x_1^2x_2^2$. When we substitute $x_1 \leftarrow a+b$ and $x_2 \leftarrow a^2+b^2$ we produce all the choices of choosing a color $(a+b)(a+b)(a^2+b^2)(a^2+b^2)$ for 3, 6, $\{2,4\}$ and $\{1,5\}$ respectively. So the expansion

$$a^6 + 2a^5b + 3a^4b + 2a^3b + 3a^2b + 2ab^5 + b^6$$

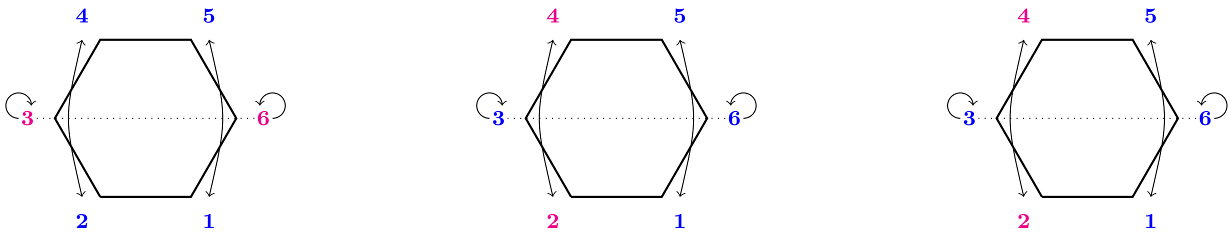
Gives us a refinement of counting the colorings that are fixed by s_5 taking into account how many times a color is used. For example the coefficient 3 in front of a^4b^2 means there are 3 coloring that will be fixed by s_5 with four a's and two b's. When we expand $(a+b)(a+b)(a^2+b^2)(a^2+b^2)$ we can obtain a^4b^2 in three different ways:

$$(a+\mathbf{b})(a+\mathbf{b})(\mathbf{a^2}+b^2)(\mathbf{a^2}+b^2)$$

$$(\mathbf{a}+b)(\mathbf{a}+b)(a+\mathbf{b^2})(\mathbf{a^2}+b^2)$$

$$(\mathbf{a}+b)(\mathbf{a}+b)(\mathbf{a^2}+b^2)(a+\mathbf{b^2})$$

Each ways of getting the monomial a^4b^2 correspond to exactly one choice of coloring that is fixed by s_5 selecting which color we put on each cycles 3, 6, $\{2,4\}$ and $\{1,5\}$ respectively. They are (listed in the same way we got the monomials above)



where here I put red for a's and blue for the b's.

So if we continue our expansion of $P_{D_6,X}$

$P_{D_6,X}(a+b, a^2+b^2, a^3+b^3, a^4+b^4, a^5+b^5, a^6+b^6)$
$= \frac{1}{12}\big(12a^6 + 12a^5b + 36a^4b^2 + 24a^3b^3 + 36a^2b^4 + 12ab^5 + 12b^6\big)$
$= a^6 + a^5b + 3a^4b^2 + 2a^3b^3 + 3a^2b^4 + ab^5 + b^6$

When we do this, in front of each monomial we use Proposition 8 for each distribution of color separately. Hence

$$P_{D_6,X} = a^6 + a^5b + 3a^4b^2 + 2a^3b^3 + 3a^2b^4 + ab^5 + b^6$$

gives us the distribution of possible coloring depending on how many times each color is used. Indeed Polya's question is answered as there is exactly three possible coloring of an exagon using 2 Br's and 4 H's. It is the coefficient of a^2b^4 in $P_{D_6,X}$ after the substitution $x_i \leftarrow a^i + b^i$.

We sum up all our general understanding as follow

Theorem 11 (Polya)**.** *Given a group G acting on a set X. Let $c = |C|$ and consider the action of G on C^X.*

(a) *Compute $P_{G,X}$ and*

$$P_{G,X}(c, c, \ldots, c) = \big|C^X/G\big|$$

(b) *Compute the substitution $x_i \leftarrow a_1^i + a_2^i + \cdots + a_c^i$ in $P_{G,X}$. The coefficient of $a_1^{d_1}a_2^{d_2}\cdots a_c^{d_c}$ in the resulting polynomial is the number of coloring of X with color a_i used d_i times.*

Exercises. Now we will try to solve Laura's question. Again, lets warm up a little bit before

Ex.4.4 For each permutation representation we have encounter this week.
- (a) Give the cycle index polynomials
- (b) Count the number of coloring using 3 colors
- (c) Count the number of coloring using 3 colors, but not repeating any color in a coloring
- (d) Count the number of coloring using 3 colors, but having one color repeated once.

Ex.4.5 Count the number of way to construct the Laura's tile with no colors used more than twice.

Ex.4.6 Can you count the number of Laura's good 2×3 tiling using only the tiles in Ex.4.6?

Ex.4.6 Can you do coloring in higher dimension? (coloring of polytopes)?

Nantel Bergeron, Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, Canada
E-mail address: bergeron@mathstat.yorku.ca
URL: http://www.math.yorku.ca/bergeron/