Representation, Tower of Algebras and Combinatorial Hopf Algebras.

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(with **C. Benedetti, N. Thiem**

... and many more)

Outline

- What is a Combinatorial Hopf Algebra.
- Sym is a strong CHA.
- Restriction on original definition of strong CHA
- How to make NCSym a strong CHA.

Combinatorial Hopf Algebra

$H = \bigoplus_{n \geq 0} H_n$ a graded connected Hopf algebra is CHA if

(weak) There is a distinguished basis with positive integral structure coefficients.

(med) it is weak CHA with a distinguish character.

(strong) it is med CHA such that the structure is obtained from representation operation

(Bonus) It is the functorial image of a Hopf monoid.

Sym is a strong CHA

Sym is the space of symmetric functions $\mathbb{Z}[h_1, h_2, \ldots]$, with $\deg(h_k) = k$ and

$$\Delta(h_k) = \sum_{i=0}^{k} h_i \otimes h_{k-i}.$$

It is a strong CHA:

- Basis: Schur functions s_λ
- Representation of symmetric groups and Frobenius map:

$$\mathcal{F}\colon \bigoplus_{n\geq 0} K_0(S_n) \to Sym$$

is an isomorphism of graded Hopf algebra where $\mathcal{F}(M^\lambda) = s_\lambda$

Hopf structure on $\bigoplus_{n\geq 0} K_0(S_n)$

$K_0(S) = \bigoplus_{n\geq 0} K_0(S_n)$ is the space of S_n-modules up to isomorphism

- Basis: Irreducible modules M^λ
- Structure:

$$M * N = \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}} M \otimes N$$

$$\Delta M = \bigoplus_{k=0}^{n} \mathrm{Res}_{S_k\times S_{n-k}}^{S_n} M$$

- $\mathcal{F}$: $K_0(S) \to Sym$ is an isomorphism of graded Hopf algebra where $\mathcal{F}(M^\lambda) = s_\lambda$

Sym is a strong CHA

Original definition of strong CHA

Consider a graded algebra $A = \bigoplus_{n \geq 0} A_n$

- Each A_n is an algebra.
- $\dim A_0 = 1$ and $\dim A_n < \infty$.
- $\rho_{n,m} : A_n \otimes A_m \hookrightarrow A_{n+m}$; injective algebra homomorphism
- A_{n+m} is projective bilateral submodule of $A_m \otimes A_m$.
- Right and left projective structure of A_{n+m} are compatible.
- There is a Makey formula linking induction and restriction

A is a tower of algebra

Original definition of strong CHA

Consider a tower of algebras $A = \bigoplus_{n\geq 0} A_n$

Let $K_0(A) = \bigoplus_{n\geq 0} K_0(A_n)$ is the space of projective A_n-modules up to isomorphism and modulo short exact sequences

- $K_0(A)$ is a graded Hopf algebra:

$$M * N = \mathrm{Ind}_{A_n\otimes A_m}^{A_{n+m}} M \otimes N$$

$$\Delta M = \bigoplus_{k=0}^{n} \mathrm{Res}_{A_k\otimes A_{n-k}}^{A_n} M$$

- H is a strong CHA if there is an A an isomorphism

$$\mathcal{F}\colon K_0(A) \to H$$

- Sym, $QSym$, $NSym$, are strong CHA.

Original definition of strong CHA

Consider a tower of algebras $A = \bigoplus_{n\geq 0} A_n$

Let $K_0(A) = \bigoplus_{n\geq 0} K_0(A_n)$ is the space of projective A_n-modules up to isomorphism and modulo short exact sequences

- $K_0(A)$ is a graded Hopf algebra:
- H is a strong CHA if there is an A an isomorphism

$$\mathcal{F}\colon K_0(A) \to H$$

THEOREM[B-Lam-Li]

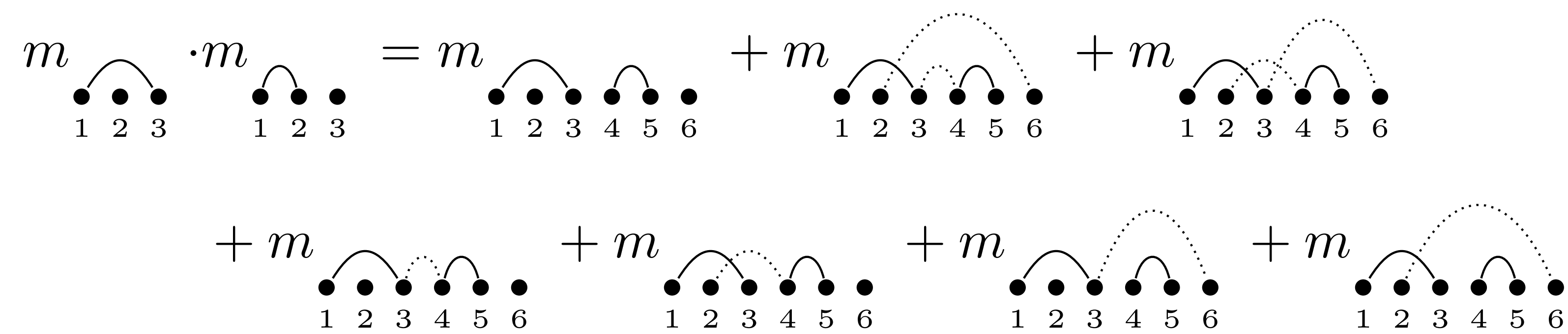
> if A is a tower of algebras, then $\dim(A_n) = r^n n!$

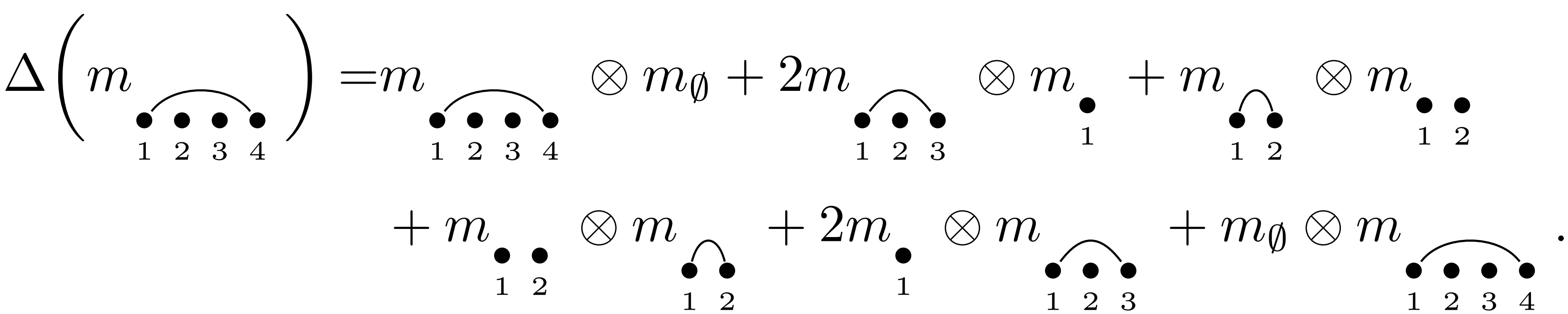
this is very restrictive... For example, it seams hopeless to find a tower of algebras for $NCSym$ (symmetric functions in non-commutative variables) .

How to make $NCSym$ a strong CHA

$NCSym$ symmetric functions in non-commutative variables.

• Basis monomial m_λ basis (sum of orbit of a word) indexed by set partitions





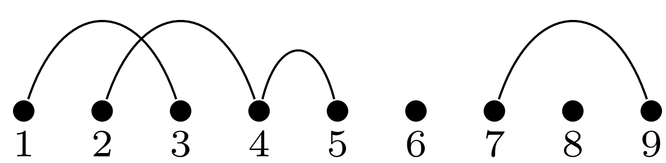
Supercharacter theory of $U_n(q)$

lumping conjugacy classes and characters together to get a more tame theory André, Diaconis-Isaac.

- Unipotent upper triangular matrices over finite Fields $\mathbf{F}_q$: $U_n(q)$.
- Superclasses in $U_n(q)$:

$$A \cong B \quad \leftrightarrow \quad (A - I) = DM(B - I)N$$

superclass representative has at most one 1 in each row and column (strictly above the diagonal).

$\lambda =$ 

Supercharacter theory of $U_n(q)$

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- Unipotent upper triangular matrices over finite Fields $\mathbf{F}_q$: $U_n(q)$.
- Superclasses in $U_n(q)$ λ
- Supercharacters χ^λ Hopf algebra structure see ArXive 28 author paper:

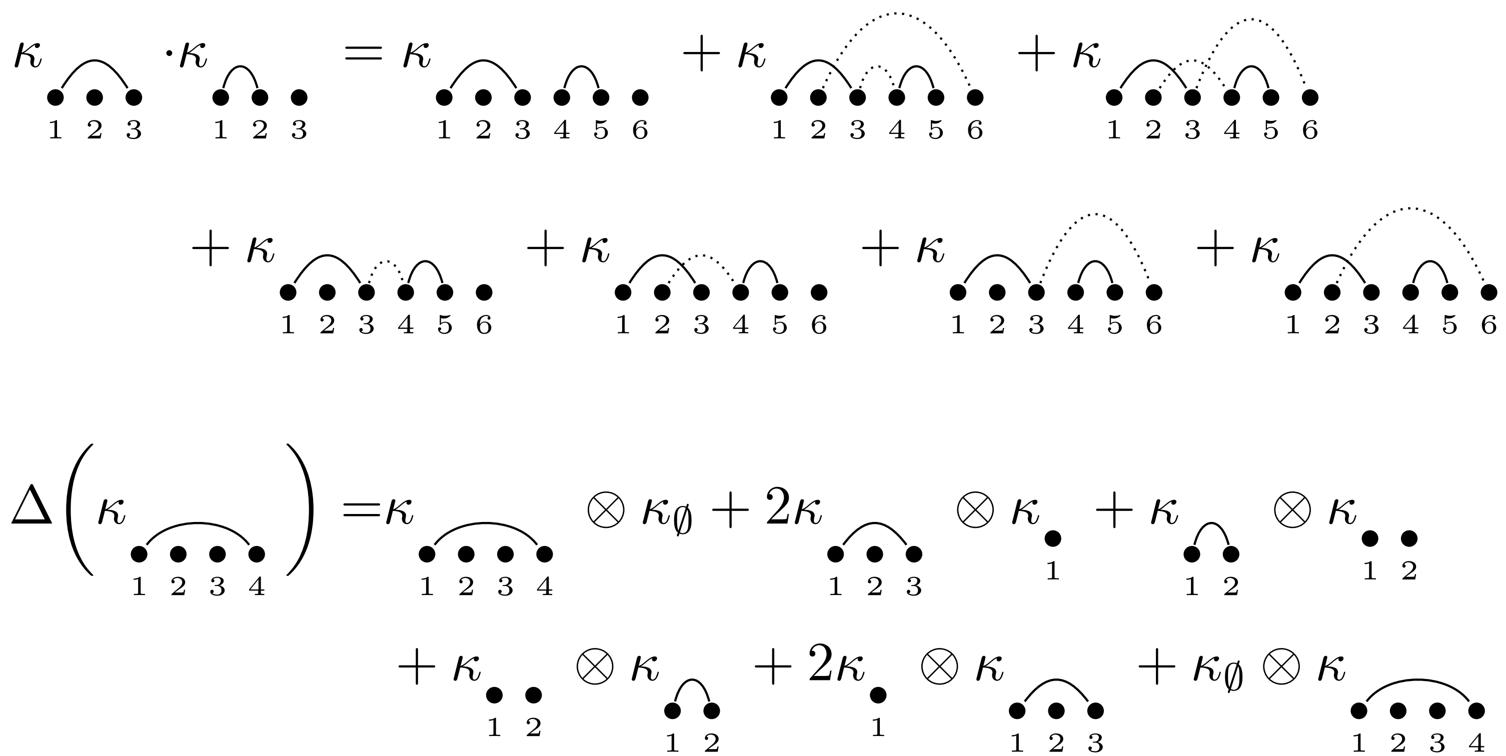
$$\Delta(\chi) = \sum_{A+B=[n]} \mathrm{Res}^{U_n(q)}_{U_{|A|}(q)\times U_{|B|}(q)} \chi$$

$$\chi \cdot \psi = \mathrm{Inf}^{U_{n+m}(q)}_{U_n(q)\times U_m(q)} \chi \otimes \psi = (\chi \otimes \psi) \circ \pi$$

where $\pi : U_{n+m}(q) \to U_n(q) \times U_m(q)$.

Supercharacter theory of $U_n(q)$

- Superclass functions κ_λ basis Hopf algebra structure is nice:



Isomorphism

• the Hopf algebra of symmetric functions in noncommutative variables is isomorphic to the Hopf algebra of superclass functions.

• Where is q? [see nice paper by Bergeron-Thiem to appear in the volume dedicated to CR]

• • •

Conclusions

What is the right definition of strong CHA?

(A) Tower of algebra A (not nesc. same xioms)

(B) K-theory of super-module theory

(C) Harish-Chandra Induction/restriction as operation:

$\mathrm{Ind} \circ \mathrm{Inf}$

$\mathrm{Def} \circ \mathrm{Res}$

Here we try to maximize Inf and Def (in the case of symmetric group, there is no possible Inf, for U_n, it is all Inf)

Gl_n is an example where we see a combination

Thank You!