

Combinatorial Inverse System (commutative and Non-commutative)

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(avec **J. C. Aval** and **H. Li**)

Combinatorial Hopf Algebra

$$H = \bigoplus_{n \geq 0} H_n$$

- **Distinguish homogeneous basis:** $\bigcup_{n \geq 0} \{b^\lambda\}_{\lambda \vdash n}$,
- **Positive graded multiplication:** $b_\lambda * b_\mu = \sum_{\nu} c_{\lambda, \mu}^{\nu} b_\nu$,
 - $c_{\lambda, \mu}^{\nu} \in \mathbb{Z}_{\geq 0}$ et $\deg(b_\nu) = \deg(b_\lambda) + \deg(b_\mu)$.
- **Positive graded comultiplication:** $\Delta(b_\nu) = \sum_{\mu, \lambda} d_{\lambda, \mu}^{\nu} b_\lambda \otimes b_\mu$,
 - $d_{\lambda, \mu}^{\nu} \in \mathbb{Z}_{\geq 0}$ et $\deg(b_\nu) = \deg(b_\lambda) + \deg(b_\mu)$.
- **Compatibility:** Usual axioms of Hopf algebras.

Some Combinatorial Hopf Algebra

Commutative

Sym *QSym* *DSym* *DQSym* ...

Some Combinatorial Hopf Algebra

Commutative

Sym

QSym

DSym

DQSym

...

Some Combinatorial Hopf Algebra

Commutative

Sym $Q\text{Sym}$ $D\text{Sym}$ $DQ\text{Sym}$...

related to tower of representation of symmetric group

nice coinvariant: quotient of polynomial ring by ideal generated by symmetric polynomials

- $\dim = n!$
- left regular representation
- isomorphic to $\{P(\partial)\Delta_n : P \text{ polynomials}\}$
- [...]

Some Combinatorial Hopf Algebra

Commutative

Sym

QSym

DSym

DQSym

...

Some Combinatorial Hopf Algebra

Commutative

Sym *QSym* *DSym* *DQSym* ...

related to tower of representation of Hecke algebra of type A at $q=0$

nice coinvariant: quotient of polynomial ring by ideal generated by quasisymmetric polynomials

- $\dim = C_n$ Catalan number
- representation ???
- isomorphic to $\{P(\partial)\Delta_J : P \text{ polynomials}, J \in C_{n-1}\}$?
- ???

Some Combinatorial Hopf Algebra

Commutative

Sym

QSym



DSym

DQSym

...

Some Combinatorial Hopf Algebra

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Sym *QSym* *DSym* *DQSym* ...

Tower of algebras ???

nice coinvariant: quotient of polynomial ring by ideal generated by diagonally symmetric polynomials

- $\dim = (n - 1)^{n+1}$
- parking functions representation
- isomorphic to $\{P(\partial)E_\lambda\Delta_n : P \text{ polynomials}, \lambda\}$
- basis ???

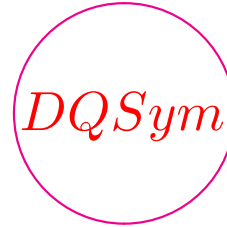
Some Combinatorial Hopf Algebra

Commutative

Sym

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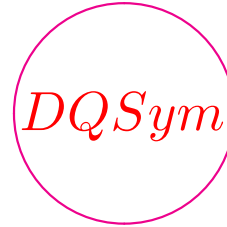
DQSym

...

Some Combinatorial Hopf Algebra

Commutative

Sym *QSym* *DSym* *DQSym* ...



Tower of algebras ???

nice coinvariant: quotient of polynomial ring by ideal generated by diagonally quasisymmetric polynomials

- dim = ???
- representation ???
- ???
- ???

Some Combinatorial Hopf Algebra

Commutative

Sym *QSym* *DSym* *DQSym* ...

NON-Commutative

Some Combinatorial Hopf Algebra

Commutative

Sym *QSym* *DSym* *DQSym* ...

NON-Commutative

NSym *NCSym* *NCQSym* *DNCSym* ...

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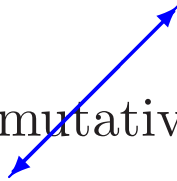
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related to tower of supertheory representation of invertible Upper triangular with entry in $\text{GF}(2)$ AIM [B-D-T-T + A, H, L, ...]

nice coinvariant: • $\dim < \infty?$ or finite G-basis ??

• representation ???

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related to tower of supertheory representation of ???

nice coinvariant: • $\dim < \infty?$ or finite G-basis ??

• representation ???

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Sym *QSym* *DSym* *DQSym* ...

NON-Commutative

NSym *NCSym* *NCQSym* *DNCSym* ...

???

???

???

Inverse System (coinvariants)

$$R = \mathbb{Q}[x_1, x_2, \dots, x_n]$$

$$J = \langle b_\lambda \rangle \text{ homogeneous ideal in } R$$

with scalar product $\langle P, Q \rangle = P(\partial)Q \Big|_{x_1=x_2=\dots=x+n=0}$

$$J^\perp = \{P : \langle Q, P \rangle = 0 \ \forall Q \in J\}$$

$$J^\perp = \{P : b_\lambda(\partial)P = 0\}$$

- $J^\perp \cong R/J$
- Solution of PDE, closed under partial derivatives
- representation...

Combinatorial Inverse System (commutative)

- $H = \bigoplus_{n \geq 0} H_n$ a **Combinatorial Hopf Algebra**.
- **Distinguish homogeneous basis:** $\bigcup_{n \geq 0} \{b^\lambda\}_{\lambda \vdash n}$,
- **Realization:** $H \hookrightarrow \mathbb{Q}[[X]]$ (as subalgebras + comultiplication)
- **Evaluation** $X_1 \subset X_2 \subset \dots$ and $\lim_{n \rightarrow \infty} X_n = X$
$$H \hookrightarrow \mathbb{Q}[[X]] \rightarrow \mathbb{Q}[X_n]$$
- **Sequence of ideals** $J_n = \langle b_\lambda(X_n) \rangle$ ideal of $\mathbb{Q}[X_n]$
- **Combinatorial Inverse System:** $\{J_n^\perp\}$

Combinatorial Inverse System (commutative)

- $H = \bigoplus_{n \geq 0} H_n$ with $\bigcup_{n \geq 0} \{b^\lambda\}_{\lambda \vdash n}$,
- **Evaluation** $X_1 \subset X_2 \subset \dots$ and $\lim_{n \rightarrow \infty} X_n = X$

$$H \hookrightarrow \mathbb{Q}[[X]] \rightarrow \mathbb{Q}[X_n]$$

- **Sequence of ideals** $J_n = \langle b_\lambda(X_n) \rangle$ ideal of $\mathbb{Q}[X_n]$
- **Combinatorial Inverse System:** $\{J_n^\perp\}$
 - $J_n^\perp \subseteq J_{n+1}^\perp$
 - J_n has finite G-basis.
 - $\dim(J_n^\perp) < \infty$?
 - Socle, Representation, ... ?

Non-commutative Inverse System

$$R = \mathbb{Q}\langle x_1, x_2, \dots, x_n \rangle$$

$J = \langle b_\lambda \rangle$ homogeneous two-sided ideal in R

with scalar product $\langle P, Q \rangle = \overleftarrow{P}(d)Q \Big|_{x_1=x_2=\dots=x+n=0}$

$$J^\perp = \{P : \langle Q, P \rangle = 0 \ \forall Q \in J\}$$

$$J^\perp = \{P : d_u \overleftarrow{b_\lambda}(d)P = 0\}$$

- $J^\perp \cong R/J$
- Solution of "PDE", closed under "d"-derivatives
- ...

Combinatorial Inverse System (non-commutative)

- $H = \bigoplus_{n \geq 0} H_n$ with $\bigcup_{n \geq 0} \{b^\lambda\}_{\lambda \vdash n}$,
- **Realization+Evaluation** $X_1 \subset X_2 \subset \dots$ and $\lim_{n \rightarrow \infty} X_n = X$

$$H \hookrightarrow \mathbb{Q}\langle\langle X \rangle\rangle \rightarrow \mathbb{Q}\langle X_n \rangle$$

- **Sequence of ideals** $J_n = \langle b_\lambda(X_n) \rangle$ ideal of $\mathbb{Q}\langle X_n \rangle$
- **Combinatorial Inverse System:** $\{J_n^\perp\}$
 - $J_n^\perp \subseteq J_{n+1}^\perp$
 - J_n has finite G-basis?
 - $\dim(J_n^\perp) < \infty$?
 - Socle, Representation, ... ?

Combinatorial Inverse System (NCSym)

- *NCSym* with $\bigcup_{n \geq 0} \{M^\lambda\}_{\lambda \vdash [n]}$,
- **Realization+Evaluation** $X_1 \subset X_2 \subset \dots$ and $\lim_{n \rightarrow \infty} X_n = X$

$$H \hookrightarrow \mathbb{Q}\langle\langle X \rangle\rangle \rightarrow \mathbb{Q}\langle X_n \rangle$$

- **Sequence of ideals** $J_n = \langle M_\lambda(X_n) \rangle$ ideal of $\mathbb{Q}\langle X_n \rangle$
- **Combinatorial Inverse System:** $\{J_n^\perp\}$
 - $J_n^\perp \subseteq J_{n+1}^\perp$
 - J_n has finite G-basis? [Conjecture: yes!]
 - $\dim(J_n^\perp) < \infty$? [Conjecture??? yes]
 - Socle, Representation, ... ?

Combinatorial Inverse System (NCSym)

- *NCSym* with $\bigcup_{n \geq 0} \{M^\lambda\}_{\lambda \vdash [n]}$,
- **Realization+Evaluation** $X_1 \subset X_2 \subset \dots$ and $\lim_{n \rightarrow \infty} X_n = X$

$$H \hookrightarrow \mathbb{Q}\langle\langle X \rangle\rangle \rightarrow \mathbb{Q}\langle X_n \rangle$$

- **Sequence of ideals** $J_n = \langle M_\lambda(X_n) \rangle$ ideal of $\mathbb{Q}\langle X_n \rangle$
- **Combinatorial Inverse System:** $\{J_n^\perp\}$

$$\dim_t(J_1^\perp) = 1$$

$$\dim_t(J_2^\perp) = 1 + t$$

$$\dim_t(J_3^\perp) = 1 + 2t + 3t^2 + 3t^3$$

$$\dim_t(J_4^\perp) =$$

$$1 + 3t + 8t^2 + 20t^3 + 47t^4 + 102t^5 + 197t^6 + 308t^7 + 248t^8 + 12t^9$$

$$\dim_t(J_5^\perp) = 1 + 4t + 15t^2 + 55t^3 + 199t^4 + 712t^5 + 2520t^6 + \dots$$

Combinatorial Inverse System (NCQSym)

- *NCSym* with $\bigcup_{n \geq 0} \{M^\alpha\}_{\alpha \models [n]}$,
- **Realization+Evaluation** $X_1 \subset X_2 \subset \dots$ and $\lim_{n \rightarrow \infty} X_n = X$

$$H \hookrightarrow \mathbb{Q}\langle\langle X \rangle\rangle \rightarrow \mathbb{Q}\langle X_n \rangle$$

- **Sequence of ideals** $J_n = \langle M_\lambda(X_n) \rangle$ ideal of $\mathbb{Q}\langle X_n \rangle$
- **Combinatorial Inverse System:** $\{J_n^\perp\}$
 - J_n has finite G-basis? [Conjecture: YES!] (~~Theorem~~)
 - $\dim(J_n^\perp) < \infty$? [Conjecture? yes]
 - Socle, Representation, ... ?