Combinatorial Inverse System (commutative and Non-commutative)

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(avec **J. C. Aval and H. Li**)

Combinatorial Hopf Algebra

$H = \bigoplus_{n\geq 0} H_n$

- Distinguish homogeneous basis: $\displaystyle\bigcup_{n\geq 0} \{b^\lambda\}_{\lambda\vdash n}$,
- *Positive* graded multiplication: $\displaystyle b_\lambda * b_\mu = \sum_\nu c^\nu_{\lambda,\mu} b_\nu$,

.

$$c^\nu_{\lambda,\mu} \in \mathbb{Z}_{\geq 0} \quad \text{et} \quad \deg(b_\nu) = \deg(b_\lambda) + \deg(b_\mu).$$

- *Positive* graded comultiplication: $\displaystyle \Delta(b_\nu) = \sum_{\mu,\lambda} d^\nu_{\lambda,\mu} b_\lambda \otimes b_\mu$,

.

$$d^\nu_{\lambda,\mu} \in \mathbb{Z}_{\geq 0} \quad \text{et} \quad \deg(b_\nu) = \deg(b_\lambda) + \deg(b_\mu).$$

- Compatibility: Usual axioms of Hopf algebras.

Some Combinatorial Hopf Algebra

Commutative

Sym $QSym$ $DSym$ $DQSym$ ...

Some Combinatorial Hopf Algebra

Commutative

Sym $\quad QSym \quad DSym \quad DQSym \quad \ldots$

Some Combinatorial Hopf Algebra

Commutative

Sym $QSym$ $DSym$ $DQSym$ ...

related to tower of representation of symmetric group

nice coinvariant: quotient of polynimial ring by ideal generated by symmetric polynomials

- $\dim = n!$
- left regular representation
- isomorphic to $\{P(\partial)\Delta_n : P \ polynomials\}$
- [...]

Some Combinatorial Hopf Algebra

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Sym $QSym$ $DSym$ $DQSym$ ...

Some Combinatorial Hopf Algebra

Commutative

Sym *QSym* *DSym* *DQSym* ...

related to tower of representation of Hecke algebra of type A at q=0

nice coinvariant: quotient of polynimial ring by ideal generated by quasisymmetric polynomials

- dim $= C_n$ Catalan number
- representation ???
- isomorphic to $\big\{P(\partial)\Delta_J : P\ polynomials,\ J \in C_{n-1}\big\}$?
- ???

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Tower of algebras ???

nice coinvariant: quotient of polynimial ring by ideal generated by diagonally symmetric polynomials

- $\dim = (n-1)^{n+1}$
- parking functions representation
- isomorphic to $\{P(\partial)E_\lambda\Delta_n : P\ polynomials,\ \lambda\}$
- basis ???

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Tower of algebras ???

nice coinvariant: quotient of polynimial ring by ideal generated by diagonally quasisymmetric polynomials

- dim = ???
- representation ???
- ???
- ???

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NON-Commutative

Some Combinatorial Hopf Algebra

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related to tower of supertheory representation of invertible Upper triangular with entry in GF(2) AIM [B-D-T-T + A, H, L, ...]

nice coinvariant: • dim $< \infty$? or finite G-basis ??

• representation ???

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related to tower of supertheory representation of ???

nice coinvariant: • dim $< \infty$? or finite G-basis ??

• representation ???

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??? ??? ???

Inverse System (coinvariants)

$R = \mathbb{Q}[x_1, x_2, \ldots, x_n]$

$J = \langle b_\lambda \rangle$ homogeneous ideal in R

with scalar product $\langle P, Q \rangle = P(\partial)Q \Big|_{x_1 = x_2 = \cdots = x+n = 0}$

$J^\perp = \left\{ P : \langle Q, P \rangle = 0 \ \forall Q \in J \right\}$

$$J^\perp = \left\{ P : b_\lambda(\partial) P = 0 \right\}$$

- $J^\perp \cong R \big/ J$
- Solution of PDE, closed under partial derivatives
- representation...

Combinatorial Inverse System (commutative)

- $H = \bigoplus_{n \geq 0} H_n$ a Combinatorial Hopf Algebra.
- Distinguish homogeneous basis: $\bigcup_{n \geq 0} \{b^\lambda\}_{\lambda \vdash n}$,
- Realization: $H \hookrightarrow \mathbb{Q}[\![X]\!]$ (as subalgebras + comultiplication)
- Evaluation $X_1 \subset X_2 \subset \cdots$ and $\lim_{n \to \infty} X_n = X$

$$H \hookrightarrow \mathbb{Q}[\![X]\!] \to \mathbb{Q}[X_n]$$

- Sequence of ideals $J_n = \langle b_\lambda(X_n) \rangle$ ideal of $\mathbb{Q}[X_n]$
- Combinatorial Inverse System: $\{J_n^\perp\}$

Combinatorial Inverse System (commutative)

- $H = \bigoplus_{n\geq 0} H_n$ with $\bigcup_{n\geq 0} \{b^\lambda\}_{\lambda\vdash n}$,
- Evaluation $X_1 \subset X_2 \subset \cdots$ and $\lim_{n\to\infty} X_n = X$

$$H \hookrightarrow \mathbb{Q}[\![X]\!] \to \mathbb{Q}[X_n]$$

- Sequence of ideals $J_n = \langle b_\lambda(X_n)\rangle$ ideal of $\mathbb{Q}[X_n]$
- Combinatorial Inverse System: $\{J_n^\perp\}$
  - $J_n^\perp \subseteq J_{n+1}^\perp$
  - J_n has finite G-basis.
  - $\dim(J_n^\perp) < \infty$?
  - Socle, Representation, ... ?

Non-commutative Inverse System

$R = \mathbb{Q}\langle x_1, x_2, \ldots, x_n \rangle$

$J = \langle b_\lambda \rangle$ homogeneous two-sided ideal in R

with scalar product $\langle P, Q \rangle = \overleftarrow{P}(d)Q\Big|_{x_1=x_2=\cdots=x+n=0}$

$J^\perp = \{P : \langle Q, P \rangle = 0 \ \forall Q \in J\}$

$$J^\perp = \{P : d_u \overleftarrow{b_\lambda}(d)P = 0\}$$

- $J^\perp \cong R/J$
- Solution of "PDE", closed under "d"-derivatives
- ...

Combinatorial Inverse System (non-commutative)

- $H = \bigoplus_{n\geq 0} H_n$ with $\bigcup_{n\geq 0} \{b^\lambda\}_{\lambda\vdash n}$,
- Realization+Evaluation $X_1 \subset X_2 \subset \cdots$ and $\lim_{n\to\infty} X_n = X$

$$H \hookrightarrow \mathbb{Q}\langle\langle X\rangle\rangle \to \mathbb{Q}\langle X_n\rangle$$

- Sequence of ideals $J_n = \langle b_\lambda(X_n)\rangle$ ideal of $\mathbb{Q}\langle X_n\rangle$
- Combinatorial Inverse System: $\{J_n^\perp\}$
  - $J_n^\perp \subseteq J_{n+1}^\perp$
  - J_n has finite G-basis?
  - $\dim(J_n^\perp) < \infty$?
  - Socle, Representation, ... ?

Combinatorial Inverse System (NCSym)

- $NCSym$ with $\bigcup_{n\geq 0} \{M^\lambda\}_{\lambda\vdash[n]}$,
- Realization+Evaluation $X_1 \subset X_2 \subset \cdots$ and $\lim_{n\to\infty} X_n = X$

$$H \hookrightarrow \mathbb{Q}\langle\langle X\rangle\rangle \to \mathbb{Q}\langle X_n\rangle$$

- Sequence of ideals $J_n = \langle M_\lambda(X_n)\rangle$ ideal of $\mathbb{Q}\langle X_n\rangle$
- Combinatorial Inverse System: $\{J_n^\perp\}$
  - $J_n^\perp \subseteq J_{n+1}^\perp$
  - J_n has finite G-basis? [Conjecture: yes!]
  - $\dim(J_n^\perp) < \infty$? [Conjecture??? yes]
  - Socle, Representation, ... ?

Combinatorial Inverse System (NCSym)

- $NCSym$ with $\bigcup_{n\geq 0} \{M^\lambda\}_{\lambda\vdash[n]}$,
- Realization+Evaluation $X_1 \subset X_2 \subset \cdots$ and $\lim_{n\to\infty} X_n = X$

$$H \hookrightarrow \mathbb{Q}\langle\langle X\rangle\rangle \to \mathbb{Q}\langle X_n\rangle$$

- Sequence of ideals $J_n = \langle M_\lambda(X_n)\rangle$ ideal of $\mathbb{Q}\langle X_n\rangle$
- Combinatorial Inverse System: $\{J_n^\perp\}$

$\dim_t(J_1^\perp) = 1$

$\dim_t(J_2^\perp) = 1 + t$

$\dim_t(J_3^\perp) = 1 + 2t + 3t^2 + 3t^3$

$\dim_t(J_4^\perp) =$
$1 + 3t + 8t^2 + 20t^3 + 47t^4 + 102t^5 + 197t^6 + 308t^7 + 248t^8 + 12t^9$

$\dim_t(J_5^\perp) = 1 + 4t + 15t^2 + 55t^3 + 199t^4 + 712t^5 + 2520t^6 + \cdots$

Combinatorial Inverse System (NCQSym)

- $NCSym$ with $\bigcup_{n\geq 0} \{M^\alpha\}_{\alpha\models[n]}$,
- Realization+Evaluation $X_1 \subset X_2 \subset \cdots$ and $\lim_{n\to\infty} X_n = X$

$$H \hookrightarrow \mathbb{Q}\langle\langle X\rangle\rangle \to \mathbb{Q}\langle X_n\rangle$$

- Sequence of ideals $J_n = \langle M_\lambda(X_n)\rangle$ ideal of $\mathbb{Q}\langle X_n\rangle$
- Combinatorial Inverse System: $\{J_n^\perp\}$
  - J_n has finite G-basis? [Conjecture: YES!] (~~Theorem~~)
  - $\dim(J_n^\perp) < \infty$? [Conjecture? yes]
  - Socle, Representation, ... ?