COMBINATORIAL HOPF ALGEBRAS - ECCO'12
EXERCISES LECTURE 2

(1). Can you come up with a product and/or coproduct structure on your favorite combinatorial object (graphs, trees, matroids, ranked posets,...)? Can you use those operations to induce a Hopf algebra structure? If you succeed, in which sense is this a CHA?

(2). Define the Schur function $s_\lambda(\mathbf{x})$ as

$$s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T$$

summing over all the semistandard Young tableaux of shape λ. Give an expansion for the Schur function s_λ in monomials, where:

- $\lambda = (2)$
- $\lambda = (2,1)$
- $\lambda = (2,2)$
- $\lambda =$ any partition. Give a combinatorial description of the coefficients appearing on this expansion.

(3). Compute $s_{(2)}(\mathbf{x})s_{(2)}(\mathbf{x})$ in terms of Schur functions. What is the coefficient of $s_{(4)}(\mathbf{x})$?

(4).
- Let V be a finite dimensional vector space and let $GL(V)$ the space of invertible linear transformations of V. A representation of a group G is a homomorphism $\phi : G \to GL(V)$.
  Let $G = S_n$, $V = \mathbb{C}$ and ϕ such that $\phi(\sigma) = (1)$ for all $\sigma \in S_n$.
  Show that this is a representation of S_n, called the trivial representation $\mathbb{I}_{S_n}$. Show that $\mathbb{I}_{S_n}$ is irreducible in the sense that it does not contain non-trivial subspaces invariant under the action of S_n.
- Using the fact that the trivial representation corresponds to the Schur function $s_{(n)}(\mathbf{x})$ and knowing that

$$\mathrm{Ind}_{S_2\times S_2}^{S_4}(\mathbb{I}_{S_2}\times\mathbb{I}_{S_2}) := \bigoplus_{\sigma_i\in S_4/(S_2\times S_2)} k\{\sigma_i\cdot(S_2\times S_2)\}$$

  where the sum is over a set of coset representatives $\{\sigma_i\}$, find a subspace of $\mathrm{Ind}_{S_2\times S_2}^{S_4}(\mathbb{I}_{S_2}\times\mathbb{I}_{S_2})$ that is isomorphic to the trivial representation of S_4. What is the decomposition (in irreducibles) of this induced representation? (hint: use (3)).
- What is the multiplicity of the trivial representation $\mathbb{I}_{S_4}$ in this decomposition?