COMBINATORIAL HOPF ALGEBRAS - ECCO'12
EXERCISES LECTURE 4

(1). Recall:

SPECIES	$\rightsquigarrow$	SYMBOLIC METHOD	$\rightsquigarrow$	E.G.F.
A		$A[n]$		$A(z) = \sum \dim A[n] z^n/n!$

(a) Show that $(A \cdot B)(z) = A(z) \times B(z)$

Now,

SPECIES	$\rightsquigarrow$	SYMBOLIC METHOD	$\rightsquigarrow$	O.G.F.
A		$A[n]_{S_n}$		$\tilde{A}(z) = \sum \dim A[n]_{S_n} z^n$

($\star$) Can you prove that $\widetilde{A \cdot B}(z) = \tilde{A}(z) \times \tilde{B}(z)$?.

(2). Compute the antipode for the Hopf monoid L.

(3). Can you see E as a Hopf monoid? How about Π?

(4). What is $K[L]$ and $\overline{K}[L]$?

($\star$). Let H be the Hopf algebra of ranked poset (up to isomorphism). Multiplication is given by cartesian product and comultiplication is given by

$$\Delta(P) = \sum_{0 \leq x \leq 1} [0, x] \otimes [x, 1]$$

Now, given a specific character ζ for H and the character φ for $Qsym$, there exists a unique homomorphism $\Psi : H \rightarrow Qsym$ such that $\zeta = \Psi \circ \varphi$. Recall that $\varphi(M_\alpha) = M_\alpha(1, 0, 0, \dots)$.
Consider $\zeta(P) = 1$ for P a single poset.

(a). Let $\Psi(Q) = \sum_\alpha f_\alpha(P) M_\alpha$, where $f_\alpha(Q)$ is the number of chains in Q of the form $0 = x_0 \leq x_1 \leq \cdots \leq x_l = 1$ where $rk(x_i) - rk(x_{i-1}) = \alpha_i$. Show that Ψ is the Hopf morphism satisfying $\zeta = \Psi \circ \varphi$.

(b). Expand Ψ in the F basis which is given by

$$F_\alpha = \sum_{\beta \geq \alpha} M_\beta$$

where $\alpha \leq \beta$ whenever α is a refinement of β.