Affine Crystal Structures and Promotion Operators

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Outline

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1. Type A Crystals
   - Classical Crystals
   - Affine Crystals
2. A Question
3. Outline of Proof

What is a crystal?

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I will say: for $\mathfrak{g}$ a Kac-Moody algebra, a *crystal* is a collection of data which encodes the structure of an integrable, highest-weight $U_q(\mathfrak{g})$-module. For this talk $\mathfrak{g}$ will always be $\mathfrak{sl}_n$ or $\widehat{\mathfrak{sl}}_n$.

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Applications:

- Representation theory
- Topology
- Mathematical physics

Combinatorics of $\mathfrak{sl}_n$ crystals

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Combinatorics of $\mathfrak{sl}_n$ crystals

A *combinatorial* model:

The $\mathfrak{sl}_n$ crystal indexed by the partition λ is

- A directed, edge-colored graph
- Vertex set: all semistandard tableaux of shape λ on the alphabet $\{1, \ldots, n\}$
- Edges: An i-colored edge changes a single tableau entry $i \to i + 1$.

Edges in the crystal graph

1. Consider i's and $i+1$'s in the reading word of the tableau
2. "Bracket" pairs of the form $(i+1, i)$
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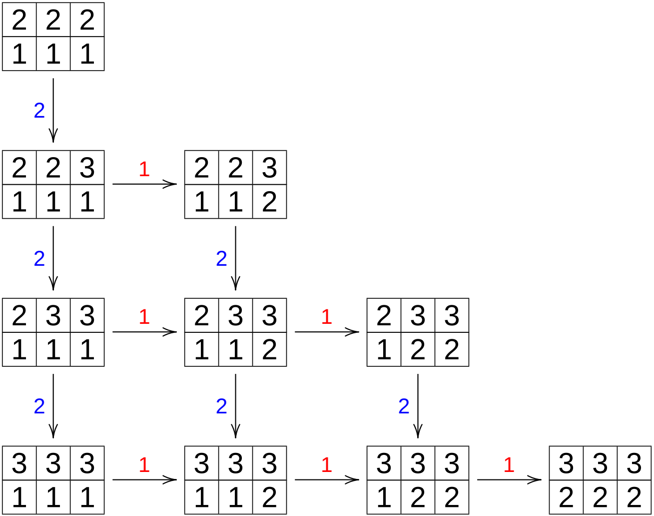
Example

The $\mathfrak{sl}_2$ crystal indexed by the partition $(3, 3)$:



Tensor products

The tensor product $B(\lambda) \otimes B(\mu)$

- Vertex set: $v \otimes w$ with $v \in B(\lambda), w \in B(\mu)$
- Edges: Described exactly as before (concatenate the reading word of v with that of w).

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$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|cc}\hline 2 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \xrightarrow{2}$$

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- Finite dimensional, irreducible $U_q'(\widehat{\mathfrak{sl}}_n)$ modules classified by Chari-Pressley (Drinfeld polynomials)
- Kirillov-Reshetikhin modules: A subset of these *with* a crystal basis

Kirillov-Reshetikhin Crystals

A combinatorial description of the KR crystal for $(\widehat{\mathfrak{sl}}_n)$, $B^{r,s}$

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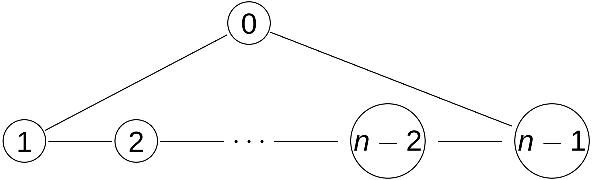
The 0-edges have a wonderful combinatorial description due to Mark Shimozono.

A Dynkin diagram automorphism

The Dynkin diagram for $(\widehat{\mathfrak{sl}}_n)$ is



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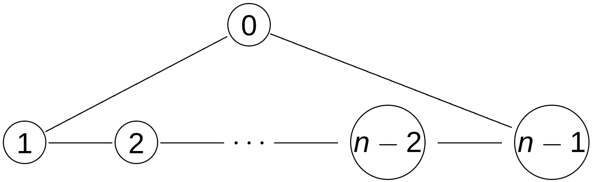

There is an automorphism of this graph (call it Ω) given by rotating all entries one unit counter-clockwise.

The KR crystals have a corresponding automorphism.

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- *pr* shifts arrows: $pr \circ e_i = e_{i+1} \circ pr$ and $pr \circ f_i = f_{i+1} \circ pr$ for $i \in \{1, 2, \ldots, n-1\}$;
- *pr* shifts content: If $wt(b) = (m_1, \ldots, m_n)$ is the content of the crystal element $b \in B$, then $wt(pr(b)) = (m_n, m_1, \ldots, m_{n-1})$;
- $pr^n = id$.

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Properties can be verified from classical data.

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- Remove all n's.
- Play *jeu de taquin.*
- *Increase all entries by 1.*
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Theorem (Shimozono)

This operation satisfies the definition of a promotion operator.

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Back to crystals

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The edge $b \xrightarrow{0} b'$ exists if and only if the edge $\mathrm{pr}(b) \xrightarrow{1} \mathrm{pr}(b')$ exists.

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therefore

$$\begin{array}{|c|c|c|}\hline 3&4&4\\\hline 2&3&3\\\hline 1&2&2\\\hline\end{array} \xrightarrow{0} \begin{array}{|c|c|c|}\hline 3&3&4\\\hline 2&2&3\\\hline 1&1&2\\\hline\end{array}$$

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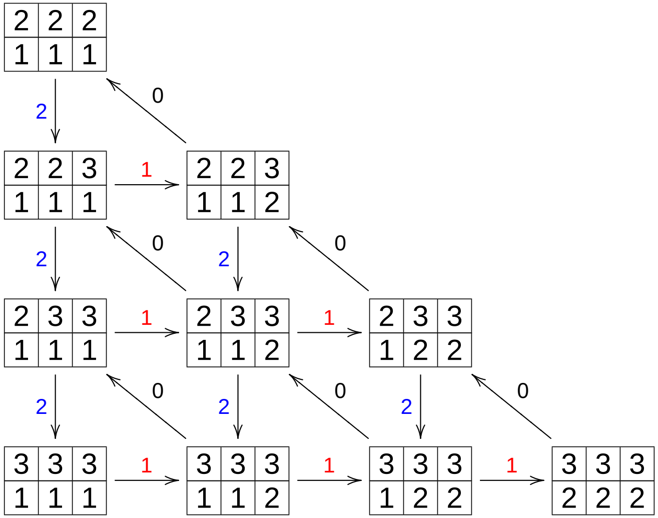
2 2 2
1 1 1
2
0
2 2 3
1 1 1
1
2 2 3
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2
0
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1 1 1
1
2 3 3
1 1 2
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- What can we say about tensor products?

Theorem (BST)

There is a **unique** *promotion operator on the tensor product of two distinct irreducible $\mathfrak{sl}_n$ crystals.*

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- It is conjectured that the only finite-dimensional affine crystals are the KR crystals $B^{r,s}$, and tensor products of these.
- Understand the correspondence with rigged configurations.

Main Ingredients

- Content consideration/“bracketing” arguments
- Reduction to special elements
- Induction
- Duality

Bracketing Arguments

In $B^{1,2} \otimes B^{1,1}$ (as $\mathfrak{sl}_3$ crystals) consider:

$$23 \otimes 1 \xrightarrow{pr} 13 \otimes 2$$

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Fixed by bracketing.

$$13 \otimes 2 \rightarrow ? \rightarrow 23 \otimes 1$$

So we must have

$$13 \otimes 2 \rightarrow 12 \otimes 3$$

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Lemma

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Example:

$$\begin{array}{ccc} 13 \otimes 2 & \xrightarrow{pr} & 12 \otimes 3 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 2} \\ 23 \otimes 2 & \overset{pr}{\dashrightarrow} & 13 \otimes 3 \end{array}$$

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In the crystal $B^{r,s} \otimes B^{r',s'}$, there is a only one possibility for promotion on any element with $s + s'$ occurences of 1.

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Set C_i = set of elements with $s + s'$ copies of i, along with all "internal" edges (those edges not colored i or $i - 1$).

Induction

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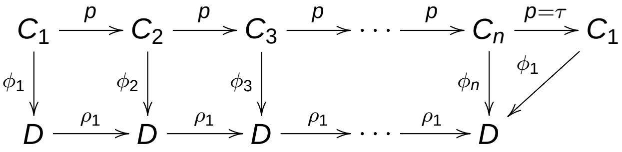
Set $D = B^{r-1,s} \otimes B^{r'-1,s'}$ as an $\mathfrak{sl}_{n-1}$ crystal. There is a *unique* crystal map $\phi_1 : C_1 \to D$.

Given a proposed promotion p, define $\phi_2 : C_2 \to D$ by

$$\phi_2 = \mathrm{pr}_D \circ \phi_1 \circ \mathrm{pr}_C^{-1}$$



Duality

There is an isomorphism of $\mathfrak{sl}_n$ crystals $B^{r,s} \cong B^{n-r,s}$. Thus for a fixed tensor product $B^{r,s} \otimes B^{r',s'}$, it is sufficient to consider $n \leq r + r'$.