Filtrations of (q, t)-Catalan numbers

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The operator ∇

The modified Macdonald polynomial $\widetilde{H}_\lambda(X;t,q)$ is defined by

$$\widetilde{H}_\lambda(X;q,t) = t^{n(\lambda)} J_\lambda\left(\frac{X}{1-1/t}; q, 1/t\right)$$

Definition (Nabla operator)

The operator ∇ is the linear operator defined on the $\widetilde{H}_\lambda(X;q,t)$ basis by

$$\nabla \widetilde{H}_\lambda(X;q,t) = t^{n(\lambda)} q^{n(\lambda')} \widetilde{H}_\lambda(X;q,t)$$

(q,t)-Catalan numbers

Definition

Let n be a positive integer. The (q,t)-Catalan number $C_n(q,t)$ is defined by

$$C_n(q,t) = \langle \, \nabla e_n(X) \, , \, e_n(X) \, \rangle = \langle \, \nabla e_n(X) \, , \, s_{1^n}(X) \, \rangle$$

Example (Catalan numbers for $n = 3$)

$$C_3(q,t) = q^3 + q^2 t + q t^2 + q t + t^3$$

Some generalizations of (q, t)-Catalan numbers

We consider special cases of k-Schur functions.

Let k and n be two integers and λ the partition defined by

$$\lambda = (k^{n \text{ div } k}, n \bmod k)$$

Proposition

$$s_{1^n}^{(k)}(X; t) = t^{n(\lambda)}\omega\left(Q'_{\lambda}(X; 1/t)\right)$$

Definition (Bergeron-D.-Zabrocki)

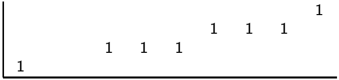
Generalization of (q, t)-Catalan numbers

$$C_n^{(n-k)}(q, t) = \left\langle \nabla s_{1^n}^{(n-k)}(X; 1/t)\ ,\ s_{1^n}(X) \right\rangle$$

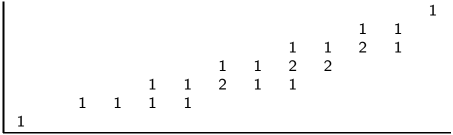
Exemple of generalized (q, t)-Catalan numbers for $n = 6$

$C_6^{(1)}$

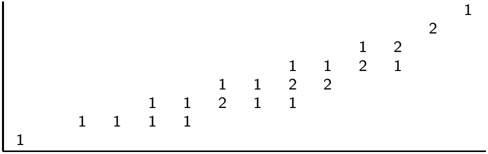
$C_6^{(2)}$



$C_6^{(3)}$



$C_6^{(4)}$



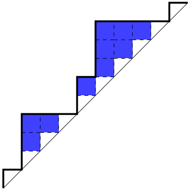
Combinatorial interpretation of (q, t)-Catalan numbers

Theorem (Garsia-Haglund)

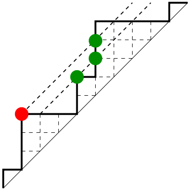
The combinatorial interpretation of $C_n(q,t)$ is given by

$$C_n(q,t) = \sum_{g \in DP_n} q^{\mathrm{area}(g)} \; t^{\mathrm{dinv}(g)}$$

$$g = (0,0,1,2,0,1,1,2,3,0)$$



$\mathrm{area}(g) = 10$



$\mathrm{dinv}(g_3) = 3$ and $\mathrm{dinv}(g) = 15$

How to find a combinatorial interpretation of our generalizations ?

Theorem

For the special case of $k \leq n/2$, we have

$$s_{1^n}^{(n-k)}(X;t) = s_{1^n} + t\ s_{21^{n-2}} + t^2\ s_{221^{n-4}} + \ \ldots\ + t^k\ s_{2^k 1^{n-2k}}$$

Example

$$s_{1111}^{(2)}(X;t) = s_{1111}(X) + t\ s_{211}(X) + t^2\ s_{22}(X)$$

$\longrightarrow$ **Study the combinatorial interpretation of**

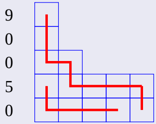
$$\langle\ \nabla s_{\lambda}(\mathbf{X})\ ,\ s_{1^n}(\mathbf{X})\ \rangle$$

(Conjectural interpretation of Loehr - Warrington)

Step 1 : Combinatorial interpretation of $\langle \nabla(s_\lambda), s_{1^n} \rangle$
The global sign

Conjecture (Loehr-Warrington)

For $\lambda = (53222)$ —*conjugate*⟶



- *The global sign is given by sgn*$(\lambda) = (-1)^{\sum_R (h(R)-1)}$

- Dissection sequence for $\lambda = (53222)$

$$n(\lambda) = (9, 0, 0, 5, 0)$$

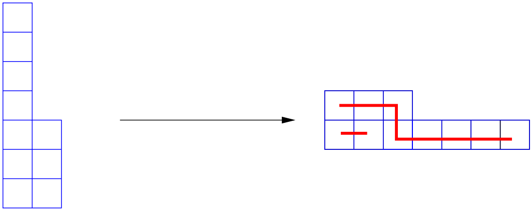
- Adjustement for $\lambda = (53222)$

$$\mathrm{adj}(\lambda) = \sum_{i=0}^{\lambda_1 - 1} (\lambda_1 - i - 1)\ \chi(n_i > 0) = \underbrace{\lambda_1 - 1 - 0}_{i=0} + \underbrace{\lambda_1 - 1 - 3}_{i=3} = 5$$

The global sign in our special case

$$\underbrace{\langle \nabla s_{1^n}^{(n-k)}(X;1/t), s_{1^n}\rangle}_{\geq 0} = \underbrace{\langle \nabla s_{1^n}, s_{1^n}\rangle}_{\geq 0} + \frac{1}{t}\underbrace{\langle \nabla s_{21^{n-2}}, s_{1^n}\rangle}_{\leq 0} + \ldots + \frac{1}{t^k}\underbrace{\langle \nabla s_{2^k1^{n-2k}}, s_{1^n}\rangle}_{\leq 0}$$

We consider the special case of partitions of shape $\lambda = (2^k\ 1^{n-2k})$



- The sign is always negative : $(-1)^{(2-1)+(1-1)} = (-1)^1 = -1$
- The dissection sequence : $n(2^k1^{n-2k}) = (n-k+1,\ k-1)$
- The adjustement sequence : $\mathrm{adj}(2^k1^{n-2k}) = 1$

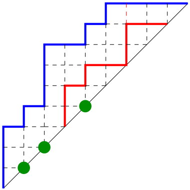
Step 2 : Combinatorial interpretation of $\langle \nabla(s_\lambda), s_{1^n} \rangle$
The Nested Dyck path (1)

Let consider the dissection sequence $n(53222) = (9, 0, 0, 5, 0)$

Definition (Nested Dyck Paths associated with $n(\lambda)$)

5 Dyck paths $G = (g^{(0)}, g^{(1)}, g^{(2)}, g^{(3)}, g^{(4)})$ of size $(9, 0, 0, 5, 0)$

and **non-crossing each others**



$$\longleftrightarrow \quad G = \begin{pmatrix} g^{(0)}: & 0 & 1 & 2 & 2 & 2 & 3 & 4 & 3 & 3 \\ g^{(1)}: & . & . & . & . & . & . & . & . & . \\ g^{(2)}: & . & . & . & . & . & . & . & . & . \\ g^{(3)}: & . & . & . & 0 & 1 & 1 & 0 & 1 & . \\ g^{(4)}: & . & . & . & . & . & . & . & . & . \end{pmatrix}$$

$$\overline{\text{area}}(G) = \sum_i \text{area}\left(g^{(i)}\right)$$

$$\overline{\text{dinv}}(G) = \text{adj}(\lambda) + \sum_i \text{dinv}\left(g^{(i)}\right) + \text{cross dinv}$$

Step 2 : Combinatorial interpretation of $\langle \nabla(s_\lambda), s_{1^n} \rangle$
The Nested Dyck path (2)

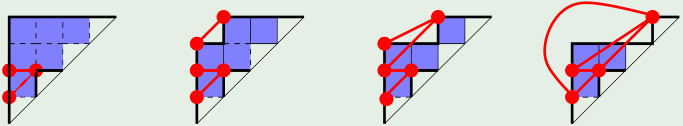
Conjecture (Combinatorial interpretation of $\langle \nabla s_\lambda(X)\ ,\ s_{1^n}(X) \rangle$)

$$\langle\ \nabla s_\lambda(X)\ ,\ s_{1^n}(X)\ \rangle = \sum_{G \in NDP_\lambda} q^{\overline{\mathrm{area}}(G)}\ t^{\overline{\mathrm{dinv}}(G)}$$

Example

$$\begin{aligned} \langle \nabla(s_{221}(X))\ ,\ s_{1^5}(X) \rangle &= -q^6t^3 - q^5t^4 - q^4t^5 - q^3t^6 \\ &= -t\ \left(q^6t^2 + q^5t^3 + q^4t^4 + q^3t^5\right) \end{aligned}$$

The dissection vector $n(221) = (4, 1)$ and $\mathrm{adj}(\lambda) = 1$



The Nested Dyck Paths in the special case $k = 1$

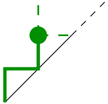
Theorem (Bergeron-D.-Zabrocki)

$$C_n^{(n-1)}(q,t) = \langle \nabla s_{1^n}^{(n-1)}(X; 1/t)\ ,\ s_{1^n} \rangle = \sum_{g \in DP_n^{(1,1)}} q^{\mathrm{area}(g)}\ t^{\mathrm{dinv}(g)}$$

where $DP_n^{(1,1)}$ *is the subset of* DP_n *going throught the point* $(1,1)$

$$\langle \nabla s_{1^n}^{(n-1)}(X; 1/t), s_{1^n} \rangle \quad = \quad \langle \nabla s_{1^n}, s_{1^n} \rangle \quad + \quad \boxed{1/t}\ \langle \nabla s_{21^{n-2}}, s_{1^n} \rangle$$

$$DP_n^{(1,1)} \quad = \quad DP_n \quad \setminus \quad NDP_{21^{n-2}} \subset DP_n \times DP_0$$



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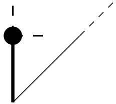

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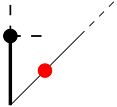


$\mathrm{area}, \ \mathrm{dinv}$ $\qquad$ $\mathrm{area}, \ \mathrm{dinv}$ $\qquad$ $\mathrm{area}(g^{(0)}) = \overline{\mathrm{area}}(G)$

$\mathrm{dinv}(g^{(0)}) = \overline{\mathrm{dinv}}\, G - \boxed{1}$

The Nested Dyck Paths in the special case $k = 2$

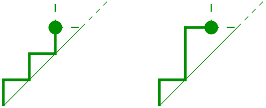
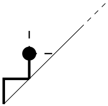
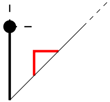
Theorem (Bergeron-D.-Zabrocki)

$$C_n^{(n-2)}(q,t) = \langle \nabla s_{1^n}^{(n-2)}(X; 1/t), s_{1^n} \rangle = \sum_{g \in DP_n^{(1,1),(3,2)}} q^{\mathrm{area}(g)} \; t^{\mathrm{dinv}(g)}$$

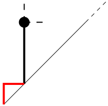
$DP_n^{(1,1),(3,2)}$ *is the subset of* DP_n *going throught* $(1,1)$ *and* $(3,2)$

$$\langle \nabla s_{1^n}^{(n-2)}(X; 1/t), s_{1^n} \rangle = \langle \nabla s_{1^n}^{(n-1)}(X; 1/t), s_{1^n} \rangle + \boxed{1/t^2} \langle \nabla s_{221^{n-4}}, s_{1^n} \rangle$$

$$DP_n^{(1,1),(3,2)} = DP_n^{(1,1)} \setminus NDP_{221^{n-4}} \subset DP_{n-1} \times DP_1$$

 =  − 

$\downarrow \psi$



$$\mathrm{area}(\psi(g^{(0)}, g^{(1)}))$$
$$\mathrm{dinv}(\psi(g^{(0)}, g^{(1)})) = \overline{\mathrm{dinv}}(G) - \boxed{2}$$

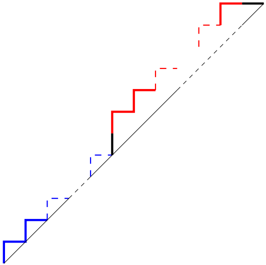
Combinatorial interpretation for $k = n - 2$

Conjecture (Bergeron-D.-Zabrocki)

$$C_n^{(2)}(q,t) = \sum_g q^{\mathrm{area}(g)} t^{\mathrm{dinv}(g)}$$

the sum is over the Dyck paths which are below the path

$$(\underbrace{1\,0\,1\,0\,\ldots\,1\,0}_{n/2-1}\,1\,\underbrace{1\,0\,1\,0\,\ldots\,1\,0}_{n/2}\,0)$$



Combinatorial interpretation at $t = 1$

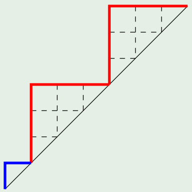
Conjecture (Bergeron-D.-Zabrocki)

$$C_n^{(k)}(q, 1) = \sum_g q^{\mathrm{area}(g)}$$

the sum is over the Dyck paths which are below the path with

- $\mathrm{n\ div\ k}$ *blocks of* k *steps up and* k *steps right*
- *a last block of* $\mathrm{n\ mod\ k}$ *steps right and up*

Example ($n = 7$ and $k = 3$)



New numbers at $t = 1$ and $q = 1$

The triangle of the generalized (q, t)-Catalan at **q=1** and **t=1** is

$n \setminus k$	1	2	3	4	5	6
1	1					
2	1	2				
3	1	2	5			
4	1	4	5	14		
5	1	4	10	14	42	
6	1	8	25	28	42	132

$$C_n^{(n)}(1,1) = C_n \quad , \quad C_n^{(n-1)}(1,1) = C_{n-1} \quad , \quad C_n^{(n-2)}(1,1) = 2C_{n-2}$$

$$\forall k > 2, \quad C_n^{(n-k)}(1,1) = \; ?$$

Subsets of parking functions

Definition (Parking functions of level k)

$$P_n^{(n-k)}(q,t) = \left\langle \nabla s_{1^n}^{(n-k)}\left(X;\frac{1}{t}\right)\ ,\ h_{1^n}(X) \right\rangle$$

$n \setminus k$	1	2	3	4	5	6
1	1					
2	2	3				
3	6	9	16			
4	24	54	64	125		
5	120	270	480	625	1296	
6	720	2430	5120	5625	7776	16807

$P_n^{(n)} = \#\{\text{Parkings functions}\} = (n+1)^{n-1}$, $P_n^{(n-1)} = (n+1)^n$

$$\forall k > 1, \quad P_n^{(n-k)} = ?$$

Algebraic interpretation (1) : The diagonal harmonics

- The Diagonal Harmonics

$$H_n = \{P(X,Y) \in \mathbb{Q}[X,Y] \ / \ \sum_{i=1}^{n} \partial x_i^h \partial y_i^k P = 0 \text{ with } h+k > 0\}$$

- The Alternating Diagonal Harmonics

$$AH_n = \{P(X,Y) \in H_n \ / \ \sigma P(X,Y) = (-1)^{\mathrm{sg}(\sigma)} P(X,Y) \ , \forall \sigma \in S_n\}$$

Theorem

- *AH_n is a bigraded vector space*

$$AH_n = \bigoplus_{i=1}^{\binom{n}{2}} \bigoplus_{j=1}^{\binom{n}{2}} (AH_n)_{i,j}$$

- *The Hilbert series of AH_n is*

$$C_n(q,t) = \mathcal{H}_{AH_n}(q,t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} q^i t^j \dim(AH_n)_{i,j}$$

Algebraic interpretation (2) : Operators conjecture

The Vandermonde determinant in X : $\Delta_n(X) = \prod_{1\leq i<j\leq n}(x_i - x_j)$

The operators E_k : $\forall k > 0,\ E_k = \sum_{i=1}^{n} y_i\, \partial x_i^p$

Theorem (The operator conjecture - Haiman)

- $H_n = \mathcal{L}_{E_1,\ldots,E_{p-1},\partial x_1,\ldots,\partial x_n}(\Delta_n(X))$
- $AH_n = \mathcal{L}_{E_1,\ldots,E_{p-1}}(\Delta_n(X))$

$C_{n|q^j t^i}(q,t)$ $\longleftarrow$ $\dim \mathcal{L}\left\{E_\mu.\Delta_n(X) \mid \mu \in \mathcal{P}_{(i,j)}\right\}$

1 $\longleftarrow$ $\dim \mathcal{L}\{$Vandermonde determinant $\Delta_n(X)\}$

$$\mu \in \mathcal{P}_{(i,j)}^{(n)} \iff |\mu| = i-1 \text{ and } l(\mu) = j-1 \text{ and } 1 \leq \mu_i \leq n-1$$

Algebraic interpretation (3) : Filtration of AH_n

Let k, n with k divides n and $d = n/k$: $AH_n^{(k)} = \mathcal{L}_{E_d,\ldots,E_n}(\Delta_n(X))$

Conjecture (Filtration of AH_n when $k|n$)

$$C_n^{(k)}(q,t) = \dim \mathcal{H}_{AH_n^{(k)}} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} q^i t^j \dim(AH_n^{(k)})_{i,j}$$

The subspace of $A_8^{(4)}$ of bidegree $q^6 t^{10}$ is of dimension 7

				1	1	2	3	5	6	4	2
q^6	$\cdots$	$\cdots$	$\cdots$	2	3	5	6	**7**	3	1	
				5	6	6	5	2	1		
								$\vdots$			
	t^{28}							t^{10}			

$$\text{Rank}\left\{\begin{array}{llll} E_{732222}.\Delta_n(X), & E_{642222}.\Delta_n(X), & E_{633222}.\Delta_n(X), & E_{552222}.\Delta_n(X), \\ E_{543222}.\Delta_n(X), & E_{533322}.\Delta_n(X), & E_{444222}.\Delta_n(X), & E_{443322}.\Delta_n(X), \\ E_{433332}.\Delta_n(X), & E_{333333}.\Delta_n(X) & & \end{array}\right\} = 7$$

Related works with N. Bergeron and M. Zabrocki

1 - Filtrations of (q, t)-Catalan numbers

Preprint : coming soon !

2 - Non-commutative analogs of k-Schur functions
(*related to non-commutative analogs of Hall-Littlewood functions defined by N. Bergeron and M. Zabrocki*)

- The level k is replaced by a composition γ
- Shared most of the properties of the commutative k-Schur functions
- Analogs of positivity properties with the non-commutative ▼

Preprint : arXiv :0804.0944