

Filtrations of (q, t) -Catalan numbers

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The operator ∇

The **modified Macdonald polynomial** $\tilde{H}_\lambda(X; t, q)$ is defined by

$$\tilde{H}_\lambda(X; q, t) = t^{n(\lambda)} J_\lambda \left(\frac{X}{1 - 1/t}; q, 1/t \right)$$

Definition (Nabla operator)

The **operator** ∇ is the linear operator defined on the $\tilde{H}_\lambda(X; q, t)$ basis by

$$\nabla \tilde{H}_\lambda(X; q, t) = t^{n(\lambda)} q^{n(\lambda')} \tilde{H}_\lambda(X; q, t)$$

(q, t) -Catalan numbers

Definition

Let n be a positive integer. The (q, t) -Catalan number $C_n(q, t)$ is defined by

$$C_n(q, t) = \langle \nabla e_n(X), e_n(X) \rangle = \langle \nabla e_n(X), s_{1^n}(X) \rangle$$

Example (Catalan numbers for $n = 3$)

$$C_3(q, t) = q^3 + q^2t + qt^2 + qt + t^3$$

Some generalizations of (q, t) -Catalan numbers

We consider **special cases of k -Schur functions**.

Let k and n be two integers and λ the partition defined by

$$\lambda = (k^{n \operatorname{div} k}, n \bmod k)$$

Proposition

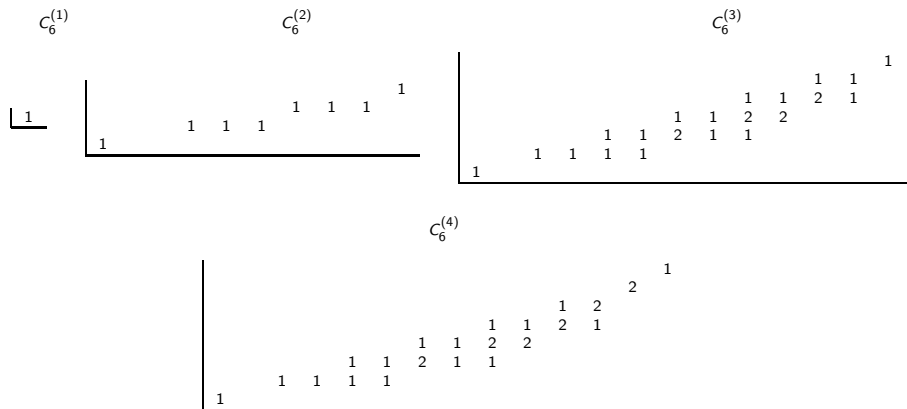
$$s_{1^n}^{(k)}(X; t) = t^{n(\lambda)} \omega(Q'_\lambda(X; 1/t))$$

Definition (Bergeron-D.-Zabrocki)

Generalization of (q, t) -Catalan numbers

$$C_n^{(n-k)}(q, t) = \left\langle \nabla s_{1^n}^{(n-k)}(X; 1/t), s_{1^n}(X) \right\rangle$$

Example of generalized (q, t) -Catalan numbers for $n = 6$



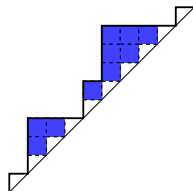
Combinatorial interpretation of (q, t) -Catalan numbers

Theorem (Garsia-Haglund)

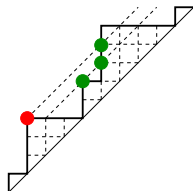
The combinatorial interpretation of $C_n(q, t)$ is given by

$$C_n(q, t) = \sum_{g \in DP_n} q^{\text{area}(g)} t^{\text{dinv}(g)}$$

$$g = (0, 0, 1, 2, 0, 1, 1, 2, 3, 0)$$



$$\text{area}(g) = 10$$



$$\text{dinv}(g_3) = 3 \text{ and } \text{dinv}(g) = 15$$

How to find a combinatorial interpretation of our generalizations?

Theorem

For the special case of $k \leq n/2$, we have

$$s_1^{(n-k)}(X; t) = s_{1^n} + t s_{21^{n-2}} + t^2 s_{221^{n-4}} + \dots + t^k s_{2^k 1^{n-2k}}$$

Example

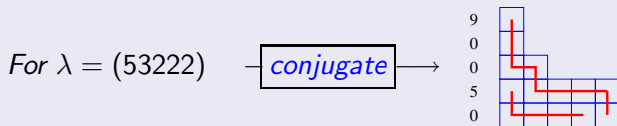
$$s_{1111}^{(2)}(X; t) = s_{1111}(X) + t s_{211}(X) + t^2 s_{22}(X)$$

→ **Study the combinatorial interpretation of**
 $\langle \nabla s_\lambda(\mathbf{X}), s_1^n(\mathbf{X}) \rangle$
(Conjectural interpretation of Loehr - Warrington)

Step 1 : Combinatorial interpretation of $\langle \nabla(s_\lambda), s_{1^n} \rangle$

The global sign

Conjecture (Loehr-Warrington)



- The *global sign* is given by $\text{sgn}(\lambda) = (-1)^{\sum_R(h(R)-1)}$

- *Dissection sequence* for $\lambda = (53222)$

$$n(\lambda) = (9, 0, 0, 5, 0)$$

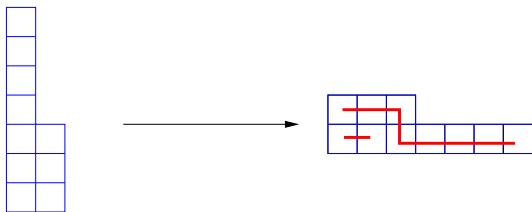
- *Adjustment* for $\lambda = (53222)$

$$\text{adj}(\lambda) = \sum_{i=0}^{\lambda_1-1} (\lambda_1 - i - 1) \chi(n_i > 0) = \underbrace{\lambda_1 - 1 - 0}_{i=0} + \underbrace{\lambda_1 - 1 - 3}_{i=3} = 5$$

The global sign in our special case

$$\underbrace{\langle \nabla_{s_1^{(n-k)}}(X; 1/t), s_1^n \rangle}_{\geq 0} = \underbrace{\langle \nabla_{s_1^n}, s_1^n \rangle}_{\geq 0} + \frac{1}{t} \underbrace{\langle \nabla_{s_{21^{n-2}}}, s_1^n \rangle}_{\leq 0} + \dots + \frac{1}{t^k} \underbrace{\langle \nabla_{s_{2^k 1^{n-2k}}}, s_1^n \rangle}_{\leq 0}$$

We consider the **special case** of partitions of shape $\lambda = (2^k 1^{n-2k})$



- The sign is always **negative** : $(-1)^{(2-1)+(1-1)} = (-1)^1 = -1$
- The **dissection sequence** : $n(2^k 1^{n-2k}) = (n - k + 1, k - 1)$
- The **adjustment sequence** : $\text{adj}(2^k 1^{n-2k}) = 1$

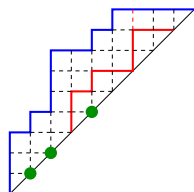
Step 2 : Combinatorial interpretation of $\langle \nabla(s_\lambda), s_{1^n} \rangle$

The Nested Dyck path (1)

Let consider the **dissection sequence** $n(53222) = (9, 0, 0, 5, 0)$

Definition (Nested Dyck Paths associated with $n(\lambda)$)

5 Dyck paths $G = (g^{(0)}, g^{(1)}, g^{(2)}, g^{(3)}, g^{(4)})$ of size $(9, 0, 0, 5, 0)$
and **non-crossing each others**



$$\longleftrightarrow G = \begin{pmatrix} g^{(0)} : & 0 & 1 & 2 & 2 & 2 & 3 & 4 & 3 & 3 \\ g^{(1)} : & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ g^{(2)} : & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ g^{(3)} : & \cdot & \cdot & \cdot & 0 & 1 & 1 & 0 & 1 & \cdot \\ g^{(4)} : & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\overline{\text{area}}(G) = \sum_i \text{area}(g^{(i)})$$

$$\overline{\text{dinv}}(G) = \text{adj}(\lambda) + \sum_i \text{dinv}(g^{(i)}) + \text{cross dinv}$$

Step 2 : Combinatorial interpretation of $\langle \nabla(s_\lambda), s_{1^n} \rangle$ The Nested Dyck path (2)

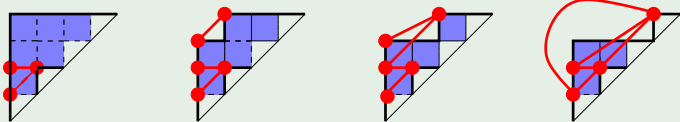
Conjecture (Combinatorial interpretation of $\langle \nabla s_\lambda(X), s_{1^n}(X) \rangle$)

$$\langle \nabla s_\lambda(X), s_{1^n}(X) \rangle = \sum_{G \in NDP_\lambda} q^{\overline{\text{area}}(G)} t^{\overline{\text{dinv}}(G)}$$

Example

$$\begin{aligned} \langle \nabla(s_{221}(X)), s_{1^5}(X) \rangle &= -q^6 t^3 - q^5 t^4 - q^4 t^5 - q^3 t^6 \\ &= -t (q^6 t^2 + q^5 t^3 + q^4 t^4 + q^3 t^5) \end{aligned}$$

The dissection vector $n(221) = (4, 1)$ and $\text{adj}(\lambda) = 1$



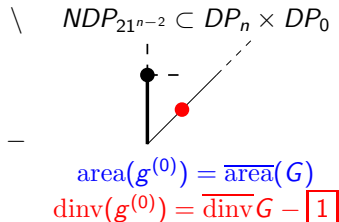
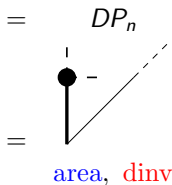
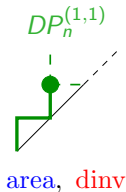
The Nested Dyck Paths in the special case $k = 1$

Theorem (Bergeron-D.-Zabrocki)

$$C_n^{(n-1)}(q, t) = \langle \nabla_{s_1^{(n-1)}}(X; \mathbf{1}/t), s_{1^n} \rangle = \sum_{g \in DP_n^{(1,1)}} q^{\text{area}(g)} t^{\text{dinv}(g)}$$

where $DP_n^{(1,1)}$ is the subset of DP_n going through the point $(1, 1)$

$$\langle \nabla_{s_1^{(n-1)}}(X; \mathbf{1}/t), s_{1^n} \rangle = \langle \nabla_{s_{1^n}}, s_{1^n} \rangle + \boxed{1/t} \langle \nabla_{s_{21^{n-2}}}, s_{1^n} \rangle$$



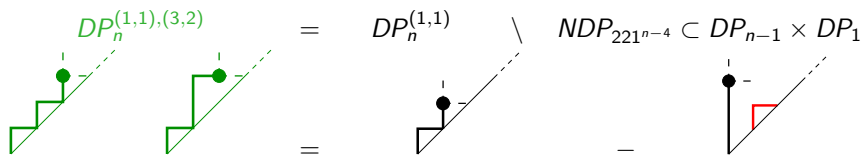
The Nested Dyck Paths in the special case $k = 2$

Theorem (Bergeron-D.-Zabrocki)

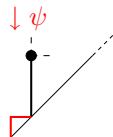
$$C_n^{(n-2)}(q, t) = \langle \nabla s_{1_n}^{(n-2)}(X; \mathbf{1}/t), s_{1_n} \rangle = \sum_{g \in DP_n^{(1,1),(3,2)}} q^{\text{area}(g)} t^{\text{dinv}(g)}$$

$DP_n^{(1,1),(3,2)}$ is the subset of DP_n going through $(1, 1)$ and $(3, 2)$

$$\langle \nabla s_{1_n}^{(n-2)}(X; \mathbf{1}/t), s_{1_n} \rangle = \langle \nabla s_{1_n}^{(n-1)}(X; \mathbf{1}/t), s_{1_n} \rangle + \boxed{1/t^2} \langle \nabla s_{221^{n-4}}, s_{1_n} \rangle$$



$$\text{dinv}(\psi(g^{(0)}, g^{(1)})) = \overline{\text{dinv}(G)} - \boxed{2}$$



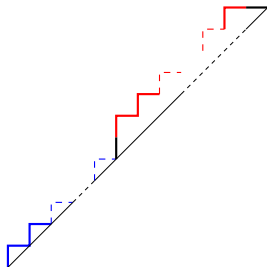
Combinatorial interpretation for $k = n - 2$

Conjecture (Bergeron-D.-Zabrocki)

$$C_n^{(2)}(q, t) = \sum_g q^{\text{area}(g)} t^{\text{dinv}(g)}$$

the sum is over the *Dyck paths* which are below the path

$$\underbrace{(1\ 0\ 1\ 0\ \dots\ 1\ 0)}_{n/2-1} 1 \underbrace{1\ 0\ 1\ 0\ \dots\ 1\ 0}_0$$



Combinatorial interpretation at $t = 1$

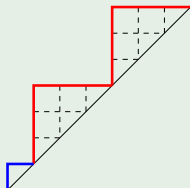
Conjecture (Bergeron-D.-Zabrocki)

$$C_n^{(k)}(q, \mathbf{1}) = \sum_g q^{\text{area}(g)}$$

the sum is over the *Dyck paths* which are below the path with

- $n \text{ div } k$ blocks of k steps up and k steps right
- a *last* block of $n \text{ mod } k$ steps right and up

Example ($n = 7$ and $k = 3$)



New numbers at $t = 1$ and $q = 1$

The triangle of the generalized (q, t) -Catalan at $q=1$ and $t=1$ is

$n \setminus k$	1	2	3	4	5	6
1	1					
2	1	2				
3	1	2	5			
4	1	4	5	14		
5	1	4	10	14	42	
6	1	8	25	28	42	132

$$C_n^{(n)}(1, 1) = C_n \quad , \quad C_n^{(n-1)}(1, 1) = C_{n-1} \quad , \quad C_n^{(n-2)}(1, 1) = 2C_{n-2}$$

$$\forall k > 2, \quad C_n^{(n-k)}(1, 1) = ?$$

Subsets of parking functions

Definition (Parking functions of level k)

$$P_n^{(n-k)}(q, t) = \left\langle \nabla_{S_{1^n}^{(n-k)}} \left(X; \frac{1}{t} \right), h_{1^n}(X) \right\rangle$$

$n \setminus k$	1	2	3	4	5	6
1	1					
2	2	3				
3	6	9	16			
4	24	54	64	125		
5	120	270	480	625	1296	
6	720	2430	5120	5625	7776	16807

$$P_n^{(n)} = \# \{\text{Parkings functions}\} = (n+1)^{n-1}, \quad P_n^{(n-1)} = (n+1)^n$$

$$\forall k > 1, \quad P_n^{(n-k)} = ?$$

Algebraic interpretation (1) : The diagonal harmonics

- The **Diagonal Harmonics**

$$H_n = \{P(X, Y) \in \mathbb{Q}[X, Y] / \sum_{i=1}^n \partial x_i^h \partial y_i^k P = 0 \text{ with } h + k > 0\}$$

- The **Alternating Diagonal Harmonics**

$$AH_n = \{P(X, Y) \in H_n / \sigma P(X, Y) = (-1)^{\text{sg}(\sigma)} P(X, Y), \forall \sigma \in S_n\}$$

Theorem

- AH_n is a **bigraded vector space**

$$AH_n = \bigoplus_{i=1}^n \bigoplus_{j=1}^n (AH_n)_{i,j}$$

- The **Hilbert series** of AH_n is

$$C_n(q, t) = \mathcal{H}_{AH_n}(q, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} q^i t^j \dim(AH_n)_{i,j}$$

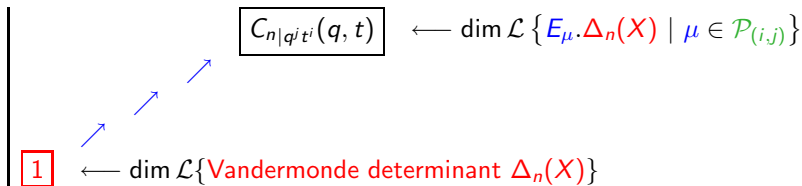
Algebraic interpretation (2) : Operators conjecture

The **Vandermonde determinant** in X : $\Delta_n(X) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$

The operators $E_k : \forall k > 0, E_k = \sum_{i=1}^n y_i \partial x_i^k$

Theorem (The operator conjecture - Haiman)

- $H_n = \mathcal{L}_{E_1, \dots, E_{p-1}, \partial x_1, \dots, \partial x_n}(\Delta_n(X))$
- $AH_n = \mathcal{L}_{E_1, \dots, E_{p-1}}(\Delta_n(X))$



$$\mu \in \mathcal{P}_{(i,j)}^{(n)} \iff |\mu| = i - 1 \text{ and } l(\mu) = j - 1 \text{ and } 1 \leq \mu_i \leq n - 1$$

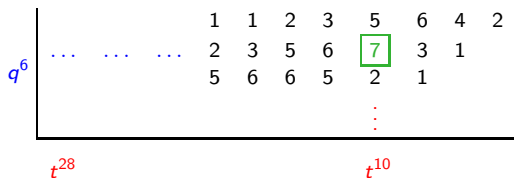
Algebraic interpretation (3) : Filtration of AH_n

Let k, n with k divides n and $d = n/k$: $AH_n^{(k)} = \mathcal{L}_{E_d, \dots, E_n}(\Delta_n(X))$

Conjecture (Filtration of AH_n when $k|n$)

$$C_n^{(k)}(q, t) = \dim \mathcal{H}_{AH_n^{(k)}} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} q^i t^j \dim(AH_n^{(k)})_{i,j}$$

The subspace of $A_8^{(4)}$ of bidegree $q^6 t^{10}$ is of dimension 7



$$\text{Rank} \left\{ \begin{array}{cccc} E_{732222} \cdot \Delta_n(X), & E_{642222} \cdot \Delta_n(X), & E_{633222} \cdot \Delta_n(X), & E_{552222} \cdot \Delta_n(X), \\ E_{543222} \cdot \Delta_n(X), & E_{533322} \cdot \Delta_n(X), & E_{444222} \cdot \Delta_n(X), & E_{443322} \cdot \Delta_n(X), \\ E_{433332} \cdot \Delta_n(X), & E_{333333} \cdot \Delta_n(X) & & \end{array} \right\} = 7$$

1 - Filtrations of (q, t) -Catalan numbers

Preprint : [coming soon!](#)

2 - Non-commutative analogs of k -Schur functions

(related to non-commutative analogs of Hall-Littlewood functions defined by N. Bergeron and M. Zabrocki)

- The **level k** is replaced by a **composition γ**
- Shared most of the properties of the commutative k -Schur functions
- Analogs of positivity properties with the non-commutative ▼

Preprint : [arXiv :0804.0944](#)