A bijection on core partitions

Brant C. Jones
brant@math.ucdavis.edu

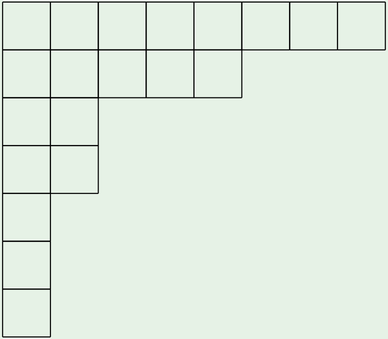
Joint with Chris Berg and Monica Vazirani
University of California, Davis

May 3, 2008

Notation

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r)$ be a *partition* of n and $\ell \geq 2$ be an integer.

Example



The ℓ-*residue* of a box (i, j) is the least nonnegative integer $\equiv j - i \mod \ell$.

Notation

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r)$ be a *partition* of n and $\ell \geq 2$ be an integer.

Example

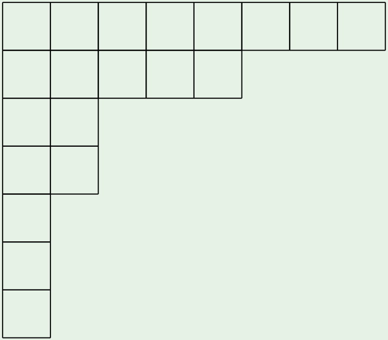
0	1	2	3	0	1	2	3
3	0	1	2	3			
2	3						
1	2						
0							
3							
2							

The ℓ-*residue* of a box (i, j) is the least nonnegative integer $\equiv j - i \mod \ell$.

Notation

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r)$ be a *partition* of n and $\ell \geq 2$ be an integer.

Example



The *hook length* of a box (i, j) is the number of boxes to the **right and below** the box, including itself. It is denoted $h^{\lambda}_{(i,j)}$.

Notation

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r)$ be a *partition* of n and $\ell \geq 2$ be an integer.

Example

14	10	7	6	5	3	2	1
10	6	3	2	1			
6	2						
5	1						
3							
2							
1							

The *hook length* of a box (i, j) is the number of boxes to the **right and below** the box, including itself. It is denoted $h^{\lambda}_{(i,j)}$.

Definition

A partition λ is an ℓ-*core* if $\ell \nmid h^{\lambda}_{(i,j)}$ for every box (i,j) of λ.

Example

14	10	7	6	5	3	2	1
10	6	3	2	1			
6	2						
5	1						
3							
2							
1							

λ is a 4-core.

Question

Given an ℓ-core, how can we project to obtain an $(\ell - 1)$-core?

Example

9	**5**	**3**	**1**
7	3	1	
5	**1**		
3			
2			
1			

4-core

$\rightarrow$

5	**2**	**1**
2		
1		

3-core

$\rightarrow$

1

2-core

ℓ-core partitions index:

- Schubert cells in the affine Grassmannian Gr of $SL(\ell, \mathbb{C})$. ($\mathrm{Gr} \cong SL_\ell(\mathbb{C}((t)))/SL_\ell(\mathbb{C}[[t]])$.)
- k-Schur functions and dual k-Schur functions in $H_*(\mathrm{Gr}) \cong \Lambda_\ell$ and $H^*(\mathrm{Gr}) \cong \Lambda^\ell$, respectively.
- Blocks in the representation theory of the symmetric group S_n over a field of characteristic $\ell > 0$.

Let $\mathbf{F}S_n$ be the group algebra of the symmetric group of a field $\mathbf{F}$ with characteristic $\ell \geq 0$. Let M be a $\mathbf{F}S_n$-module.

Then there is a commutative subalgebra of $\mathbf{F}S_n$ generated by *Jucys-Murphy elements* $\{L_1, L_2, \ldots, L_n\}$.

By the Jordan-Chevalley decomposition, $M = \oplus_{\mathbf{i}=(i_1,\ldots,i_n)} M[\mathbf{i}]$ where $M[\mathbf{i}] = \{v \in M : (L_j - i_j)^N v = 0 \text{ for sufficiently large } N\}$.

- The weights $\mathbf{i} = (i_1, \ldots, i_n)$ correspond to (residues of) *standard tableaux on ℓ-regular partitions*.
- The S_n-orbits of the weights are parameterized by $\Gamma = \ell$-cores.

Then,

$$M = \oplus_{\gamma \in \Gamma} M[\gamma] \text{ where } M[\gamma] = \oplus_{\mathbf{i} \in \gamma} M[\mathbf{i}].$$

- $\mathcal{C}_\ell$ = The set of all ℓ-cores.
- $\mathcal{C}_\ell^k$ = The subset of $\mathcal{C}_\ell$ having first part k.
- $\mathcal{C}_{\ell-1}^{\leq k}$ = The subset of $\mathcal{C}_{\ell-1}$ having first part $\leq k$.

We will define a bijection

$$\Phi_\ell^k : \mathcal{C}_\ell^k \to \mathcal{C}_{\ell-1}^{\leq k}$$

Then,

$$\sum_{k \geq 0} |C_\ell^k| x^k = \sum_{k \geq 0} \binom{k+\ell-2}{k} x^k = \frac{1}{(1-x)^{\ell-1}}.$$

Beta numbers and Abaci

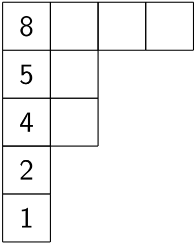
The tableau is determined by first column hooklengths. These can be generalized to β-numbers.

8			
5			
4			
2			
1			

⋯	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	⋯
⋯	○	○	○	●	○	○	●	○	○	●	●	○	●	●	●	⋯

Beta numbers and Abaci

	Runner 0	Runner 1	Runner 2
	⋮	⋮	⋮
Level -1 →	(-3)	(-2)	(-1)
Level 0 →	0	(**1**)	(**2**)
Level 1 →	3	(**4**)	(**5**)
Level 2 →	6	7	(**8**)
Level 3 →	9	10	11
	⋮	⋮	⋮

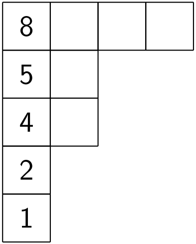


3-core

⋯	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	⋯
⋯	○	○	○	●	○	○	●	○	○	●	●	○	●	●	●	⋯

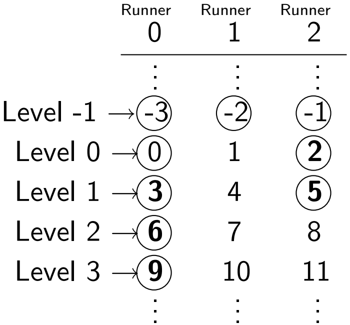
Beta numbers and Abaci

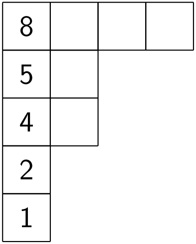
	Runner 0	Runner 1	Runner 2
	⋮	⋮	⋮
Level -1 →	**-3**	**-2**	**-1**
Level 0 →	0	**1**	**2**
Level 1 →	3	**4**	**5**
Level 2 →	6	7	**8**
Level 3 →	9	10	11
	⋮	⋮	⋮



The abacus for $\beta = (\mathbf{8}, \mathbf{5}, \mathbf{4}, \mathbf{2}, \mathbf{1}, -1, -2, -3, \ldots)$ has balance number $2 = (-1) + 1 + 2$.

Beta numbers and Abaci





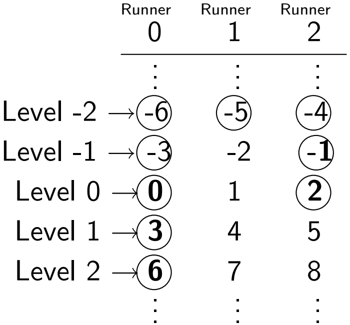
The abacus for $\beta = (\mathbf{8}, \mathbf{5}, \mathbf{4}, \mathbf{2}, \mathbf{1}, -1, -2, -3, \ldots)$ has balance number 2.
The abacus for $\beta = (9, 6, 5, 3, 2, 0, -1, -2, \ldots)$ has balance number $3 = 3 + (-1) + 1$.

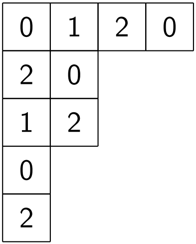
Theorem

Theorem 2.7.16, Lemma 2.7.38 in James–Kerber

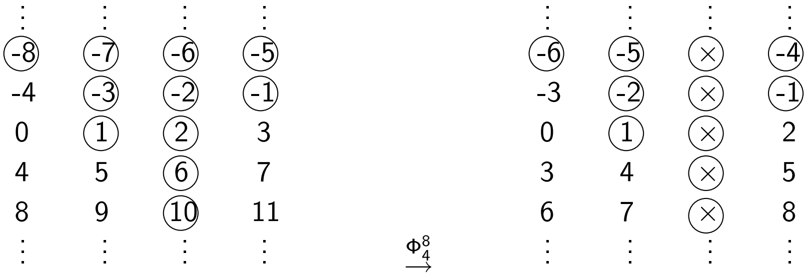
- *λ is an ℓ-core if and only if any (equivalently, every) abacus of λ on ℓ runners is flush.*
- *Moreover, in the balanced flush abacus of an ℓ-core λ, each active bead on runner i corresponds to a row of λ whose rightmost box has residue i.*

Beta numbers and Abaci

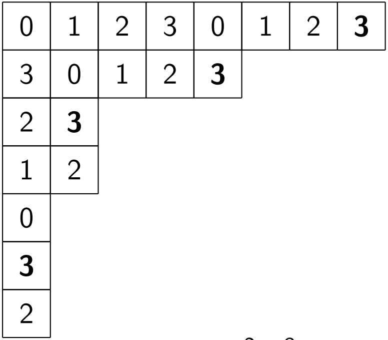




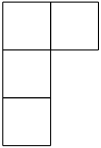
3-core



4-core $(8, 5, 2^2, 1^3)$ $\xrightarrow{\Phi_4^8}$ 3-core $(2, 1^2)$.



$\xrightarrow{\Phi_4^8}$

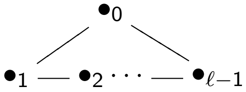


4-core $(8, 5, 2^2, 1^3)$ 3-core $(2, 1^2)$.

The affine symmetric group

The *affine symmetric group* $\widetilde{S}_\ell$ is presented as a Coxeter group by:

- Generators $s_0, s_1, \ldots, s_{\ell-1}$.
- Involution relations $s_i^2 = 1$.
- Commuting relations $s_i s_j = 1$ if $|i - j| \geq 2$.
- Braid relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_0 s_{\ell-1} s_0 = s_{\ell-1} s_0 s_{\ell-1}$.



This is an infinite Coxeter group.

- Let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell\}$ be an orthonormal basis of the Euclidean space $\mathbf{R}^\ell$ with inner product $(\cdot, \cdot)$.
- For $1 \leq i \leq \ell - 1$, let s_i be the reflection that interchanges ε_i and ε_{i+1}
- Let s_0 be the affine reflection of $\mathbf{R}^\ell$ defined on $v = \sum_{j=1}^{\ell} a_j \varepsilon_j$ by

$$s_0(v) = (a_\ell + 1)\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_{\ell-1}\varepsilon_{\ell-1} + (a_1 - 1)\varepsilon_\ell.$$

- The *simple roots* Δ *of type* $A_{\ell-1}$ are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \ldots, \quad \alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell.$$

- The $\mathbf{Z}$-span Λ_R of Δ is called the *root lattice of type* $A_{\ell-1}$.

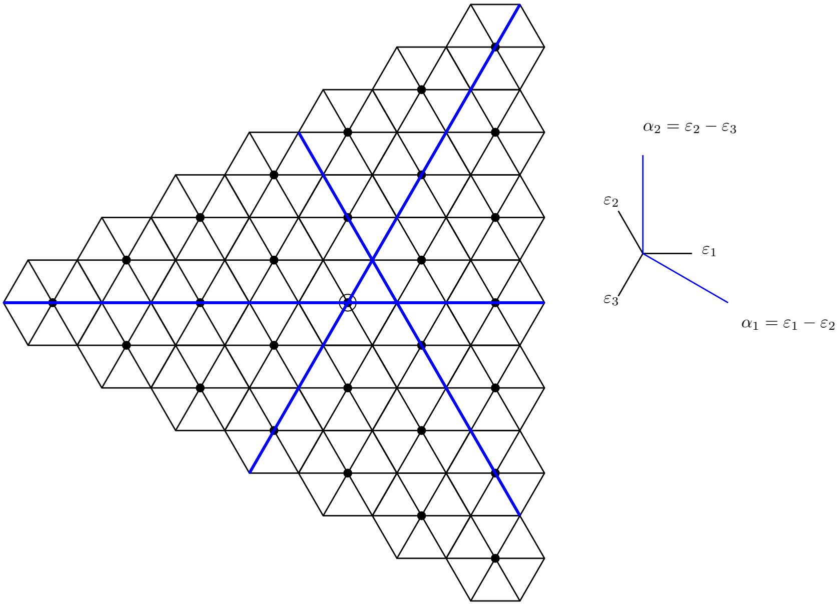
Let $V = \mathbf{R} \otimes_{\mathbf{Z}} \Lambda_R \subsetneq \mathbf{R}^\ell$. Observe that each reflection s_i preserves V. Then $\{s_0, s_1, \ldots, s_{\ell-1}\}$ are a set of Coxeter generators for the affine symmetric group $\widetilde{S}_\ell$ acting on V.

In this presentation, S_ℓ is a parabolic subgroup of $\widetilde{S}_\ell$. The "parabolic quotient"

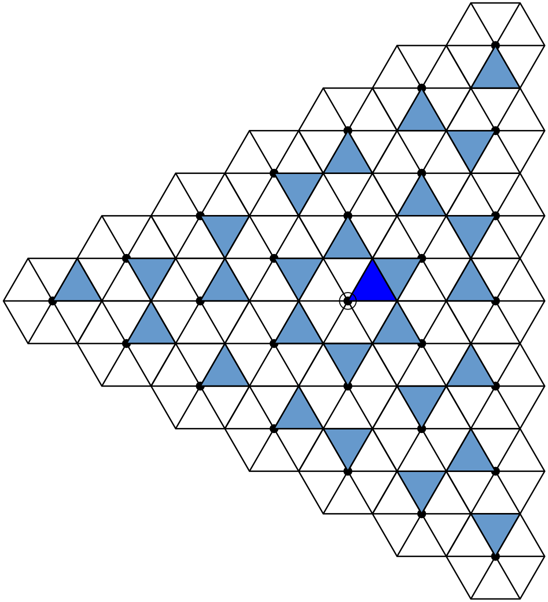
$$\widetilde{S}_\ell / S_\ell = \{w \in \widetilde{S}_\ell : l(ws_i) > l(w) \text{ for all } s_i \text{ where } 1 \leq i \leq \ell - 1\}$$

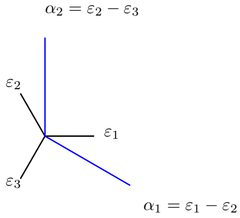
gives a unique representative of minimal length from each coset wS_ℓ of $\widetilde{S}_\ell$. Here $l(w)$ is the minimum *length* of any expression for w as a product of the s_i.

On the other hand, $\widetilde{S}_\ell = S_\ell \ltimes \Lambda_R$ so $\widetilde{S}_\ell / S_\ell$ is in bijection with Λ_R.



$\alpha_2 = \varepsilon_2 - \varepsilon_3$
ε_2
ε_1
ε_3
$\alpha_1 = \varepsilon_1 - \varepsilon_2$




$\alpha_2 = \varepsilon_2 - \varepsilon_3$
ε_2
ε_1
ε_3
$\alpha_1 = \varepsilon_1 - \varepsilon_2$

Let $\mathbf{a} = (a_1, \ldots, a_\ell) \in \Lambda_R$ written in the ε_i basis, so each $a_i \in \mathbf{Z}$ and $\sum_{i=1}^{\ell} a_i = 0$.

We form a balanced flush abacus from $\mathbf{a}$ by filling the $(i-1)^{\mathrm{st}}$ runner with beads from $-\infty$ down to level a_i.

This defines a bijection

$$\pi : \{(a_1, \ldots, a_\ell) : a_i \in \mathbf{Z}, \sum_{i=1}^{\ell} a_i = 0\} \to \{\text{balanced flush abaci}\} \to \mathcal{C}_\ell.$$

Definition

Fix $\lambda \in \mathcal{C}_\ell$ and suppose $\mathbf{b} = (b_0, \ldots, b_{\ell-1}) = \pi^{-1}(\lambda)$.

For $0 \leq i \leq \ell - 1$, we say that s_i is an *ascent* for λ if $b_{i-1} > b_i - \delta_{i,0}$, and we say that s_i is a *descent* for λ if $b_{i-1} < b_i - \delta_{i,0}$. Here, we interpret b_{-1} as $b_{\ell-1}$.

Example

The 4-core $(5,2,1,1,1)$ corresponds to $(2,0,0,-2) \in \Lambda_R$ with minimal length coset representative $w = s_0 s_1 s_2 s_3 s_2 s_1 s_0$. Hence, s_1 and s_3 are ascents for λ and s_0 is a descent for λ.

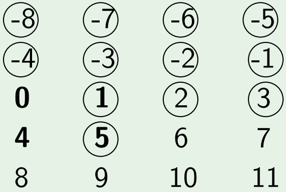
Proposition

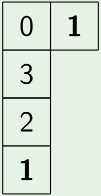
Let λ be an ℓ-core.

- *If s_i is an* **ascent** *for λ then s_i acts on λ by* **adding all boxes with residue** *i to λ such that the result is a partition.*
- *If s_i is a* **descent** *for λ then s_i acts on λ by* **removing all of the boxes with residue** *i that lie at the end of both their row and column so that their removal results in a partition.*
- *If s_i is* **neither an ascent nor a descent** *for λ then s_i* **does not change** *λ.*

Example

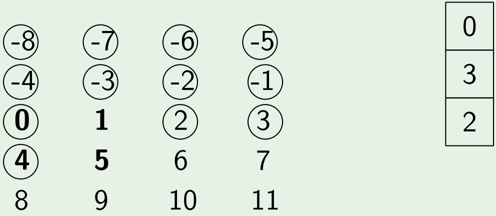
$\ell = 4,\ s_1(-1, 1, 0, 0) = (1, -1, 0, 0).$





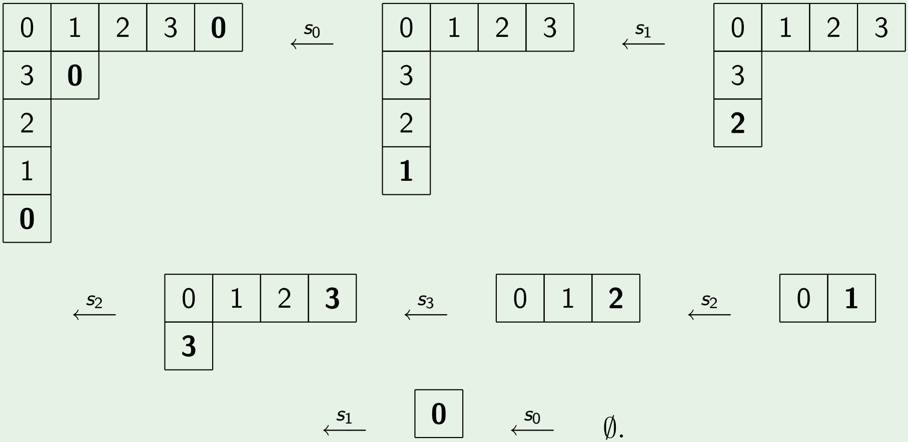
Example

$\ell = 4,\ s_1(-1, 1, 0, 0) = (1, -1, 0, 0).$



Example

The *canonical reduced expression* $w(\lambda)$ for the 4-core $\lambda = (5, 2, 1, 1, 1)$ is $s_0 s_1 s_2 s_3 s_2 s_1 s_0$.



Proposition

For $\lambda \in \mathcal{C}_\ell$ we have that $w(\lambda)$ is a reduced expression for the minimal length coset representative indexed by λ.
The Coxeter length of $w(\lambda)$ is

$$l(w(\lambda)) = \sum_{i=0}^{\ell-1} \lambda_{R(i)}$$

where $R(i)$ is the longest row of λ whose rightmost box has residue i.

Corollary

Applying Φ_ℓ^k reduces the Coxeter length by exactly k.

Example

Let $\ell = 5$.

$\lambda =$

$\dot{0}$	$\dot{1}$	$\dot{2}$	$\dot{3}$	$\dot{4}$	$\dot{0}$	$\dot{1}$	$\dot{2}$	$\dot{3}$
$\dot{4}$	$\dot{0}$	$\dot{1}$	$\dot{2}$	$\dot{3}$				
0 3	1 4	2 0						
$\dot{2}$	$\dot{3}$							
3 1	0 2							
2 0								
1 4								
$\dot{3}$								
0 2								

$\Phi_\ell^k(\lambda) =$

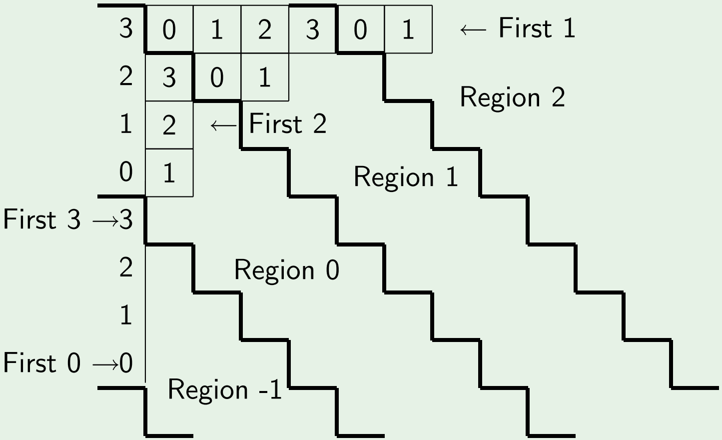
0	1	2
3	0	
2		
1		
0		

$$w(\lambda) = w = \mathbf{s_2} s_3 \mathbf{s_4} \mathbf{s_0} \mathbf{s_1} s_2 \mathbf{s_4} \mathbf{s_3} s_1 s_0 s_4 s_3 s_2 s_1 s_0 \mapsto w(\Phi_\ell^k(\lambda)) = s_0 s_1 s_2 s_3 s_1 s_0$$

The Coxeter length has been reduced by $15 - 6 = 9 = \lambda_1$.

Example

Let $\ell = 4$ and $\lambda = (6, 3, 1, 1)$. From the picture below, we see that the n-vector for λ is $(-1, 2, 0, -1)$.



Proposition

Suppose that $\pi(\mathbf{a}) = \pi(a_1, \ldots, a_\ell) = \lambda$. *Then we have*

$$\lambda_1 = (a_i - 1)\ell + i$$

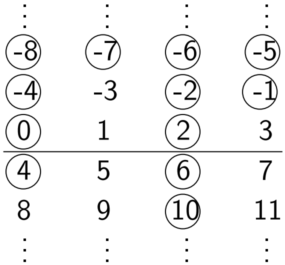
where a_i *is the rightmost occurrence of the largest coordinate in* $\mathbf{a}$.

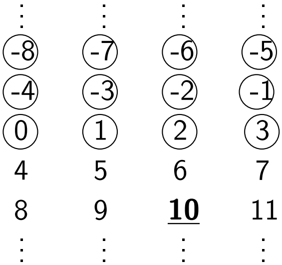
Corollary

For $k \geq 0$, *let* H_ℓ^k *denote the affine hyperplane*

$$H_\ell^k = \{\mathbf{a} = (a_1, \ldots, a_\ell) \in \mathbf{R}^\ell : (\mathbf{a}, \varepsilon_{(k \mod \ell)}) = \lceil \frac{k}{\ell} \rceil\} \cap V$$

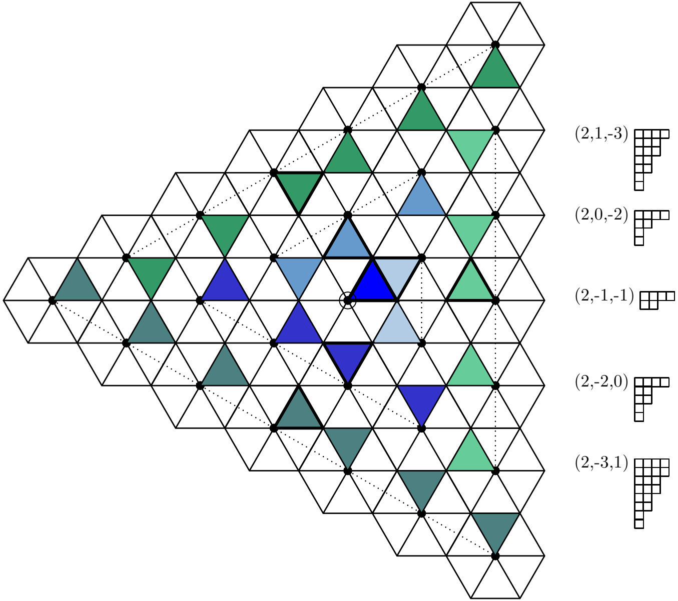
inside V, *where* $1 \leq (k \mod \ell) \leq \ell$. *Then under the correspondence* π, *the* ℓ*-cores* λ *with* $\lambda_1 = k$ *all lie inside* $H_\ell^k \bigcap \Lambda_R$.





$$7 = \lambda_1 = (a_i - 1)\ell + i = (2 - 1)4 + 3.$$

$$H_4^7 = \{(a_1, a_2, a_3, a_4) : a_3 = 2\} \cap V$$

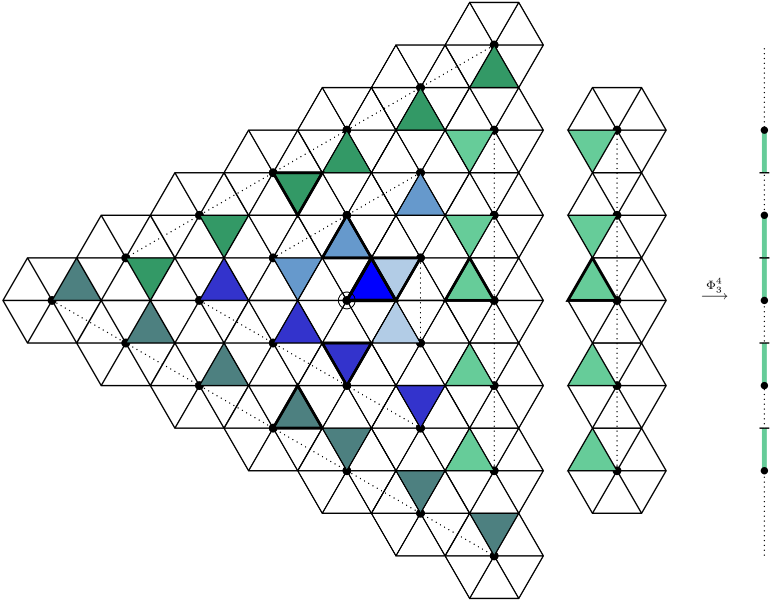

(2,1,-3)
(2,0,-2)
(2,-1,-1)
(2,-2,0)
(2,-3,1)

Theorem

Let ψ_ℓ *be the affine map defined by* $\psi_\ell(a_1, \ldots, a_\ell) = (a_\ell + 1, a_1, a_2, \ldots, a_{\ell-1})$. *Then,*

$$\pi^{-1} \circ \Phi_\ell^k \circ \pi(a_1, \ldots, a_\ell) = \psi_{\ell-1}^{a_i}(a_1, \ldots, \widehat{a_i}, \ldots, a_\ell)$$

where a_i *is the rightmost occurrence of the largest entry among* $\{a_1, \ldots, a_\ell\}$ *and the circumflex indicates omission.*


Φ_3^4

Lapointe–Morse correspondence

ℓ-cores $\leftrightarrow$ $(\ell - 1)$-bounded partitions.

Example

$\ell = 5$.

$\lambda =$

12	9	7	**4**	**2**	**1**
9	6	**4**	**1**		
7	**4**	**2**			
6	**3**	**1**			
4	**1**				
2					
1					

$\rho_\ell(\lambda) =$

9	**6**	**1**
7	**4**	
6	**3**	
5	**2**	
4	**1**	
2		
1		

$$
\begin{array}{ccccc}
\left\{\lambda \in \mathcal{C}_\ell^k\right\} & \xrightarrow{tr} & \left\{\begin{matrix}\lambda \in \mathcal{C}_\ell,\ len(\lambda) = \\ k\end{matrix}\right\} & \xrightarrow{\rho_\ell} & \left\{\begin{matrix}\text{partitions } \nu \\ \text{with } \nu_1 \leq \ell - 1 \\ \text{and } len(\nu) = k\end{matrix}\right\} \\
\Phi_\ell^k \big\downarrow & & \big\downarrow \widetilde{\Phi_\ell^k} & & \big\downarrow \Upsilon_\ell^k \\
\left\{\begin{matrix}\mu \in \\ \mathcal{C}_{\ell-1}^{\leq k}\end{matrix}\right\} & \xrightarrow{tr} & \left\{\begin{matrix}\mu \in \mathcal{C}_{\ell-1},\ len(\mu) \leq \\ k\end{matrix}\right\} & \xrightarrow[\rho_{\ell-1}]{} & \left\{\begin{matrix}\text{partitions } \sigma \\ \text{with } \sigma_1 \leq \ell - 2 \\ \text{and } len(\sigma) \leq k\end{matrix}\right\}
\end{array}
$$

$\lambda =$

$\underline{12}$	9	$\underline{7}$	**4**	$\underline{\mathbf{2}}$	**1**
$\underline{9}$	6	$\underline{\mathbf{4}}$	**1**		
$\underline{7}$	**4**	$\underline{\mathbf{2}}$			
$\underline{6}$	**3**	$\underline{\mathbf{1}}$			
$\underline{\mathbf{4}}$	**1**				
$\underline{\mathbf{2}}$					
$\underline{\mathbf{1}}$					

$\rho(\lambda) =$

9	**6**	**1**
7	**4**	
6	**3**	
5	**2**	
4	**1**	
2		
1		

$\widetilde{\Phi^k_\ell}(\lambda) =$

7	**3**	**1**
5	**1**	
3		
2		
1		

$\rho(\widetilde{\Phi^k_\ell}(\lambda)) =$

6	**1**
4	
3	
2	
1	