

A bijection on core partitions

Brant C. Jones

`brant@math.ucdavis.edu`

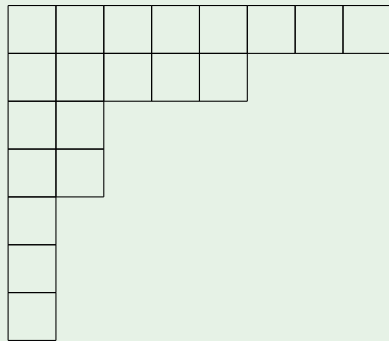
Joint with Chris Berg and Monica Vazirani
University of California, Davis

May 3, 2008

Notation

Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_r)$ be a *partition* of n and $\ell \geq 2$ be an integer.

Example



The ℓ -*residue* of a box (i, j) is the least nonnegative integer $\equiv j - i \pmod{\ell}$.

Notation

Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_r)$ be a *partition* of n and $\ell \geq 2$ be an integer.

Example

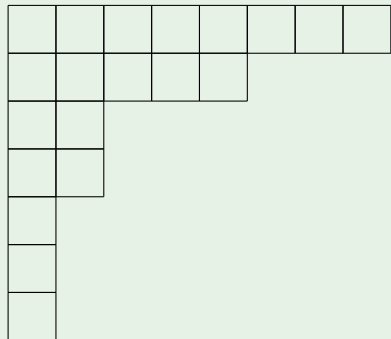
0	1	2	3	0	1	2	3
3	0	1	2	3			
2	3						
1	2						
0							
3							
2							

The ℓ -*residue* of a box (i, j) is the least nonnegative integer $\equiv j - i \pmod{\ell}$.

Notation

Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_r)$ be a *partition* of n and $\ell \geq 2$ be an integer.

Example



The *hook length* of a box (i, j) is the number of boxes to the **right and below** the box, including itself. It is denoted $h_{(i,j)}^\lambda$.

Notation

Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_r)$ be a *partition* of n and $\ell \geq 2$ be an integer.

Example

14	10	7	6	5	3	2	1
10	6	3	2	1			
6	2						
5	1						
3							
2							
1							

The *hook length* of a box (i, j) is the number of boxes to the **right and below** the box, including itself. It is denoted $h_{(i,j)}^\lambda$.

Definition

A partition λ is an ℓ -core if $\ell \nmid h_{(i,j)}^\lambda$ for every box (i,j) of λ .

Example

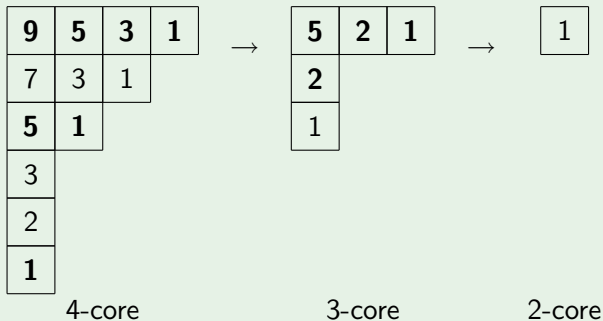
14	10	7	6	5	3	2	1
10	6	3	2	1			
6	2						
5	1						
3							
2							
1							

λ is a 4-core.

Question

Given an l -core, how can we project to obtain an $(l - 1)$ -core?

Example



ℓ -core partitions index:

- Schubert cells in the affine Grassmannian Gr of $SL(\ell, \mathbb{C})$.
($\text{Gr} \cong SL_\ell(\mathbb{C}((t)))/SL_\ell(\mathbb{C}[[t]])$.)
- k -Schur functions and dual k -Schur functions in $H_*(\text{Gr}) \cong \Lambda_\ell$ and $H^*(\text{Gr}) \cong \Lambda^\ell$, respectively.
- Blocks in the representation theory of the symmetric group S_n over a field of characteristic $\ell > 0$.

Let $\mathbf{F}S_n$ be the group algebra of the symmetric group of a field \mathbf{F} with characteristic $\ell \geq 0$. Let M be a $\mathbf{F}S_n$ -module.

Then there is a commutative subalgebra of $\mathbf{F}S_n$ generated by *Jucys-Murphy elements* $\{L_1, L_2, \dots, L_n\}$.

By the Jordan-Chevalley decomposition, $M = \bigoplus_{\mathbf{i}=(i_1, \dots, i_n)} M[\mathbf{i}]$ where $M[\mathbf{i}] = \{v \in M : (L_j - i_j)^N v = 0 \text{ for sufficiently large } N\}$.

- The weights $\mathbf{i} = (i_1, \dots, i_n)$ correspond to (residues of) *standard tableaux on ℓ -regular partitions*.
- The S_n -orbits of the weights are parameterized by $\Gamma = \ell$ -cores.

Then,

$$M = \bigoplus_{\gamma \in \Gamma} M[\gamma] \text{ where } M[\gamma] = \bigoplus_{\mathbf{i} \in \gamma} M[\mathbf{i}].$$

- $\mathcal{C}_\ell =$ The set of all ℓ -cores.
- $\mathcal{C}_\ell^k =$ The subset of \mathcal{C}_ℓ having first part k .
- $\mathcal{C}_{\ell-1}^{\leq k} =$ The subset of $\mathcal{C}_{\ell-1}$ having first part $\leq k$.

We will define a bijection

$$\Phi_\ell^k : \mathcal{C}_\ell^k \rightarrow \mathcal{C}_{\ell-1}^{\leq k}$$

Then,

$$\sum_{k \geq 0} |\mathcal{C}_\ell^k| x^k = \sum_{k \geq 0} \binom{k + \ell - 2}{k} x^k = \frac{1}{(1-x)^{\ell-1}}.$$

Beta numbers and Abaci

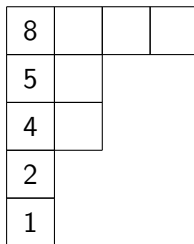
The tableau is determined by first column hooklengths. These can be generalized to β -numbers.

8			
5			
4			
2			
1			



Beta numbers and Abaci

	Runner 0	Runner 1	Runner 2
	⋮	⋮	⋮
Level -1 →	⊖3	⊖2	⊖1
Level 0 →	0	1	2
Level 1 →	3	4	5
Level 2 →	6	7	8
Level 3 →	9	10	11
	⋮	⋮	⋮

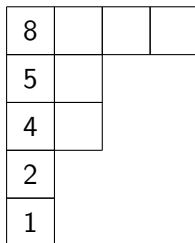


3-core



Beta numbers and Abaci

	Runner 0	Runner 1	Runner 2
	⋮	⋮	⋮
Level -1 →	⊖3	⊖2	⊖1
Level 0 →	0	⊕1	⊕2
Level 1 →	3	⊕4	⊕5
Level 2 →	6	7	⊕8
Level 3 →	9	10	11
	⋮	⋮	⋮



The abacus for $\beta = (8, 5, 4, 2, 1, -1, -2, -3, \dots)$ has balance number $2 = (-1) + 1 + 2$.

Beta numbers and Abaci

	Runner 0	Runner 1	Runner 2
	⋮	⋮	⋮
Level -1 →	⓪	⓪	⓪
Level 0 →	⓪	1	⓪
Level 1 →	⓪	4	⓪
Level 2 →	⓪	7	8
Level 3 →	⓪	10	11
	⋮	⋮	⋮

8			
5			
4			
2			
1			

The abacus for $\beta = (8, 5, 4, 2, 1, -1, -2, -3, \dots)$ has balance number 2.

The abacus for $\beta = (9, 6, 5, 3, 2, 0, -1, -2, \dots)$ has balance number

$3 = 3 + (-1) + 1$.

Theorem

Theorem 2.7.16, Lemma 2.7.38 in James–Kerber

- λ is an ℓ -core if and only if any (equivalently, every) abacus of λ on ℓ runners is flush.
- Moreover, in the balanced flush abacus of an ℓ -core λ , each active bead on runner i corresponds to a row of λ whose rightmost box has residue i .

Beta numbers and Abaci

	Runner 0	Runner 1	Runner 2
	⋮	⋮	⋮
Level -2 →	⊖6	⊖5	⊖4
Level -1 →	⊖3	-2	⊖1
Level 0 →	0	1	2
Level 1 →	3	4	5
Level 2 →	6	7	8
	⋮	⋮	⋮

0	1	2	0
2	0		
1	2		
0			
2			

3-core

⋮	⋮	⋮	⋮
(-8)	(-7)	(-6)	(-5)
-4	(-3)	(-2)	(-1)
0	(1)	(2)	3
4	5	(6)	7
8	9	(10)	11
⋮	⋮	⋮	⋮

4-core $(8, 5, 2^2, 1^3)$

Φ_4^8
→

⋮	⋮	⋮	⋮
(-6)	(-5)	(×)	(-4)
-3	(-2)	(×)	(-1)
0	(1)	(×)	2
3	4	(×)	5
6	7	(×)	8
⋮	⋮	⋮	⋮

3-core $(2, 1^2)$.

0	1	2	3	0	1	2	3
3	0	1	2	3			
2	3						
1	2						
0							
3							
2							

4-core $(8, 5, 2^2, 1^3)$

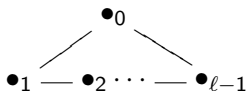
Φ_4^8
→

3-core $(2, 1^2)$.

The affine symmetric group

The *affine symmetric group* \tilde{S}_ℓ is presented as a Coxeter group by:

- Generators $s_0, s_1, \dots, s_{\ell-1}$.
- Involution relations $s_i^2 = 1$.
- Commuting relations $s_i s_j = 1$ if $|i - j| \geq 2$.
- Braid relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_0 s_{\ell-1} s_0 = s_{\ell-1} s_0 s_{\ell-1}$.



This is an infinite Coxeter group.

- Let $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_\ell\}$ be an orthonormal basis of the Euclidean space \mathbf{R}^ℓ with inner product (\cdot, \cdot) .
- For $1 \leq i \leq \ell - 1$, let s_i be the reflection that interchanges ε_i and ε_{i+1}
- Let s_0 be the affine reflection of \mathbf{R}^ℓ defined on $v = \sum_{j=1}^{\ell} a_j \varepsilon_j$ by

$$s_0(v) = (a_\ell + 1)\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_{\ell-1}\varepsilon_{\ell-1} + (a_1 - 1)\varepsilon_\ell.$$

- The *simple roots* Δ of type $A_{\ell-1}$ are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \dots, \quad \alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell.$$

- The \mathbf{Z} -span Λ_R of Δ is called the *root lattice of type* $A_{\ell-1}$.

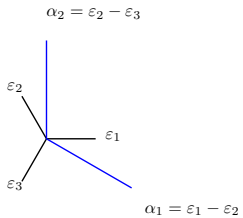
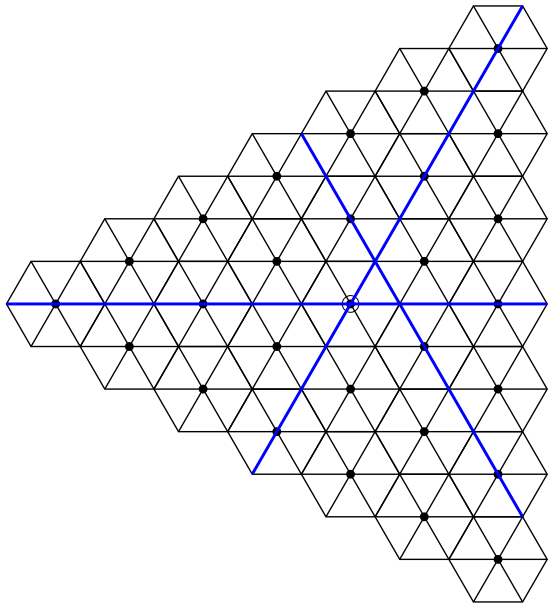
Let $V = \mathbf{R} \otimes_{\mathbf{Z}} \Lambda_R \subsetneq \mathbf{R}^\ell$. Observe that each reflection s_i preserves V . Then $\{s_0, s_1, \dots, s_{\ell-1}\}$ are a set of Coxeter generators for the affine symmetric group S_ℓ acting on V .

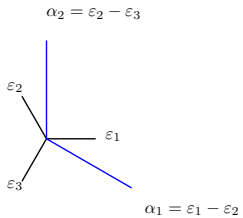
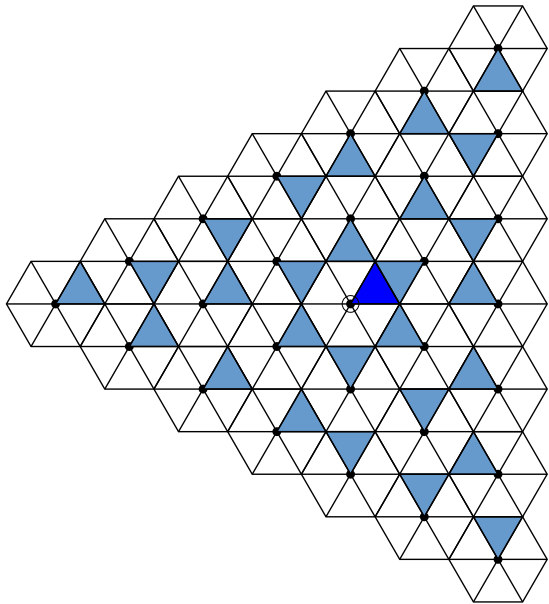
In this presentation, S_ℓ is a parabolic subgroup of \tilde{S}_ℓ . The “parabolic quotient”

$$\tilde{S}_\ell/S_\ell = \{w \in \tilde{S}_\ell : l(ws_i) > l(w) \text{ for all } s_i \text{ where } 1 \leq i \leq \ell - 1\}$$

gives a unique representative of minimal length from each coset wS_ℓ of \tilde{S}_ℓ . Here $l(w)$ is the minimum *length* of any expression for w as a product of the s_j .

On the other hand, $\tilde{S}_\ell = S_\ell \times \Lambda_R$ so \tilde{S}_ℓ/S_ℓ is in bijection with Λ_R .





Let $\mathbf{a} = (a_1, \dots, a_\ell) \in \Lambda_R$ written in the ε_i basis, so each $a_i \in \mathbf{Z}$ and $\sum_{i=1}^{\ell} a_i = 0$.

We form a balanced flush abacus from \mathbf{a} by filling the $(i - 1)^{\text{st}}$ runner with beads from $-\infty$ down to level a_i .

This defines a bijection

$$\pi : \{(a_1, \dots, a_\ell) : a_i \in \mathbf{Z}, \sum_{i=1}^{\ell} a_i = 0\} \rightarrow \{\text{balanced flush abaci}\} \rightarrow \mathcal{C}_\ell.$$

Definition

Fix $\lambda \in \mathcal{C}_\ell$ and suppose $\mathbf{b} = (b_0, \dots, b_{\ell-1}) = \pi^{-1}(\lambda)$.

For $0 \leq i \leq \ell - 1$, we say that s_i is an *ascent* for λ if $b_{i-1} > b_i - \delta_{i,0}$, and we say that s_i is a *descent* for λ if $b_{i-1} < b_i - \delta_{i,0}$. Here, we interpret b_{-1} as $b_{\ell-1}$.

Example

The 4-core $(5, 2, 1, 1, 1)$ corresponds to $(2, 0, 0, -2) \in \Lambda_R$ with minimal length coset representative $w = s_0 s_1 s_2 s_3 s_2 s_1 s_0$. Hence, s_1 and s_3 are ascents for λ and s_0 is a descent for λ .

Proposition

Let λ be an ℓ -core.

- If s_i is an **ascent** for λ then s_i acts on λ by **adding all boxes with residue i** to λ such that the result is a partition.
- If s_i is a **descent** for λ then s_i acts on λ by **removing all of the boxes with residue i** that lie at the end of both their row and column so that their removal results in a partition.
- If s_i is **neither an ascent nor a descent** for λ then s_i **does not change** λ .

Example

$$\ell = 4, s_1(-1, 1, 0, 0) = (1, -1, 0, 0).$$

(-8)	(-7)	(-6)	(-5)
(-4)	(-3)	(-2)	(-1)
0	1	2	3
4	5	6	7
8	9	10	11

0	1
3	
2	
1	

Example

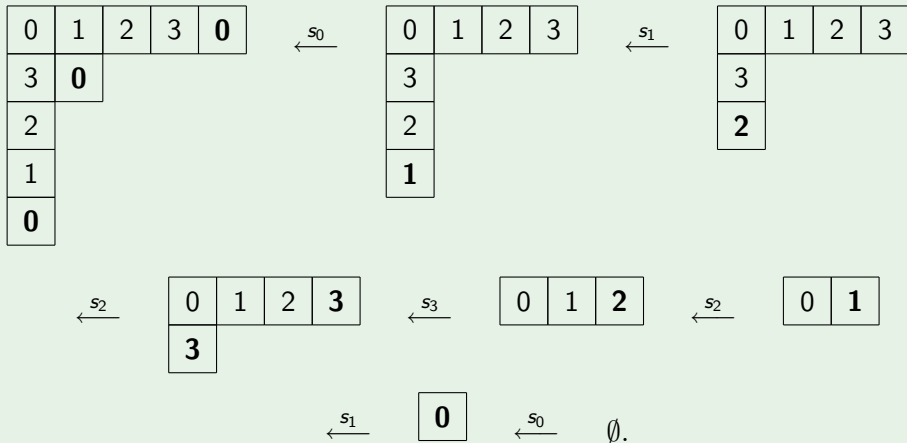
$$\ell = 4, s_1(-1, 1, 0, 0) = (1, -1, 0, 0).$$

$\ominus 8$	$\ominus 7$	$\ominus 6$	$\ominus 5$
$\ominus 4$	$\ominus 3$	$\ominus 2$	$\ominus 1$
0	1	2	3
4	5	6	7
8	9	10	11

0
3
2

Example

The *canonical reduced expression* $w(\lambda)$ for the 4-core $\lambda = (5, 2, 1, 1, 1)$ is $s_0 s_1 s_2 s_3 s_2 s_1 s_0$.



Proposition

For $\lambda \in \mathcal{C}_\ell$ we have that $w(\lambda)$ is a reduced expression for the minimal length coset representative indexed by λ .

The Coxeter length of $w(\lambda)$ is

$$l(w(\lambda)) = \sum_{i=0}^{\ell-1} \lambda_{R(i)}$$

where $R(i)$ is the longest row of λ whose rightmost box has residue i .

Corollary

Applying Φ_ℓ^k reduces the Coxeter length by exactly k .

Example

Let $\ell = 5$.

$$\lambda =$$

$\dot{0}$	$\dot{1}$	$\dot{2}$	$\dot{3}$	$\dot{4}$	$\dot{0}$	$\dot{1}$	$\dot{2}$	$\dot{3}$
$\dot{4}$	$\dot{0}$	$\dot{1}$	$\dot{2}$	$\dot{3}$				
0	1	2						
3	4	0						
$\dot{2}$	$\dot{3}$							
3	0							
1	2							
2								
0								
1								
4								
$\dot{3}$								
0								
2								

$$\Phi_{\ell}^k(\lambda) =$$

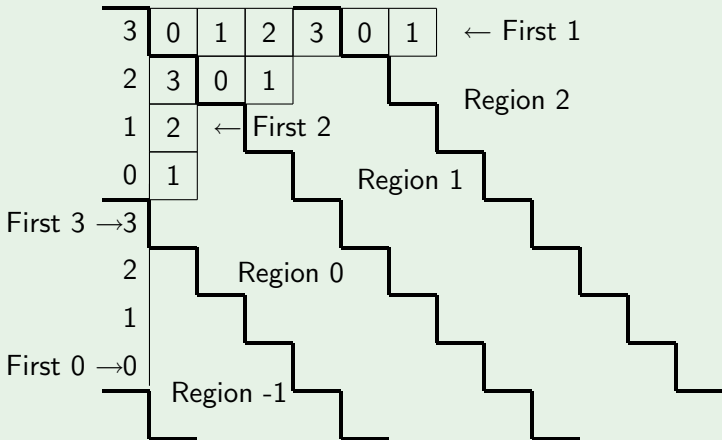
0	1	2
3	0	
2		
1		
0		

$$w(\lambda) = w = \mathbf{s_2 s_3 s_4 s_0 s_1 s_2 s_4 s_3 s_1 s_0 s_4 s_3 s_2 s_1 s_0} \mapsto w(\Phi_{\ell}^k(\lambda)) = s_0 s_1 s_2 s_3 s_1 s_0$$

The Coxeter length has been reduced by $15 - 6 = 9 = \lambda_1$.

Example

Let $\ell = 4$ and $\lambda = (6, 3, 1, 1)$. From the picture below, we see that the n -vector for λ is $(-1, 2, 0, -1)$.



Proposition

Suppose that $\pi(\mathbf{a}) = \pi(a_1, \dots, a_\ell) = \lambda$. Then we have

$$\lambda_1 = (a_i - 1)\ell + i$$

where a_i is the rightmost occurrence of the largest coordinate in \mathbf{a} .

Corollary

For $k \geq 0$, let H_ℓ^k denote the affine hyperplane

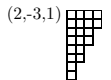
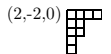
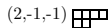
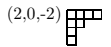
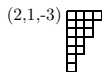
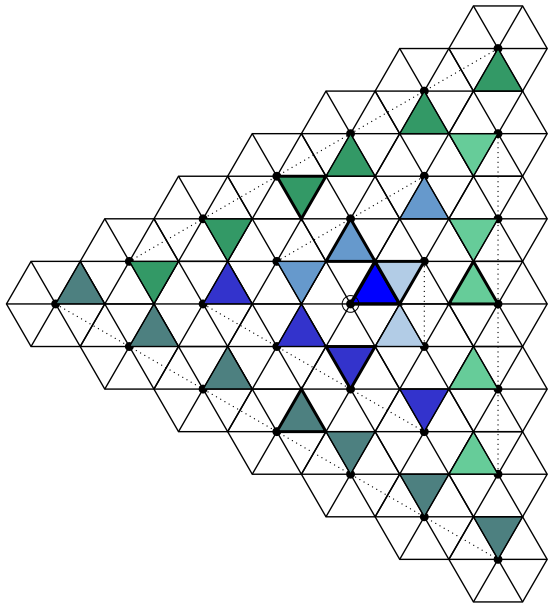
$$H_\ell^k = \{\mathbf{a} = (a_1, \dots, a_\ell) \in \mathbf{R}^\ell : (\mathbf{a}, \varepsilon_{(k \bmod \ell)}) = \lceil \frac{k}{\ell} \rceil\} \cap V$$

inside V , where $1 \leq (k \bmod \ell) \leq \ell$. Then under the correspondence π , the ℓ -cores λ with $\lambda_1 = k$ all lie inside $H_\ell^k \cap \Lambda_R$.

⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
⊖8	⊖7	⊖6	⊖5	⊖8	⊖7	⊖6	⊖5
⊖4	-3	⊖2	⊖1	⊖4	⊖3	⊖2	⊖1
⊖0	1	⊖2	3	⊖0	1	⊖2	⊖3
⊖4	5	⊖6	7	4	5	6	7
8	9	⊖10	11	8	9	<u>10</u>	11
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

$$7 = \lambda_1 = (a_i - 1)\ell + i = (2 - 1)4 + 3.$$

$$H_4^7 = \{(a_1, a_2, a_3, a_4) : a_3 = 2\} \cap V$$



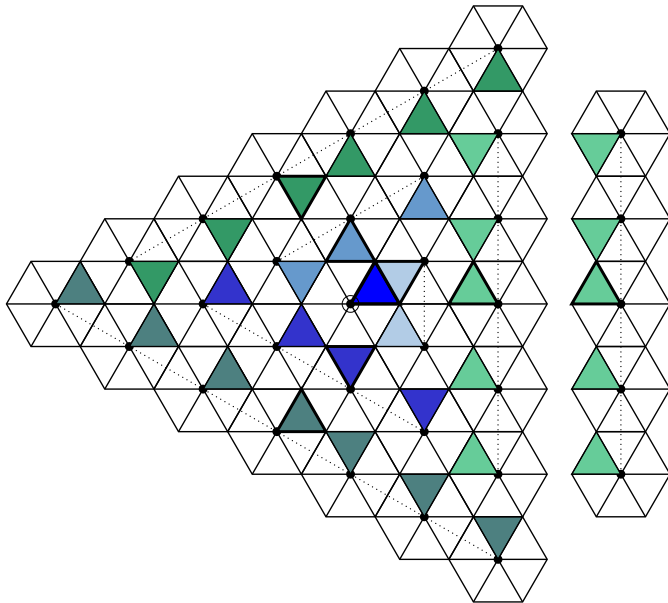
Theorem

Let ψ_ℓ be the affine map defined by

$\psi_\ell(a_1, \dots, a_\ell) = (a_\ell + 1, a_1, a_2, \dots, a_{\ell-1})$. Then,

$$\pi^{-1} \circ \Phi_\ell^k \circ \pi(a_1, \dots, a_\ell) = \psi_{\ell-1}^{a_i}(a_1, \dots, \widehat{a_i}, \dots, a_\ell)$$

where a_i is the rightmost occurrence of the largest entry among $\{a_1, \dots, a_\ell\}$ and the circumflex indicates omission.



Lapointe–Morse correspondence

ℓ -cores \leftrightarrow $(\ell - 1)$ -bounded partitions.

Example

$\ell = 5$.

$$\lambda =$$

12	9	7	4	2	1
9	6	4	1		
7	4	2			
6	3	1			
4	1				
2					
1					

$$\rho_\ell(\lambda) =$$

9	6	1
7	4	
6	3	
5	2	
4	1	
2		
1		

$$\begin{array}{ccccc}
\{\lambda \in \mathcal{C}_\ell^k\} & \xrightarrow{tr} & \left\{ \lambda \in \mathcal{C}_\ell, \text{len}(\lambda) = \right. & \xrightarrow{\rho_\ell} & \left. \begin{array}{l} \text{partitions } \nu \\ \text{with } \nu_1 \leq \ell - 1 \\ \text{and } \text{len}(\nu) = k \end{array} \right\} \\
\Phi_\ell^k \downarrow & & \downarrow \widetilde{\Phi}_\ell^k & & \downarrow \Upsilon_\ell^k \\
\left\{ \begin{array}{l} \mu \\ \mathcal{C}_{\ell-1}^{\leq k} \end{array} \right\} & \xrightarrow{tr} & \left\{ \mu \in \mathcal{C}_{\ell-1}, \text{len}(\mu) \leq \right. & \xrightarrow{\rho_{\ell-1}} & \left. \begin{array}{l} \text{partitions } \sigma \\ \text{with } \sigma_1 \leq \ell - 2 \\ \text{and } \text{len}(\sigma) \leq k \end{array} \right\}
\end{array}$$

$$\lambda =$$

<u>12</u>	9	<u>7</u>	4	<u>2</u>	1
<u>9</u>	6	<u>4</u>	1		
<u>7</u>	4	<u>2</u>			
<u>6</u>	3	<u>1</u>			
<u>4</u>	1				
<u>2</u>					
<u>1</u>					

$$\rho(\lambda) =$$

9	6	1
7	4	
6	3	
5	2	
4	1	
2		
1		

$$\widetilde{\Phi}_\ell^k(\lambda) =$$

7	3	1
5	1	
3		
2		
1		

$$\rho(\widetilde{\Phi}_\ell^k(\lambda)) =$$

6	1
4	
3	
2	
1	