Recent applications of combinatorial Hopf algebras

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Introduction

Lots of applications of combinatorial Hopf algebras to different contexts. Today:

- Factorizations of normal braids,
- q-enumeration of permutation tableaux.

Our goal: use the *numerous* constraints of Hopf morphisms to

- find the right intermediate questions,
- solve the algebraic part with the little finger,
- leave us with (un)easy combinatorial lemmas.

Normal braids

(Black/White)board example ...

Sequences of permutations $(\sigma_1, \ldots, \sigma_k)$ s.t.

$$\forall i, \quad \mathrm{Des}(\sigma_{i+1}^{-1}) \subset \mathrm{Des}(\sigma_i).$$

Ex.:

$$(31524,\ 24513,\ 12435,\ 41523,\ \ldots)$$

Adjacency matrices

$$M_2 = \begin{pmatrix} 1 & . \\ 1 & 1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & 1 & . \\ 1 & . & 1 & 1 & . & . \\ 1 & 1 & . & . & 1 & . \\ 1 & . & 1 & 1 & . & . \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Dehornoy's conjecture

Conjecture

For all n,

$$CharPol(M_n) \mid CharPol(M_{n+1}).$$

In the sequel, those matrices are endomorphisms ϕ_n of **FQSym**, the Hopf algebra of permutations (b/w board).

Which strategy?

Let's assume we find a surjective map ∂ (from $\mathbf{FQSym}_n$ to $\mathbf{FQSym}_{n-1}$) commuting with ϕ.

Then if $K := \ker \partial$, K is stable by ϕ.

So the matrix of ϕ is block triangular on $K \oplus K'$ and since $\mathbf{FQSym}_n / K \sim \mathbf{FQSym}_{n-1}$, the result holds.

The map

$$\sigma = 8\,4\,7\,5\,1\,6\,9\,3\,2.$$

compute the *middle increases/decreases* of $\sigma' = (n+1).\sigma.0$:

$$\color{red}{8}\,4\,7\,\color{red}{5}\,1\,\color{red}{6}\,9\,\color{red}{3}\,\color{red}{2}.$$

Then remove the red numbers one at a time and standardize.

47516932	47516832
84716932	74615832
84751932	74651832
84751692	73641582
84751693	73641582

Hence

$$\partial \mathbf{F}_{847516932} = -\mathbf{F}_{47516832} - \mathbf{F}_{74615832} + \mathbf{F}_{74651832} - 2\mathbf{F}_{73641582}.$$

Proofs (not even sketched)

- Surjective?

$$\partial((n+1).\sigma) = \sigma + \text{greater terms.}$$

- Map is a derivation?

$$\partial_i(\mathbf{F}_\sigma \mathbf{F}_\mu) = \partial_i(\mathbf{F}_\sigma)\ \mathbf{F}_\mu, \partial_j(\mathbf{F}_\sigma \mathbf{F}_\mu) = \mathbf{F}_\sigma \partial_{j-n}(\mathbf{F}_\mu)$$

for $i < n$ and $j > n$ resp. and where $\partial = \sum_i \partial_i$.

- Commutation?

$$\partial(\phi(\mathbf{F}_\sigma)) = \text{explicit.}$$

Main question adressed now: how to guess ∂ from scratch?

CHA: from **FQSym** to **NCSF**

First observations on M_n.

- Many equal rows and columns: only depends on the *descent composition* of permutations.
- Image set inside noncommutative symmetric functions **NCSF**:

$$\partial(\mathbf{F}_\sigma) = S^I,$$

where $I = DC(\sigma)$.

So ϕ has same char. pol. up to powers of X as ϕ':

$$\phi' : \begin{cases} \mathrm{QSym} & \rightarrow & \mathbf{NCSF} \\ F_I & \mapsto & S^I. \end{cases}$$

Removing the kernel

Note that ϕ' is a morphism from **NCSF** to QSym...

But it can be seen as an endomorphism:

$$\phi'' := \mathrm{Id}_{\mathbf{NCSF}\to\mathrm{QSym}} \circ \phi'.$$

Since the morphism goes through symmetric functions, can take it from Sym to Sym. This one has no kernel left.

First values of ∂

Since it is a derivation, only needs to describe it on generators!
Experiments over eigenspaces then gives:

$$\partial h_i = (i-2)h_{i-1}.$$

Thanks to Hopf, only remains to lift this, first to **NCSF** and QSym, and then to **FQSym**...

Permutation tableaux, their shapes, and the universe

Combinatorial objects coming from the study of the totally non-negative part of the Grassmannian. No need to know even the defn!

The only part we need: through a bijection,
PT *shape* $\sim$ permutations *descent tops*.

Descent tops:

3	21	12	111
123	132 231 312	213	321

Hopf algebras context: refine descent tops thanks to recoils

$\mathfrak{M}_3 =$

DT \ Rec	3	21	12	111
3	123			
21		132 312	231	
12			213	
111				321

This last matrix describes a transition matrix in **NCSF** between ribbon Schur functions and *Tevlin's fundamental L* basis.

$$R_{21} = 2L_{21} \qquad R_{12} = L_{21} + L_{12}.$$

Question 1: enumerate permutation tabx w.r.t. shape?

Let's define *Tevlin's monomial* Ψ basis

$$L_I = \sum_{J \text{ finer } I} \Psi_J.$$

so that

$$RL_4 = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ . & 3 & 2 & . & 1 & 1 & . & . \\ . & . & 2 & . & 1 & . & . & . \\ . & . & 1 & 3 & . & 2 & 1 & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & 2 & 1 & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \end{pmatrix} \qquad R\Psi_4 = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ 1 & 3 & 2 & . & 1 & 1 & . & . \\ 1 & . & 2 & . & 1 & . & . & . \\ 1 & 3 & 5 & 3 & 2 & 3 & 1 & . \\ 1 & . & . & . & 1 & . & . & . \\ 1 & 3 & 2 & . & 2 & 3 & 1 & . \\ 1 & . & 2 & . & 2 & . & 1 & . \\ 1 & 3 & 5 & 3 & 3 & 5 & 3 & 1 \end{pmatrix}$$

Q. equiv. to compute E_I the entries sum over rows of RL_n.
Definitely easier to compute C_I over $R\Psi_n$...

A first (easy) result

I	4	31	22	211	13	121	112	1111
C_I	1	8	4	18	2	12	6	24

Easy to guess *and* prove

$$I = (i_1, \ldots, i_r) \mapsto C_I := r^{i_1}(r-1)^{i_2} \ldots 1^{i_r}.$$

Then, by inclusion-exclusion, one gets E_I.

Ex.:

$$E_{211} = C_{211} - C_{22} - C_{31} + C_4 = 7.$$

More combinatorial coefficients

C_I is the expansion of $\sum_J R_J$ in the Ψ. But $\sum_J R_J = S_1^n$!

What about the other S^I?

$$S\Psi_4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 3 & 2.3 & 2 & 2.3 & 2.2 & 2.2.2 \\ 1 & 1 & 3 & 3 & 2 & 2 & 2.2 & 2.2 \\ 1 & 4 & 4.3/2 & 3.4 & 3 & 3.3 & 3.3 & 2.3.3 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 4 & 3 & 2.3 & 3 & 3.3 & 2.3 & 2.2.3 \\ 1 & 1 & 3 & 3 & 3 & 3 & 2.3 & 2.3 \\ 1 & 4 & 4.3/2 & 3.4 & 4 & 3.4 & 3.4 & 2.3.4 \end{pmatrix}$$

General formula: product of binomial coefficients.

From the PTs world: q-analogs!

For us: power of q equals: # patterns $31-2$ in σ.

$$R_q L_{q4} = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ . & 1+q+q^2 & 1+q & . & 1 & q & . & . \\ . & . & 1+q & . & 1 & . & . & . \\ . & . & q & 1+q+q^2 & . & 1+q & 1 & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & 1+q & 1 & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \end{pmatrix}$$

Ex.: the two contributors for q^2 are 4123 and 4132.

Question 2: q-enumerate permutation tabx w.r.t. shape?

Almost the same solution except one has to compute a q-transition matrix between Ψ and L:

$$L_q\Psi_4 = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ 1 & q & . & . & . & . & . & . \\ 1 & . & q & . & . & . & . & . \\ 1 & q & q^2 & q^3 & . & . & . & . \\ 1 & . & . & . & q & . & . & . \\ 1 & q & . & . & q^2 & q^3 & . & . \\ 1 & . & q & . & q^2 & . & q^3 & . \\ 1 & q & q^2 & q^3 & q^3 & q^4 & q^5 & q^6 \end{pmatrix}$$

Essentially the major index of the *refinement composition* between I and J.

From R to Ψ

Up to the previous q-transition, one gets

$$R_q\Psi_4 = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ 1 & q[3] & q+q^2 & . & q & q^2 & . & . \\ 1 & . & q+q^2 & . & q & . & . & . \\ 1 & q[3] & q+2q^2+q^3+q^4 & q^3[3] & q+q^2 & q^2+q^3+q^4 & q^3 & . \\ 1 & . & . & . & q & . & . & . \\ 1 & q[3] & q+q^2 & . & q+q^2 & q^2+q^3+q^4 & q^3 & . \\ 1 & . & q+q^2 & . & q+q^2 & . & q^3 & . \\ 1 & q[3] & q+2q^2+q^3+q^4 & q^3[3] & q[3] & q^2+q^3+2q^4+q^5 & q^3[3] & q^6 \end{pmatrix}$$

The sum is then the direct q-analog of the first one:

$$I = (i_1, \ldots, i_r) \mapsto C_I(q) := [r]^{i_1}[r-1]^{i_2} \ldots 1^{i_r}.$$

More q-combinatorial coefficients

Same idea as before: why concentrate only on the last row of the q-matrix S to Ψ?

A little miracle occurs:

$$S_q\Psi_4 = \begin{pmatrix} [1] & [1] & [1] & [1] & [1] & [1] & [1] & [1] \\ [1] & [4] & [3] & [2][3] & [2] & [2][3] & [2][2] & [2][2][2] \\ [1] & [1] & [3] & [3] & [2] & [2] & [2][2] & [2][2] \\ [1] & [4] & [4][3]/[2] & [3][4] & [3] & [3][3] & [3][3] & [2][3][3] \\ [1] & [1] & [1] & [1] & [2] & [2] & [2] & [2] \\ [1] & [4] & [3] & [2][3] & [3] & [3][3] & [2][3] & [2][2][3] \\ [1] & [1] & [3] & [3] & [3] & [3] & [2][3] & [2][3] \\ [1] & [4] & [4][3]/[2] & [3][4] & [4] & [3][4] & [3][4] & [2][3][4] \end{pmatrix}$$

Proofs

How to fill the (Hopf) gaps?

- Combinatorial interpretation of $S\Psi_n$: statistics on *integer packed matrices M*
  - Column = column-evaluation of M,
  - Row = letters last occurrences composition of the reading word of M,
  - Power of q = *special inversions* of the reading word of M.
- Natural combinatorial interpretation of $R\Psi_n$: statistics on *packed words w*
  - Column = descent composition of w,
  - Row = letters last occurrences composition of w.
  - Power of q = *special inversions* of w.

That was the easy part.

Proofs (2/2)

About $S_q L_{q_n}$ and $R_q L_{q_n}$, the answer from the PT's world is of no use: doesn't fit in the Hopf stuff.

One solution: reformulate it!

- Combinatorial interpretation of SL_n: statistics on permutations σ
  - Column = finer than Des(σ^{-1}),
  - Row = Lehmer composition of σ,
  - Power of q = inversions of σ (up to a global row power)
- Natural combinatorial interpretation of RL_n: statistics on permutations σ
  - Column = descent composition of σ^{-1},
  - Row = Lehmer composition of σ.
  - Power of q = inversions of σ (up to ...).

Example of the Lehmer composition

$$\begin{array}{rcccccccccc} \sigma & = & 6 & 3 & 7 & 1 & 2 & 4 & 9 & 8 & 5 \\ Lehmer(\sigma^{-1}) & = & 3 & 3 & 1 & 2 & 4 & 0 & 0 & 1 & 0 \end{array}$$

Compute LC:

$$\emptyset \to \{1\} \to \{1,4\}, \to \{1,4,2\}.$$

Want to guess how to compute LC?

Too early?

Type B

Cominuscule grassmannian for type B worked out by Lam-Williams.

- The q formula is the type A formula up to a factor $\prod(1+q^i)$,
- As in the A case, formulas can be proved by brute force,
- Need new (type B) Hopf algebras... but preliminary results are encouraging: only needs to fix the basis Ψ^B!
- Combinatorial generalization to any number of colors.

Conjecture

Satisfactory to get the inversions statistic involved when computing q-binomial coefficients... However, there seems to be something about patterns!

Main conjecture: get back to RL_n:

- Column $=$ descent composition of σ^{-1},
- Row $=$ descent tops composition of σ,
- Power of q $=$ number of patterns $31-2$ of σ.

Strategies: either direct bijection (inside recoil classes!) or find the adapted algebraic environment!