Finite Automata and Wilf Equivalence for the Generalized Factor Order

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Wilf Equivalence for Permutations

Given a sequence of distinct numbers $s = s_1 \cdots s_n$, we let $red(s)$ denote the permutation of S_n whose elements have the same relative order at $s_1 \cdots s_n$. For example,

$$red(5276) = 2143.$$

Then we say that a permutation $\sigma = \sigma_1 \cdots \sigma_m$ **occurs** in permutation $\tau = \tau_1 \cdots \tau_n$ if there is a subsequence $1 \leq i_1 < \cdots < i_m \leq n$ such that $red(\tau_{i_1} \cdots \tau_{i_m}) = \sigma$.

For example, if $\tau = 7\ 1\ 4\ 2\ 3\ 5\ 6$, then $2\ 1\ 3$ occurs in τ.

$$7\ 1\ 4\ 2\ 3\ 5\ 6$$

.

We say that τ **avoids** σ if σ does not occur in τ.

We let

$$S_n(\sigma) = |\{\tau \in S_n : \tau \text{ avoids } \sigma\}|. \tag{1}$$

We say that $\sigma, \tau \in S_m$ are **Wilf equivalent** if $S_n(\sigma) = S_n(\tau)$ for all n.

Given a permutation $\alpha = \alpha_1 \cdots \alpha_n \in S_n$, let

$$\begin{aligned} \alpha^r &= \alpha_n \cdots \alpha_n \quad \textbf{reverse} \\ \alpha^c &= (n+1-\alpha_1) \cdots (n+1-\alpha_n) \quad \textbf{complement} \end{aligned}$$

Clearly, if σ occurs in τ, then σ^r occurs in τ^r and
if σ occurs in τ, then σ^c occurs in τ^c. Thus
(i) τ avoids σ iff τ^r avoids σ^r and
(ii)τ avoids σ iff τ^c avoids σ^c.

Thus

1. $S_n(1\ 2\ 3) = S_n(3\ 2\ 1)$.
2. $S_n(2\ 1\ 3) = S_n(3\ 1\ 2) = S_n(1\ 3\ 2) = S_n(2\ 3\ 1)$.

Lemma 0.1. $S_n(1\ 2\ 3) = S_n(1\ 3\ 2)$.

Proof Simion and Schmidt (1985)

We define a bijection

$$f : \{\sigma \in S_n : \sigma \text{ avoids } 1\ 3\ 2\} \longrightarrow \{\sigma \in S_n : \sigma \text{ avoids } 1\ 2\ 3\}.$$

We say that an entry in a permutation σ is *left-to-right mininum* of σ if it is smaller than all the entries of σ which precede it.

$$\sigma = 6\ 7\ 3\ 4\ 1\ 2\ 5\ 8$$

The map f keeps the left-to-right minimum the same and writes the remaining elements in decreasing order.

$$f(\sigma) = 6\ 8\ 3\ 7\ 1\ 5\ 4\ 2$$

Theorem 0.2.

$$S_n(1\ 3\ 2) = C_n = \frac{\binom{2n}{n}}{n+1}. \tag{2}$$

Stanley-Wilf Conjecture (1980). Let σ be any permutation, then there exists a constant c_σ such that

$$S_n(\sigma) \leq c_\sigma^n.$$

Recently proved by Marcos and Tardos.

Definition

Let σ and τ be permutations in S_m.

The we say that σ is **Wilf equivalent** to τ if $S_n(\sigma) = S_n(\tau)$ for all n.

Variations

1) You repeat the definitions for consecutive occurrences.

2) You can consider so-called dashed patterns. $1\ 2 - 4 - 3\ 5$.

The Generalized Factor Order

Let $\mathcal{P} = (P, \leq_P)$ be a partially ordered set.

Let $P^* = \{w = w_1 \cdots w_n : w_i \in P \text{ for all } i\}$.
We let ϵ denote the empty word
For $w, u \in P^*$, we say that u is a **factor** of w if $w = vuv'$ for some words $v, v' \in P^*$. If $v = \epsilon$, then we say u is **prefix** of w and if $v' = \epsilon$, then we say u is **suffix** of w.

Then we define the **generalized factor order** on P^* by declaring the $u = u_1 \cdots u_m \leq_{\mathcal{P}} w = w_1 \cdots w_n$ if and only if there is and $i \geq 0$ such that $u_j \leq_P w_{i+j}$ for $j = 1, \ldots, m$.
In such a case, we will say that w has an embedding of u starting at position $i + 1$.

We say that w **avoids** u if $u \not\leq_{\mathcal{P}} w$.

Rational Generating Functions

Let $\mathbb{Z}\langle\langle P\rangle\rangle$ be the algebra of formal power series with integer coefficients and having the elements of P as noncommuting variables.

$$\mathbb{Z}\langle\langle P\rangle\rangle = \{f = \sum_{w \in P^*} c(w)w : c(w) \in \mathbb{Z} \text{ for all } w\}.$$

If $f \in \mathbb{Z}\langle\langle P\rangle\rangle$ has no constant term, i.e. $c(\epsilon) = 0$, then we let

$$f^* = \epsilon + f + f^2 + f^3 + \cdots = \frac{1}{\epsilon - f}.$$

We say that f is **rational** if it can be constructed for the elements of P using only a finite number of applications of the algebra operations and the star operation.

Regular Languages

A **language** is any $\mathcal{L} \subseteq P^*$ and it has an associated generating function

$$f_{\mathcal{L}} = \sum_{w \in \mathcal{L}} w.$$

The language $\mathcal{L}$ is **regular** iff $f_{\mathcal{L}}$ is rational.

Definition 0.3. *A* **Deterministic Finite Automaton** *(DFA) M is specified by a quintuple $M = (Q, \Sigma, \delta, s, F)$ where*

1. *Q is a finite alphabet of state symbols;*
2. *Σ is an alphabet of input symbols;*
3. *$\delta : Q \times \Sigma \to Q$ is a transition function;*
4. *$s \in Q$ is the start state; and*
5. *$F \subseteq Q$ is a set of accepting or final states.*

Definition 0.4. *Let $M = (Q, \Sigma, \delta, s, F)$ be a DFA.*

1. *We say that a word $Q\Sigma^*$ is a configuration. (A configuration represents a current state of M and remaining unread input.)*
2. *If px and qy are configurations, then we write $px \vdash qy$ if $x = ay$ with $a \in \Sigma$ and $\delta(p, a) = q$. (Here the idea is that if we are in state p reading the symbol a, then the machine moves to a new state q and uses up the symbol a if $\delta(p, a) = q$.)*
3. *If px and qy are configurations, then we say $px \vdash^1 qy$ if $ps \vdash qy$ and, for $k > 1$, we say that $px \vdash^k qy$ if there exists a configuration rz such that $px \vdash rz$ and $rz \vdash^{k-1} qy$.*
4. *If px and qy are configurations, then we say $px \vdash^* qy$ if there exists a $k \in \mathbb{P}$ such that $ps \vdash^k qy$.*
5. *We say that a word $w \in \Sigma^*$ is* **accepted** *by M if and only if $sw \vdash^* q\epsilon$ where ϵ is the empty word and $q \in F$.*

The definition of a **Nondeterministic Finite Automaton** (NFA) is similar. That is, an NFA N is again a 5-tuple $N = (Q, \Sigma, \delta, s, F)$ where Q, Σ, s and F are as in the definition of a DFA but now

$$\delta \subseteq Q \times \Sigma \times Q.$$

In a NFA, δ is called the *transition relation*.
The definition of configurations for NFA's is the same for as for DFA's, but now we say that $px \vdash qy$ if $x = ay$ with $a \in \Sigma$ and $(p, a, q) \in \delta$. The definitions of $px \vdash^k qy$, $px \vdash^* qy$ then remain the same.

The main difference that there may be more than one sequence of configurations that leads from px to qy in this case.

A word $w \in \Sigma^*$ is **accepted** by N if and only if $sw \vdash^* q\epsilon$ where ϵ is the empty word and $q \in F$.

$L(M)$ is the set of all words accepted by M.

Theorem 0.5. *Over a finite alphabet, the following are equivalent.*

1. *$\mathcal{L}$ is regular.*

2. *$\mathcal{L}$ is the set of all words accepted by a DFA.*

3. *$\mathcal{L}$ is the set of all words accepted by a NFA.*

Given $u \in P^*$, we define three languages.

$\mathcal{F}(u) = \{w \in P^* : u \leq w\}$ and $F(u) = \sum_{w \in \mathcal{F}(u)} w$.

$\mathcal{A}(u) = \{w \in P^* : w \text{ avoids } u\}$ and $A(u) = \sum_{w \in \mathcal{A}(u)} w$.

$\mathcal{S}(u)$ equals the set of all $w \in P^*$ such that the only embedding of u into w is a suffix of w and $S(u) = \sum_{w \in \mathcal{S}(u)} w$.

Theorem 0.6. *Let $\mathcal{P} = (P, \leq p)$ be any poset and let $u \in P^*$. Then*

1. *$\mathcal{F}(u) = \mathcal{S}(u)P^*$ and $F(u) = S(u)(\epsilon - P)^{-1}$ and*
2. *$\mathcal{A}(u) = P^* - \mathcal{F}(u)$ and $A(u) = (\epsilon - P)^{-1} - F(u)$.*

Hence if one of $\mathcal{F}(u)$, $\mathcal{A}(u)$, or $\mathcal{S}(u)$, then all of $\mathcal{F}(u)$, $\mathcal{A}(u)$, and $\mathcal{S}(u)$ are rational.

Weight Generating Functions

Assume that P is a subset of the natural numbers, then for any word $w = w_1 \cdots w_n$ in P^*, we define

$$\begin{aligned} |w| &= n \\ \Sigma(w) &= w_1 + \cdots + w_n \\ wt(w) &= t^{|w|} x^{\Sigma(w)}. \end{aligned}$$

$F(u; x, t) = \sum_{w \in \mathcal{F}(u)} wt(w)$.

$A(u; x, t) = \sum_{w \in \mathcal{A}(u)} wt(w)$.

$S(u; x, t) = \sum_{w \in \mathcal{S}(u)} wt(w)$.

Wilf Equivalence for the Generalized Factor Order

We say that words u and v in P^* are **Wilf Equivalent** relative $\mathcal{P} = (P, \leq_P)$, written $u \backsim v$, if

$$A(u; x, t) = A(v; x, t).$$

The NFA for $\mathcal{S}(u)$

Assume $|u| = \ell$.

States: All subsets of $\{1, \ldots, \ell\}$.

Initial State: $s = \emptyset$.

Alphabet: $\Sigma = P$.

Transition Relation:
If $T \subseteq \{1, \ldots, \ell\}$ and $w = w_1 \cdots w_n$ is path that leads from $\emptyset$ to T, then the NFA will be constructed so that if the path is continued, the only possible possible positions in which an embeding of u can start are those in the set $\{m - t + 1 : t \in T\}$.

Final States: T such that $\ell \in T$.

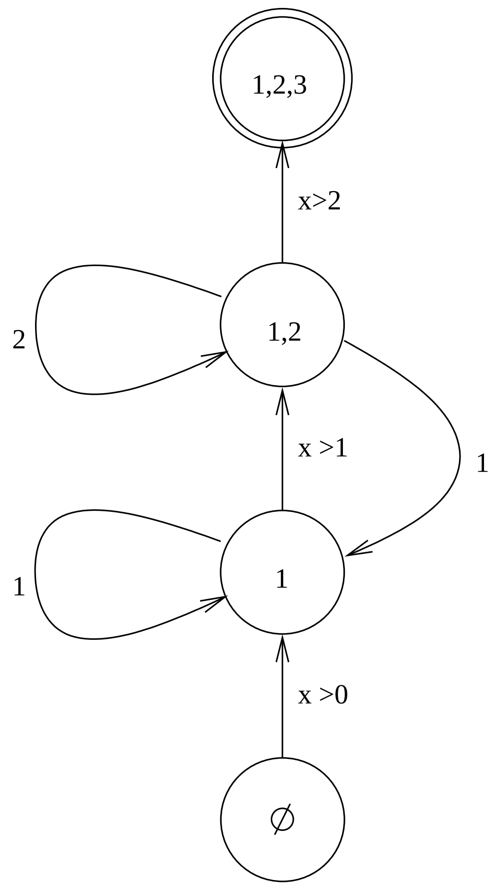

Figure 1: The DFA for $w = 1\ 2\ 3$.

L_0 denote the generating function of the set of all words v that reach an accepting state starting at state S_0,

L_1 denote the generating function of the set of all words v that reach an accepting state starting at state $S_{(1)}$,

L_2 denote the generating function of the set of all words v that reach an accepting state starting at state $S_{(1,2)}$, and

$L_3 = 1$.

Then we see that we have the following set of equations for $L_0, \ldots, L_3$.

$$
\begin{aligned}
L_3 &= 1 && (3)\\
L_2 &= t\frac{x^3}{1-x}L_3 + tx^2L_2 + txL_1 && (4)\\
L_1 &= t\frac{x^2}{1-x}L_2 + txL_1 && (5)\\
L_0 &= \frac{x}{1-x}L_1. && (6)
\end{aligned}
$$

One can easily solve this system of equations to find that

$$
L_0 = \frac{t^3x^6}{(1-x)^2(1-x-tx+tx^3-t^2x^4)} \qquad (7)
$$

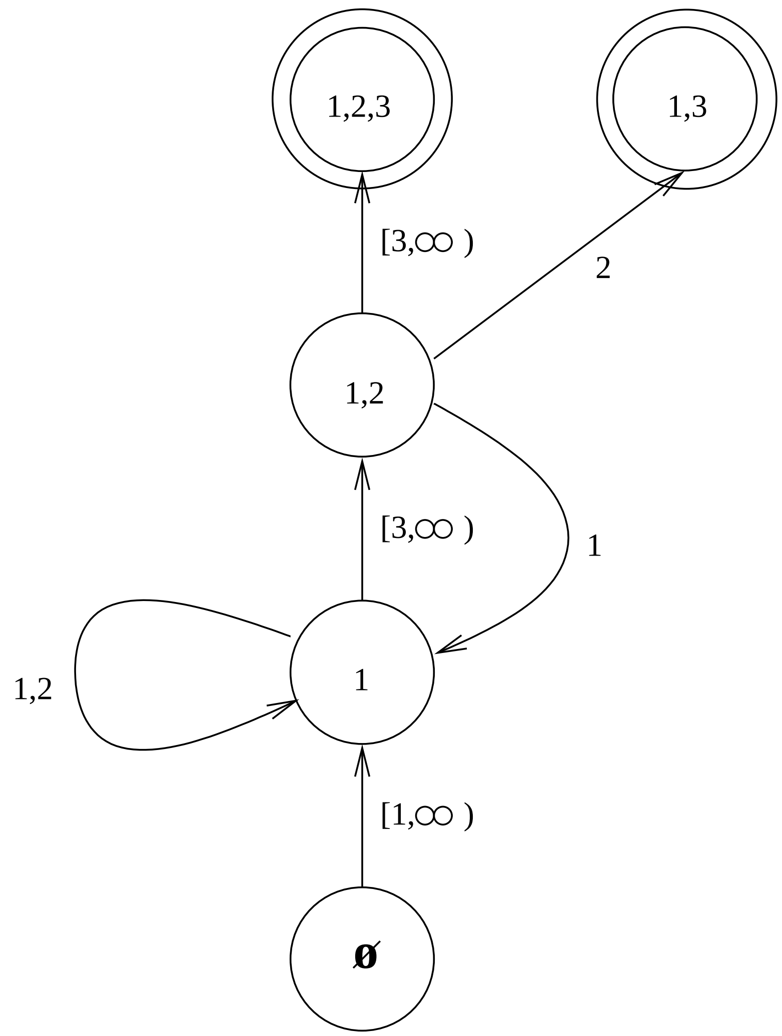

Figure 2: The DFA for $w = 1\ 3\ 2$.

L_0 denote the generating function of the set of all words v that reach an accepting state starting at state S_0,

L_1 denote the generating function of the set of all words v that reach an accepting state starting at state $S_{(1)}$,

L_2 denote the generating function of the set of all words v that reach an accepting state starting at state $S_{(1,2)}$,and

$L_3 = 1$ corresponding to state $S_{(1,2,3)}$ and

$L_4 = 1$ corresponding to state $S_{(1,3)}$.

Then we see that we have the following set of equations for $L_0, \ldots, L_3$.

$$
\begin{aligned}
L_4 &= 1 && (8) \\
L_3 &= 1 && (9) \\
L_2 &= t\frac{x^3}{1-x}L_3 + tx^2L_4 + txL_1 && (10) \\
L_1 &= t\frac{x^3}{1-x}L_2 + t(x+x^2)L_1 && (11) \\
L_0 &= \frac{x}{1-x}L_1. && (12)
\end{aligned}
$$

One can easily solve this system of equations to find that

$$
L_0 = \frac{t^3x^6}{(1-x)^2(1-x-tx+tx^3-t^2x^4)} \tag{13}
$$

Theorem 0.7. *If* $\mathcal{P} = (P, \leq_P)$ *is a finite poset such that* $P \subseteq \mathbb{N}$*, then for all* u*,*

1. $\mathcal{F}(u)$*,* $\mathcal{A}(u)$*,* $\mathcal{S}(u)$ *are all rational languages.*
2. $F(u; x, t)$*,* $A(u; x, t)$*, and* $S(u; x, t)$ *are rational.*

Corollary 0.8. *For any* $u, v \in P^*$*, it is decidable whether* u *is Wilf equivalent to* v*.*

Theorem 0.9. *Let* $\mathcal{P} = (P, \leq)$ *be a poset with a weight function* $wt : P^* \to \mathbb{Z}[x_1, \ldots, x_n]$ *and* $u = u_1 \cdots u_n \in P^*$. *If*
(i) $\sum_{a \geq u_i} wt(a)$ *for all* i*,*
(ii) $\sum_{a \not\geq u} wt(a)$*, and*
(iii)$\sum_{a \in P} wt(a)$
are rational generating functions. Then $F(u; x_1, \ldots, x_n)$*,* $S(u; x_1, \ldots, x_n)$ *and* $A(u; x_1, \ldots, x_n)$ *are rational generating functions.*

We construct an automaton Γ that accepts $\mathcal{F}(u)$. Let U equal the set of letters in u.

The set of states of Γ is the set of all $w \in U^*$ such that $|w| \leq n$ including the empty word ϵ.

The start state is the empty word ϵ. The final states are the set of all $w \in U^n$ such that $u \leq w$.

Now suppose that $w = w_1 \cdots w_k \in U^*$ and $|w| \leq n$.

If w is a final state, then there is loop at w labeled a for every $a \in P$.

If w is not a final state, then there is an edge from w to ϵ labeled with a for every a which is not comparable to any element of U.

If w is not final state and $a \geq v$ where $v \in U$, then there an edge labeled with a from $w_1 \ldots w_k v$ if $k < n$ and an edge labeled with a from $w_2 \ldots w_n v$ if $k = n$.

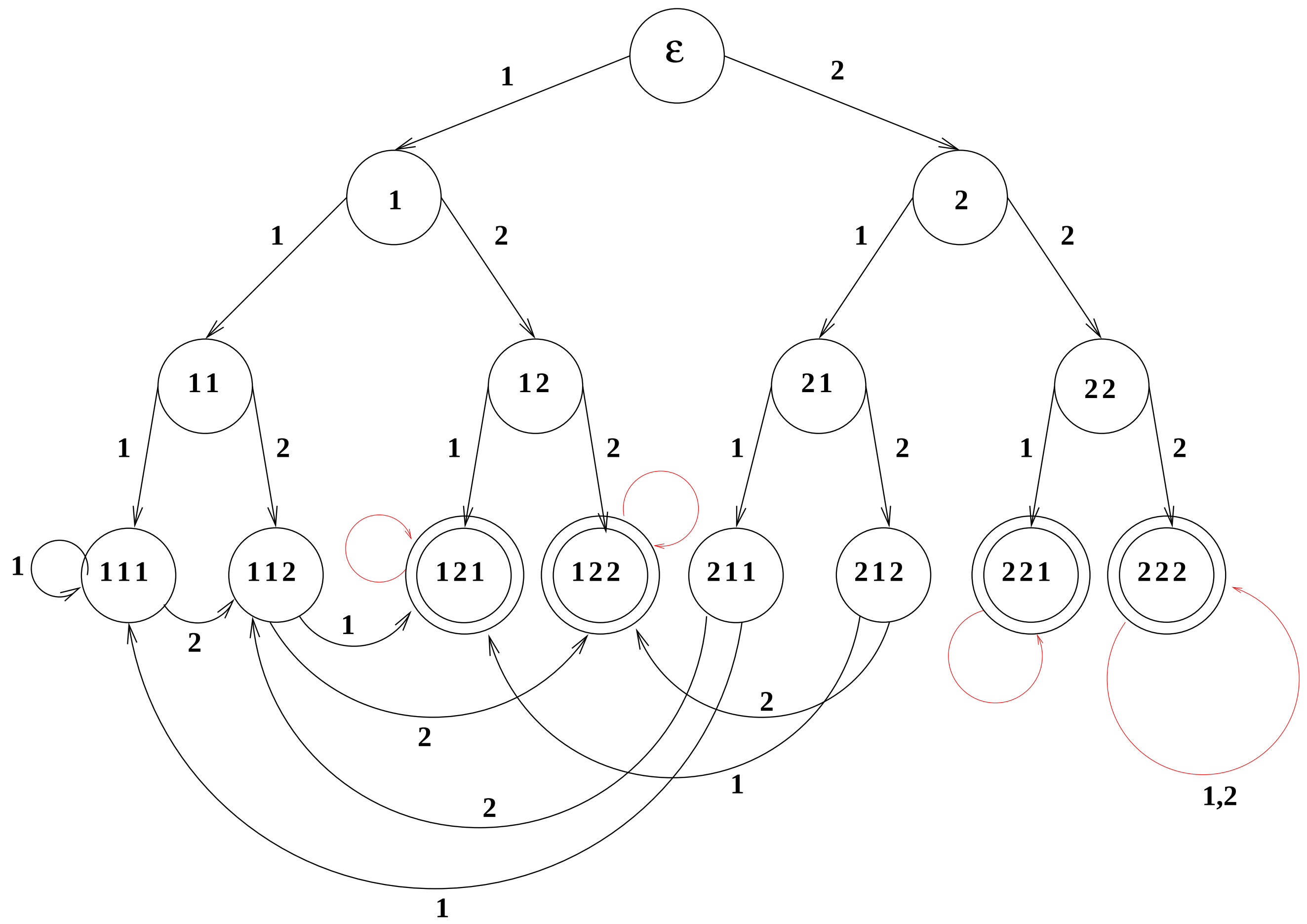

Figure 3: The NFA for $w = 1\ 2\ 1$.

1	$\frac{tx}{1-x}$
2	$\frac{tx^2}{(1-x)(1-tx)}$
3	$\frac{tx^3}{(1-x)(1-x-tx+tx^3)}$
11	$\frac{t^2x^2}{(1-x)^2}$
12,21	$\frac{t^2x^3}{(1-x)^2(1-tx)}$
13,31	$\frac{t^2x^4}{(1-x)^2(1-tx-tx^2)}$
22	$\frac{t^2x^4}{(1-x)(1-x-tx+tx^2-t^2x^3)}$
23,32	$\frac{t^2x^5}{(1-x)(1-x-tx+tx^3-t^2x^4)}$
33	$\frac{t^2x^6}{(1-x)(1-x-tx+tx^3-t^2x^4-t^2x^5)}$
111	$\frac{t^3x^3}{(1-x)^3}$
112,121,211	$\frac{t^3x^4}{(1-x)^3(1-tx)}$
122,221	$\frac{t^3x^5}{(1-x)^2(1-x-tx+tx^2-t^2x^3)}$
212	$\frac{t^3x^5(1+tx^2)}{(1-x)(1-x+t^2x^3)(1-x-tx+tx^2-t^2x^3)}$

113,131,311	$\frac{t^3x^5}{(1-x)^3(1-tx-tx^2)}$
213,312	$\frac{t^3x^6(1+tx^3)}{(1-x)(1-x+t^2x^4)(1-x-tx+tx^3-t^2x^4)}$
123,132,231,321	$\frac{t^3x^6}{(1-x)^2(1-x-tx+tx^3-t^2x^4)}$
222	$\frac{t^3x^6}{(1-x)(1-2x-tx+x^2+2tx^2-tx^3-t^2x^3+t^2x^4-t^3x^5)}$
133,331	$\frac{t^3x^7}{(1-x)^2(1-x-tx+tx^3-t^2x^4-t^2x^5)}$
313	$\frac{t^3x^7(1+tx^3+tx^4)}{(1-x)(1-x+t^2x^4+t^2x^5)(1-x-tx+tx^3-t^2x^4-t^2x^5)}$
223,232,322	$\frac{t^3x^7}{(1-x)(1-2x-tx+x^2+tx^2+tx^3-tx^4-t^2x^4+t^2x^5-t^3x^6)}$
323	$\frac{t^3x^8(1+tx^3)}{(1-x)(1-2x-tx+x^2+tx^2+tx^3-tx^4-t^2x^4+t^2x^5-t^3x^6-t^3x^7+t^3x^8-t^4x^9-t^4x^{10})}$
233,332	$\frac{t^3x^8}{(1-x)(1-2x-tx+x^2+tx^2+tx^3-tx^4-t^2x^4+t^2x^6-t^3x^7)}$
333	$\frac{t^3x^9}{(1-x)(1-2x-tx+x^2+tx^2+tx^3-tx^4-t^2x^4+t^2x^6-t^3x^7-t^3x^8)}$

The Wilf Equivalences for permutations of length 4

I	1234,1243,1342,1432,2341,2431,3421,4321
II	1324,1423,3241,4231
III	2134,2143,3412,4312
IV	3124,3214,4123,4213
V	2314,2413,3142,4132

If σ is a permutation in class IV, then $S(\sigma; x, t) = \frac{P(x,t)}{Q((x,t)}$ where

$$P(x,t) = t^4\,x^{10}\,(-1 + x - t\,x^4 + t\,x^6 + t^2\,(1+t)\,x^{10} + t^3\,x^{11} + t^4\,x^{14} + t^4\,x^{15})$$

and

$$\begin{aligned} Q(x,t) \quad = \quad & (-1+x) \times \\ & \big(1 - (4+t)\,x + 3\,(2+t)\,x^2 - (4+3\,t)\,x^3 + (1+2\,t)\,x^4 - t\,(3+t)\,x^5 + \\ & t\,(3+2\,t)\,x^6 + t\,(-1+t^2)\,x^7 - t^2\,(2+2\,t+t^2)\,x^8 + t^2\,(1+t)\,x^9 + \\ & (-1+t)\,t^3\,x^{10} + t^3\,(1+t+t^2)\,x^{12} - t^4\,(1+t)\,x^{13} + (-1+t)\,t^4\,x^{14} + \\ & t^4\,(1+2\,t)\,x^{15} - t^5\,(1+t)\,x^{17} + t^7\,x^{18} + t^6\,(1+2\,t)\,x^{19} + \\ & t^7\,x^{20} - t^7\,x^{21} + (-1+t)\,t^7\,x^{22} + 2\,t^8\,x^{23} + t^8\,x^{24}\big) \end{aligned}$$

12345, 12354, 12453, 12543, 13452, 13542, 14532, 15432, 23451, 23541, 24531, 25431, 34521, 35421, 45321, 54321
12435, 12534, 14352, 15342, 24351, 25341, 43521, 53421
13245, 13254, 14523, 15423, 32451, 32541, 45231, 54231
21345, 21354, 21453, 21543, 34512, 35412, 45312, 54312
23145, 23154, 45132, 54132
32145, 32154, 45123, 54123
24153, 25143, 34152, 35142
14235, 14325, 15234, 15324, 42351, 43251, 52341, 53241
31425, 31524, 32415, 32514, 41523, 42513, 51423, 52413
24315, 25314, 41352, 51342
24135, 25134, 43152, 53142
34215, 35214, 41253, 51243
34125, 35124, 42153, 52143
41325, 42315, 51324, 52314
41235, 43215, 51234, 53214
42135, 43125, 52134, 53124
13425, 13524, 14253, 15243, 34251, 35241, 42531, 52431
21435, 21534, 43512, 53412
24513, 25413, 31452, 31542
23415, 23514, 41532, 51432
31245, 31254, 45213, 54213

21 classes.

Wilf equivalence relative to the poset $\mathcal{P} = (\mathbb{P}, \leq)$.

For any word $u \in \mathbb{P}^*$ and integer $k \geq 1$, we let u^{+k} be the result of adding k to each letter.

For example, if $u = 1\ 3\ 1\ 2$, then $u^{+3} = 4\ 6\ 4\ 5$.

Similarly, if $w \in \{k+1, k+2, \cdots\}^*$, then we let w^{-k} denote the result of subtracting k from each letter.

If $u = u_1 \cdots u_n$, we let $u^r = u_n \cdots u_1$ denote the reverse of u.

Theorem 0.10. *Let* $\mathcal{P} = (\mathbb{P}, \leq)$ *and* $u, v \in \mathbb{P}$. *Then*

(i) $u \backsim u^r$,

(ii) $u \backsim v$ *implies* $u1^a \backsim v1^a$ *for all* $a \geq 1$,

(iii) $u1^a \backsim 1^a u$ *for all* $a \geq 1$, *and*

(iv) $u \backsim v$ *implies* $u^{+k} \backsim v^{+k}$ *for any* $k \geq 1$.

Corollary 0.11. *Let* $\mathcal{P} = (\mathbb{P}, \leq)$ *and* $u, v \in \mathbb{P}$*. Then*

(i) $1^a u 1^b \backsim 1^c u 1^d$ *whenever* $a + b = c + d$ *and*

(ii) *for any of pair increasing words* $\alpha, \alpha' \in \{1, \ldots, k+1\}^*$ *and decreasing words* $\beta, \beta' \in \{1, \ldots, k+1\}^*$ *such that for all* $1 \leq i \leq k$*, the number of occurrences of* i *in* $\alpha\beta$ *equals the number of occurrences of* $i \in \alpha'\beta'$*,* $u \backsim v$ *implies* $\alpha u^{+k} \beta \backsim \alpha' v^{+k} \beta'$*.*

Theorem 0.12. *Let $n \geq 3$, $u = \alpha(n-1)\beta n\gamma$, and $v = \alpha n\beta(n-1)\gamma$ where $\alpha, \beta, \gamma \in \{1, \ldots, n-1\}^*$. Then $u \backsim v$.*

We should also observe that interchanging the the positions of the top two letters, n and $n-1$, in a word does not necessarily preserve Wilf equivalence when there is more than one occurrence of n. For example, $122 \not\backsim 212$.

Example

We need a bijection $\Theta : \mathcal{A}(123) \to \mathcal{A}(132)$.

Case 1. Use the idenity on $\mathcal{A}(123) \cap \mathcal{A}(132)$.

Case 2 Thus you need to define

$\Theta : \mathcal{A}(123) - \mathcal{A}(132) \to \mathcal{A}(132) - \mathcal{A}(123)$.

1322211

1232211

1223211

1222311

Using all of the previous results, we explain all but two cases of the Wilf equivalences for S_5.

That is, our results imply that $21345 \backsim 21354 \backsim 45312 \backsim 54312$ and $21453 \backsim 21543 \backsim 34512 \backsim 35412$, but they do not tell why these two groups are Wilf equivalent to each other.

Similarly our results imply that $31425 \backsim 31524 \backsim 42513 \backsim 52413$ and $32415 \backsim 32514 \backsim 41532 \backsim 51432$ but they do not tell us why these two groups are Wilf equivalent to each other.

There is another phenomenon that we observed from our computations of the generating functions $S(u;x,t)$. It seems to be the case that if $u_1 \cdots u_n \backsim v_1 \cdots v_n$, then $u_1^k \cdots u_n^k \backsim v_1^k \cdots v_n^k$ for all $k \geq 2$. That is, the operation of replacing each letter in word by k copies of itself seems to preserve Wilf equivalence. For example, we know $132 \backsim 321$. Then one can compute that

$$S(113322;x,t) = S(332211;x,t) =$$

$$\frac{x^{18}(1+x)^2(1+x-x^4)(1+x^3-x^6)^2}{(-1+x)^3(-1+2x+x^4-x^5+x^6+x^7+x^9+2x^{12}-x^{14}+3x^{15}+6x^{16}+4x^{17}-3x^{19}-3x^{20}-x^{21})}$$

so that $113322 \backsim 332211$. Similarly, we have computed that

$$S(111333222;x,t) = S(333222111;x,t)$$

so that $111333222 \backsim 333222111$.

Strong Wilf Equivalence

Given words u and w, we let $\mathcal{EM}(u,w)$ equal the set of all i such that u has an embedding in w which starts at position i. For example if $u = 123$ and $w = 11334112456$, then $\mathcal{EM}(u,w) = \{2,3,7,8,9\}$.

Then we say that u **is strongly Wilf equivalent to** v, written $u \backsim_s v$, if there is a weight preserving bijection from $\Gamma : \mathbb{P}^* \longrightarrow \mathbb{P}^*$ such that for all $w \in \mathbb{P}^*$, $\mathcal{EM}(u,w) = \mathcal{EM}(v,\Gamma(w))$.

Theorem 0.13. *If* $u = u_1 \cdots u_n$ *and* $v = v_1 \cdots v_n$ *are words in* $\mathbb{P}^*$, *then* $u \backsim_s w$ *implies* $u_1^k \cdots u_n^k \backsim_s v_1^k \cdots v_n^k$.

Fix $k \geq 2$ and assume that $u \sim_s w$. Thus there is a weight preserving bijection $\Gamma : \mathbb{P}^* \to \mathbb{P}^*$ such that for all $w \in \mathbb{P}^*$, $\mathcal{EM}(u, w) = \mathcal{EM}(v, \Gamma(w))$.

For any word $w \in \mathbb{P}$ and any $1 \leq i \leq k$, we let $w^{(i)}$ denote the word that results by taking the letters in w that are in positions of the form $i + jk$. For example, if $w = 3\ 3\ 4\ 1\ 2\ 4\ 6\ 2\ 1\ 2$ and $k = 3$, then $w^{(1)} = 3\ 1\ 6\ 2$, $w^{(2)} = 3\ 2\ 2$, and $w^{(3)} = 4\ 4\ 1$.

It is easy see that the positions which start an embedding of $u_1^k \cdots u_n^k$ in w are completely determined by the positions which start an embedding of $u_1 \cdots u_n$ in each of the $w^{(i)}$'s. We then define a map $\Gamma^{(k)} : \mathbb{P}^* \to \mathbb{P}^*$ such that $\Gamma^{(k)}(w) = \bar{w}$ if and only if for each $1 \leq i \leq k$, $\bar{w}^{(i)} = \Gamma(w^{(i)})$.

Theorem 0.14. *Let u and v be words of length n in $\mathbb{P}^*$ and let $a \geq 1$. Then*

(1) $1^a u \backsim_s u1^a$.

(2) $u \backsim_s v$ *implies* $1^a u \backsim_s 1^a v$ *and* $u1^a \backsim_s v1^a$.

(3) *For all* $k \geq 1$, $u \backsim_s v$ *implies* $u^{+k} \backsim_s v^{+k}$

Corollary 0.15. *Let* $\mathcal{P} = (\mathbb{P}, \leq)$ *and* $u, v \in \mathbb{P}$. *Then*

(1) $1^a u 1^b \backsim_s 1^c u 1^d$ *whenever* $a + b = c + d$ *and*

(2) *for any pair increasing words* $\alpha, \alpha' \in \{1, \ldots, k\}^*$ *and decreasing words* $\beta, \beta' \in \{1, \ldots, k\}^*$ *such that for all* $1 \leq i \leq k$, *the number of occurrences of* i *in* $\alpha\beta$ *equals the number of occurrences of* $i \in \alpha'\beta'$, *we have that* $u \backsim_s v$ *implies* $\alpha u^{+k} \beta \backsim_s \alpha' v^{+k} \beta'$.

We claim that 2 1 4 3 is not strongly Wilf equivalent to 3 4 1 2. Consider how we can construct of minimum weight word w of length 7 such that $\mathcal{EM}(2\ 1\ 4\ 3, w) = \{1, 3, 4\}$. That is, consider the following calculations.

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2 1 4 3
    2 1 4 3
      2 1 4 3
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2 1 4 3 4 4 3
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3 4 1 2
    3 4 1 2
      3 4 1 2
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3 4 3 4 4 2 2
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Figure 4: Calculations of minimum weight words for embeddings.

Explicit Formulas

Theorem 0.16. *For any* $s \geq 2$, $\ell \geq 1$, *and* $k \geq 0$,

$$S(1^k s^\ell : x, t) = \tag{14}$$

$$\frac{x^{k+\ell s} t^{k+\ell}}{(1-x)^{k+1}((x^s t)^{\ell-1}(1-xt[s-1]_x) + \sum_{r=1}^{\ell-2}(1-x)^r(1-x-xt)(x^s t)^{l-2-r})}$$

where $[t]_x = 1 + x + x^2 + \cdots + x^{t-1}$.

Define a sequence of polynomials $B_n(x,t)$ for $n \geq 2$ which are defined by recursion as follows.

$$\begin{aligned} B_2(x,t) &= xt \qquad (15) \\ B_{k+1}(x,t) &= x^{k+1}t(B_k(x,t) - (1-x)^{k-2}) + xt(1-x)^{k-2} \text{ for } k \geq 2. \end{aligned}$$

Theorem 0.17. *For $n \geq 2$,*

$$S(12\cdots n; x,t) = \frac{x^{\binom{n+1}{2}}}{(1-x)^2 C_n(x,t)} \qquad (16)$$

where $C_n(x,t) = (1-x)^{n-2} - B_n(x,t)$.

The Mobius Function of the Factor Order

For any words u and v, we say $u \ll w$ if u is a factor of w. (This is the generalized factor order for the antichain.)

We say that u is a *left factor* (*right factor*) of w is u prefix (suffix) of w.

The *dominant outer factor* of w, $o(w)$, is the longest word other than w which is both a left factor and a right factor.

The *dominant inner factor* of w, $i(w)$, is $w_2 \cdots w_{n-1}$ if $w = w_1 \cdots w_n$.

A word *trivial* of w if all of its letters are equal, i.e. $w = k^n$.

Example: $w = abbaabb$, then $o(w) = abb$ and $i(w) = bbaab$.

Theorem 0.18. *(Björner). In the factor order, if $u \ll w$, then*

$$\mu(u,v) = \begin{cases} \mu(u,o(w)) & \text{if } |w|-|u| \geq 2 \text{ and } u \ll o(w) \not\ll i(w) \\ 1 & \text{if } |w|-|u| = 2,\ w \text{ is not trivial,} \\ & \text{and } u = o(w) \text{ or } u = i(w) \\ (-1)^{|w|-|u|} & \text{if } |w|-|u| < 2, \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

Question: Is $\{w : \mu(u,w) \neq 0\}$ rational?

Answer No!

Lemma 0.19. *(***Pumping Lemma** *Let $M = (Q, \Sigma, \delta, s, F)$ be a DFA and $p = |Q|$. For all words $w \in L(M)$ such that $|w| \geq p$, w can be factored as $w = xyz$ for some words x, y, and z such that*

1. *$|xy| \leq p$,*
2. *$|y| \geq 1$, and*
3. *for all $i \geq 0$, $xy^i z \in L(M)$.*

Idea of the proof that $\{w : \mu(u, w) \neq 0\}$.

Suppose we are give p from the pumping lemma.

Then $w = ab^{p+1}ab^{2p+2}ab^{p+1}a$ and $u = a$.

$o(w) = ab^{p+1}a \quad i(w) = b^{p+1}ab^{2p+2}ab^{p+1}$

Thus $u \ll o(w) \not\ll i(w)$ and, hence, $\mu(u, w) = \mu(a, o(w))$.

If $v = ab^{n+1}a$, then $o(v) = a$ and $i(v) = b^{p+1}$. Hence $\mu(a, v) = \mu(a, o(v)) = \mu(a, a) = 1$.

However, then pumping lemma fails!

Case 1: $x = \epsilon$ and $y = a$.
Then $v = xy^3z$ equals $a^3b^{p+1}ab^{2p+2}ab^{p+1}a$ which implies $o(w) = a$ and $a \ll i(w)$ so $\mu(u, v) = 0$.

Case 2: $x = \epsilon$ and $y = ab^i$ for some $1 \leq i \leq p-1$.
Then $v = xy^2z = ab^iab^ib^{p+1-i}ab^{2p+2}ab^{p+1}a$.

$o(v) = a$ and $i(v) = b^iab^ib^{p+1-i}ab^{2p+2}ab^{p+1}$ so $\mu(a, v) = 0$.

Case 3: $x \neq \epsilon$ and $x = ab^i$ for some $1 \leq i \leq p-2$ and $y = b^j$. for some j.
Then $v = xy^{p+1}z = ab^ib^{(p+1)j}b^{p+1-i-j}ab^{2p+2}ab^{p+1}a$.

$o(v) = a$ and $i(v) = b^ib^{(p+1)j}b^{p+1-i-j}ab^{2p+2}ab^{p+1}$ so $\mu(a, v) = 0$.

Open Questions

(1) **If $u \sim v$, then must v be a rearrangement of u?**

(2) **Does $u \sim v$ imply that there is a bijection $\Theta : \mathbb{P}^* \rightarrow \mathbb{P}^*$ such that for all $w \in \mathbb{P}^*$, $w \in \mathcal{F}(u) \iff \Theta(w) \in \mathcal{F}(v)$ and $\Theta(w)$ is a rearrangement of w?**

This is true for all the examples in the tables.

That is, suppose that $[m]$ is the finite poset consisting of the integers $[m] = \{1, \ldots, m\}$ under the standard order. For any word $w \in [m]^*$ and $i \in [m]$, let $c_i(w)$ equal the number of occurrences of i in w. Then we can define the weight of of w,

$W_{[m]}(w) = \prod_{i=1}^{m} x_i^{c_i(w)}$ and set

$$\begin{aligned}
S(u; x_1, \ldots, x_m) &= \sum_{w \in \mathcal{S}(u)} W_{[m]}(w), \\
F(u; x_1, \ldots, x_m) &= \sum_{w \in \mathcal{F}(u)} W_{[m]}(w), \text{ and} \\
A(u; x_1, \ldots, x_m) &= \sum_{w \in \mathcal{A}(u)} W_{[m]}(w).
\end{aligned}$$

(3) **Find a theorem which, together with the results already proved, explains all the Wilf equivalences in S_5.** In particular, our bijective results show that

$$21345 \sim 21354 \sim 45312 \sim 54312 \text{ and}$$
$$21453 \sim 21543 \sim 34512 \sim 35412$$

but not why a permutation of the first group is Wilf equivalent to one of the second. The other row of Table 1 which breaks into two groups is

$$31425 \sim 31524 \sim 42513 \sim 52413 \text{ and}$$
$$32415 \sim 32514 \sim 41523 \sim 51423.$$

(4) **Is it always the case that the number of elements of** S_n **Wilf equivalent to a given permutation is a power of 2?**

(5) **Is it true that** $312 \sim_s 213$**?**