Affine Schubert News

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AMS meeting Claremont, May 3, 2008

Outline

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Affine Schubert calculus

Non-symmetric Macdonald polynomials

Affine crystals

Stanley symmetric functions

Schubert calculus

- **Enumerative Geometry:** counting subspaces satisfying certain intersection conditions (Hilbert's 15th problem) Schubert, Pieri, Giambelli,... 1874
- **Cohomology:** computations in cohomology ring of the Grassmannian $H^*(G/P)$ with $G = SL_n(\mathbb{C})$ and $P \subset G$ maximal parabolic 1950's
- **Symmetric Functions:** cohomology ring of Grassmannian (with its natural Schubert basis) same as the algebra of symmetric functions (with Schur basis) 1950's
- **Combinatorics:** multiplication of Schubert basis governed by Littlewood-Richardson rule 1970's

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Affine Schubert calculus

Definition

G affine Kac–Moody group
$P \subset G$ maximal parabolic subgroup
G/P affine Grassmannian Gr

Example: $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$
affine Grassmannian $Gr = SL_{k+1}(\mathcal{K})/SL_{k+1}(\mathcal{O})$

Theorem (Lam)

Schubert bases of $H_*(Gr)$ and $H^*(Gr)$ are given by k-Schur functions and dual k-Schur functions of Lascoux, Lapointe, Morse

Structure constants include genus zero Gromov-Witten invariants or fusion coefficients

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Topics related to affine Schubert calculus

- **Symmetric functions:**
  - Macdonald polynomials
  - k-Schur functions and k-branching (Thomas Lam's talk)
- **Representation theory:**
  - representation theoretic interpretation of k-Schur functions (Mark Haiman's talk)
  - Hecke algebras (Nicolas Thiery's talk)
  - Crystals (Jason Bandlow's talk)
- **Combinatorics:**
  - Core partitions (Brant Jones's talk)

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Symmetric group

Definition (Symmetric group)

The symmetric group S_n

- generators $s_1, \ldots, s_{n-1}$
- relations

$$
\begin{aligned}
s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2 \\
s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \\
s_i^2 &= 1
\end{aligned}
$$

Affine symmetric group

Definition

The affine symmetric group $\tilde{S}_n$

- generators $s_0, s_1, \ldots, s_{n-1}$
- relations

$$s_i s_j = s_j s_i \qquad \text{for } |i-j| \geq 2$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$
$$s_i^2 = 1$$

Remark

All indices $i \in [0, n-1]$ are taken modulo n.

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Affine Hecke algebra

Definition

Affine Hecke algebra $\mathcal{H}$

- generated by $T_0, \ldots, T_{n-1}$
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$$\begin{aligned} T_i T_j &= T_j T_i && |i-j| \geq 2 \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \\ (T_i - t)(T_i + 1) &= 0 \end{aligned}$$

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Action on polynomials

$$s_i f(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$$

$$s_0 f(x_1, \ldots, x_n) = f(qx_n, x_2, \ldots, x_{n-1}, x_1/q)$$

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$$(T_i - t)(T_i + 1) = 0$$

with indices modulo n

Action on polynomials

$$T_i = ts_i + \frac{(1-t)x_{i+1}}{x_i - x_{i+1}}(1 - s_i)$$

$$T_0 = ts_0 + \frac{(1-t)x_1}{qx_n - x_1}(1 - s_0)$$

Non-symmetric Macdonald polynomials

Definition (Cherednik)

$E_\alpha(x_1,\ldots,x_n;q,t)$ are characterized by

1. $E_{s_i(\alpha)}(X;q,t) = \left(T_i + \frac{1-t}{1-q^{\langle \mu\,,\,\alpha_i^\vee \rangle} t^{\langle w_\mu(\rho)\,,\,\alpha_i^\vee\rangle}}\right) E_\alpha(X;q,t)$, $i \neq 0$
2. $E_{\pi(\alpha)}(X;q,t) = q^{\alpha_n} x_1 \Psi E_\alpha(X;q,t)$

$$s_i(\alpha_1,\ldots,\alpha_n) = (\alpha_1,\ldots,\alpha_{i+1},\alpha_i,\ldots,\alpha_n)$$
$$\pi(\alpha_1,\ldots,\alpha_n) = (\alpha_n+1,\alpha_1,\ldots,\alpha_{n-1})$$
$$\Psi f(x_1,\ldots,x_n) = x_1 f(x_2,\ldots,x_n,x_1/q)$$
$$\rho = (n-1,n-2,\ldots,1)$$

w_μ maximum length permutation such that $w_\mu^{-1}(\mu)$ is dominant

Non-symmetric Macdonald as eigenfunctions

Definition (Haglund, talk at MSRI)

$E_{\alpha,\omega}$ depending on composition α and permutation ω

1. $T_i E_{\alpha,\omega}(X; q, t) = t^{\cdots} E_{\alpha,\omega.s_i}(X; q, t)$
2. $T_0 E_{\alpha,\omega}(X; q, t) = q^{\cdots} t^{\cdots} E_{\alpha,\omega.s_0}(X; q, t)$

If $s_{i_1} \cdots s_{i_k} = 1$ in $W_0 = S_n$, then

$$T_{i_1} \cdots T_{i_k} E_{\alpha,\omega} = q^{\cdots} t^{\cdots} E_{\alpha,\omega}$$

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Level-0 action

How can $s_{i_1} \cdots s_{i_k} = 1$ in W_0?

Answer: $s_{i_1} \cdots s_{i_k}$ is a translation in the affine Weyl group W!

s_1, s_2, s_0

$s_1 s_2 s_1 s_0 = 1$ in S_3

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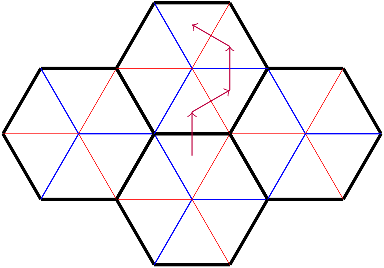

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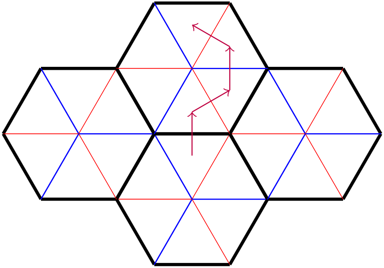

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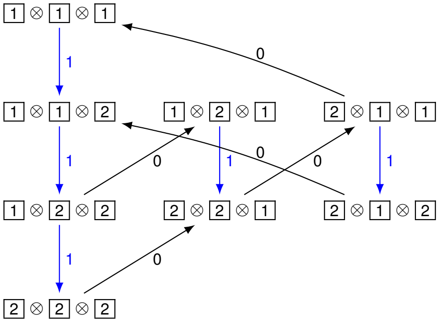
Crystal graph



Can we understand the q-statistic from this point of view?

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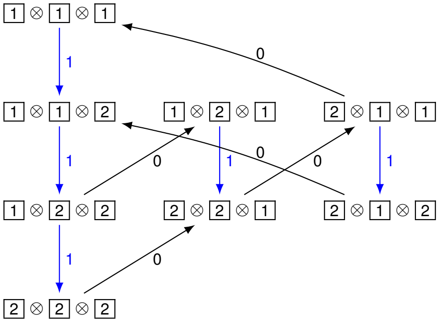

Can we understand the q-statistic from this point of view?

Motivation

$\mathfrak{g}$ Lie algebra/Kac–Moody Lie algebra

- Crystal bases are combinatorial bases for $U_q(\mathfrak{g})$ as $q \to 0$
- Affine finite crystals:
  - appear in 1d sums of exactly solvable lattice models
  - path realization of integrable highest weight $U_q(\mathfrak{g})$-modules
  - fermionic formulas
- Irreducible finite-dimensional affine $U_q(\mathfrak{g})$-modules classified by Chari-Pressley via Drinfeld polynomials
- HKOTY conjectured that the Kirillov-Reshetikhin modules $W^{r,s}$ have crystal bases

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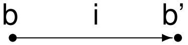
Axiomatic Crystals

A $U_q(\mathfrak{g})$-crystal is a nonempty set B with maps

$$\mathrm{wt}\colon B \to P$$
$$e_i, f_i\colon B \to B \cup \{\emptyset\} \quad \text{for all } i \in I$$

satisfying

$$f_i(b) = b' \Leftrightarrow e_i(b') = b \qquad \text{if } b, b' \in B$$
$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i \qquad \text{if } f_i(b) \in B$$
$$\langle h_i\,, \mathrm{wt}(b)\rangle = \varphi_i(b) - \varepsilon_i(b)$$

Write 

for $b' = f_i(b)$

Tensor products

Definition

B, B' crystals

$B \otimes B'$ is $B \times B'$ as sets with

$$\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$$

$$f_i(b \otimes b') = \begin{cases} f_i(b) \otimes b' & \text{if } \varepsilon_i(b) \geq \varphi_i(b') \\ b \otimes f_i(b') & \text{otherwise} \end{cases}$$

$$\underbrace{\overset{b}{- - -}}_{\varphi_i(b)} \otimes \underbrace{\overset{b'}{+ + + +}}_{\varepsilon_i(b')}$$

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Energy function

Definition

Local energy function $H : B \otimes B \to \mathbb{Z}$

$$H(e_i(b \otimes b')) = H(b \otimes b') \qquad \text{if } i \neq 0$$

$$H(e_0(b \otimes b')) = H(b \otimes b') + \begin{cases} -1 & \text{if } e_0 \text{ acts right} \\ 1 & \text{if } e_0 \text{ acts left} \end{cases}$$

Definition

Global energy function $E : B^{\otimes L} \to \mathbb{Z}$

$$E(b_1 \otimes \cdots \otimes b_L) = \sum_{1 \leq i < L} i \cdot H(b_i \otimes b_{i+1})$$

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Example of energy function

Example

Type A_{n-1}: $B = \{\boxed{1}, \ldots, \boxed{n}\}$

$$H(b \otimes b') = \begin{cases} 1 & \text{if } b > b' \\ 0 & \text{otherwise} \end{cases}$$

descents or inversions

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$$E(b) = 1 + \cdots$$

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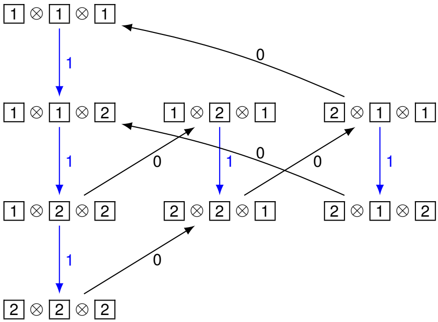

In S_2, $s_0 s_1 = 1$, but in crystal $(s_0 s_1)^3 = 1$

In general:
If $p = s_{i_1} \cdots s_{i_k} = 1$ in W_0, then $p^{kL} = 1$ in crystal $B^{\otimes L}$.

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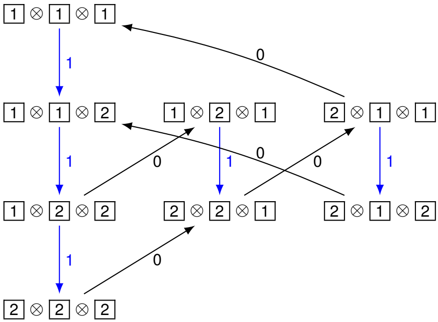

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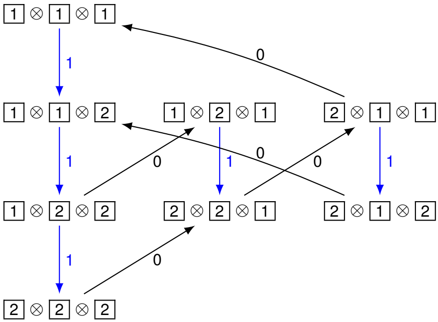

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In general:

If $p = s_{i_1} \cdots s_{i_k} = 1$ in W_0, then $p^{kL} = 1$ in crystal $B^{\otimes L}$.

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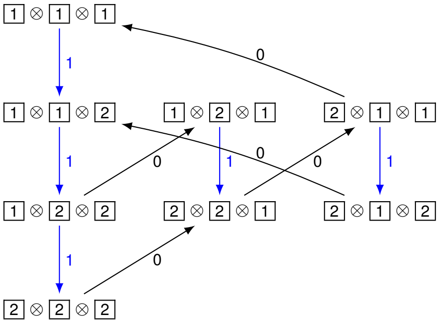

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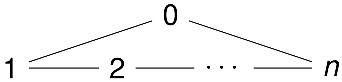
(Open) problems

- Further unfolding of alcove picture to capture crystal picture?
- Geometric interpretation of energy function?
- How do we get affine crystal structures?

Dynkin automorphism

Type $A_{n-1}^{(1)}$:

KMN[2] proved existence of crystals $B^{r,s}$ for Kirillov-Reshetikhin modules $W^{r,s}$



automorphism σ

Promotion operator pr uniquely defined by Shimozono

$$\begin{array}{ccc} B^{r,s} & \xrightarrow{\mathrm{pr}} & B^{r,s} \\ f_a \downarrow & & \downarrow f_{a+1} \\ B^{r,s} & \xrightarrow[\mathrm{pr}]{} & B^{r,s} \end{array}$$

$$\langle h_{a+1}\,,\ \mathrm{wt}(\mathrm{pr}(b))\rangle = \langle h_a\,,\ \mathrm{wt}(b)\rangle$$

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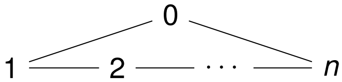

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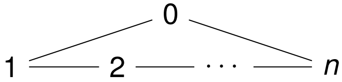

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Then $\quad e_0 = \mathrm{pr}^{-1} \circ e_1 \circ \mathrm{pr} \qquad f_0 = \mathrm{pr}^{-1} \circ f_1 \circ \mathrm{pr}$

Dynkin automorphism

Type	Dynkin diagram
$A_5^{(1)}$	1 — 0 — 5; 1 — 2 — 3 — 4 — 5
$B_5^{(1)}$	0, 1 — 2 — 3 — 4 $\xrightarrow{2}$ 5
$A_9^{(2)}$	0, 1 — 2 — 3 — 4 $\xleftarrow{2}$ 5
$D_5^{(1)}$	0, 1 — 2 — 3 — 4, 5
$C_5^{(1)}$	0 $\xrightarrow{2}$ 1 — 2 — 3 — 4 $\xleftarrow{2}$ 5
$D_5^{(2)}$	0 $\xleftarrow{2}$ 1 — 2 — 3 $\xrightarrow{2}$ 4
$A_{10}^{(2)}$	0 $\xleftarrow{2}$ 1 — 2 — 3 — 4 $\xleftarrow{2}$ 5

Existence of Kirillov-Reshetikhin crystals

Theorem (OS)

The Kirillov-Reshetikhin crystals $B^{r,s}$ exist for nonexceptional types.

Combinatorial models for these crystals can be constructed using the classical decompositions

$$B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda)$$

and the automorphism σ (i special node $\sigma(i) = 0$)

$$f_0 = \sigma^{-1} \circ f_i \circ \sigma$$
$$e_0 = \sigma^{-1} \circ e_i \circ \sigma$$

Dynkin diagram automorphisms

Question: What about several tensor factors?

Promotion operator acts on each tensor factor

$$\mathrm{pr}(b_1 \otimes \cdots \otimes b_L) = \mathrm{pr}(b_1) \otimes \cdots \otimes \mathrm{pr}(b_L)$$

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$\rightsquigarrow$ see Jason Bandlow's talk

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Outline

Stanley symmetric functions

Introduced in 1984 by Stanley

- used to study # of reduced words of $w \in S_n$
- closely related to Schubert polynomials of Lascoux and Schützenberger (related to geometry of flag varieties)

Definition

For $w \in S_n$

$$F_{w^{-1}}(x_1, x_2, \ldots) = \sum_{a_1 \cdots a_\ell \in \mathcal{R}(w)} \sum_{\substack{1 \le b_1 \le \cdots \le b_\ell \\ a_i > a_{i+1} \Rightarrow b_{i+1} > b_i}} x_{b_1} \cdots x_{b_\ell}$$

where $\mathcal{R}(w)$ is the set of reduced words for w.

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Properties

Theorem (Stanley)

1. $F_w(x)$ *is a symmetric function.*
2. $[x_1 \cdots x_{\ell(w)}]F_w(x) =$ *number of reduced words for* w

Theorem (Edelman-Greene, Lascoux-Schützenberger)

The Schur expansion of F_w *is positive.*

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nilCoxeter algebra

Definition (nilCoxeter algebra)

The nilCoxeter algebra

- generators $u_1, \ldots, u_{n-1}$
- relations

$$\begin{aligned} u_i u_j &= u_j u_i && \text{for } |i-j| \geq 2 \\ u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1} \\ u_i^2 &= 0 \end{aligned}$$

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Stanley symmetric functions for other types

- For each Weyl group W one can construct a new nilCoxeter algebra by taking the associated graded $\mathbb{C}[W]$.
- Finding Stanley symmetric functions for each W is equivalent to finding a particular commutative subalgebra of the nilCoxeter algebra.

Theorem (Lam; LSS)

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Enjoy the remaining talks !