Noncommutative Symmetric Functions which Interpolate.

Lenny Tevlin

Physics Department, Yeshiva University

500 West 185th Street, New York, N.Y. 10033

U.S.A.

AMS Spring Western Section Meeting, May 3-4, 2008

Plan of the talk:

- Noncommutative symmetric functions without parameters.

Plan of the talk:

- Noncommutative symmetric functions without parameters.
  - Compositions and operations with them.

Plan of the talk:

- Noncommutative symmetric functions without parameters.
  - Compositions and operations with them.
  - Monomial, fundamental, and ribbon Schur symmetric functions.

Plan of the talk:

- Noncommutative symmetric functions without parameters.
  - Compositions and operations with them.
  - Monomial, fundamental, and ribbon Schur symmetric functions.
  - A noncommutative Cauchy identity and noncommutative bilinear form.

Plan of the talk:

- Noncommutative symmetric functions without parameters.
  - Compositions and operations with them.
  - Monomial, fundamental, and ribbon Schur symmetric functions.
  - A noncommutative Cauchy identity and noncommutative bilinear form.
- Hall-Littlewood type noncommutative symmetric functions which interpolate.

Plan of the talk:

- Noncommutative symmetric functions without parameters.
  - Compositions and operations with them.
  - Monomial, fundamental, and ribbon Schur symmetric functions.
  - A noncommutative Cauchy identity and noncommutative bilinear form.
- Hall-Littlewood type noncommutative symmetric functions which interpolate.
- Hook Jack type noncommutative symmetric functions which interpolate.

Compositions, Their Reverses and Conjugates.

Compositions, Their Reverses and Conjugates.

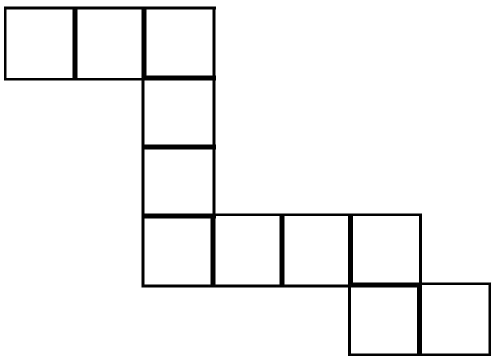
A composition is ordered set of integers: $I = (i_1, \ldots, i_n)$. The sum of all parts is denoted by $|I|$, and the number of parts – by $\ell(I)$.

Compositions, Their Reverses and Conjugates.

A composition is ordered set of integers: $I = (i_1, \dots, i_n)$. The sum of all parts is denoted by $|I|$, and the number of parts – by $\ell(I)$.

$$I = (3, 1, 1, 4, 2),\ |I| = 11,\ \ell(I) = 5$$

$I =$



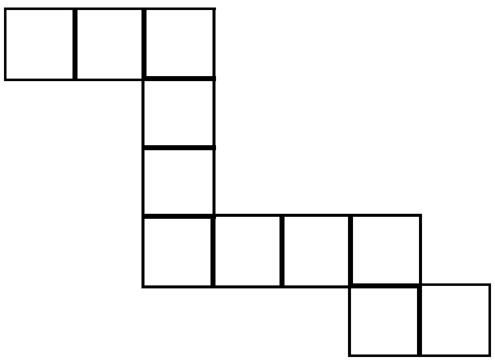
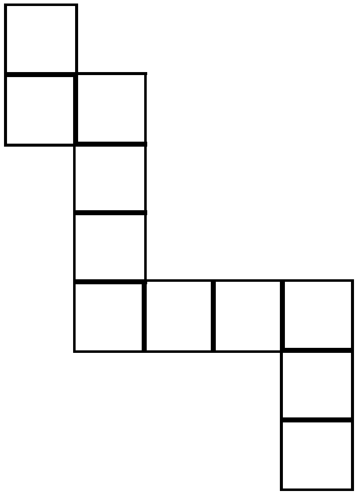
Compositions, Their Reverses and Conjugates.

A composition is ordered set of integers: $I = (i_1, \ldots, i_n)$. The sum of all parts is denoted by $|I|$, and the number of parts – by $\ell(I)$. For a composition I define a **reverse** composition $\overline{I} = (i_n, \ldots, i_1)$.
For instance, if $I = (3, 1, 1, 4, 2)$, then $\overline{I} = (2, 4, 1, 1, 3)$.

Compositions, Their Reverses and Conjugates.

A composition is ordered set of integers: $I = (i_1, \ldots, i_n)$. The sum of all parts is denoted by $|I|$, and the number of parts – by $\ell(I)$. Parts of a **conjugate** composition $\widetilde{I}$ can be read from the diagram of the composition I from left to right and from bottom to top:

$$I = (3, 1, 1, 4, 2) \quad \widetilde{I} = (1, 2, 1, 1, 4, 1, 1)$$

$I =$  $\widetilde{I} =$ 

Reverse Refinement Order.

Let $I = (i_1, \ldots, i_n), J = (j_1, \ldots, j_k)$, $|J| = |I|$ then I is said to be greater in the **reverse refinement order** (or, simply, **finer**) than J,

$$I \succ J$$

if $J = (i_1 + \ldots + i_{p_1}, \ldots, i_{p_{s-1}+1} + \ldots + i_{p_s}, \ldots, i_{p_{k-1}+1} + \ldots + i_n)$

Notation: $lp(I, J) = \prod_{s=1}^{\ell(J)} i_{p_s}$.

Reverse Refinement Order.

Let $I = (i_1, \ldots, i_n), J = (j_1, \ldots, j_k)$, $|J| = |I|$ then I is said to be greater in the **reverse refinement order** (or, simply, **finer**) than J,

$$I \succ J$$

if $J = (i_1 + \ldots + i_{p_1}, \ldots, i_{p_{s-1}+1} + \ldots + i_{p_s}, \ldots, i_{p_{k-1}+1} + \ldots + i_n)$

Notation: $lp(I, J) = \prod_{s=1}^{\ell(J)} i_{p_s}$.

For instance, $(3, 3, 2) \prec (3, 1, 2, 2)$

Reverse Refinement Order.

Let $I = (i_1, \ldots, i_n)$, $J = (j_1, \ldots, j_k)$, $|J| = |I|$ then I is said to be greater in the **reverse refinement order** (or, simply, **finer**) than J,

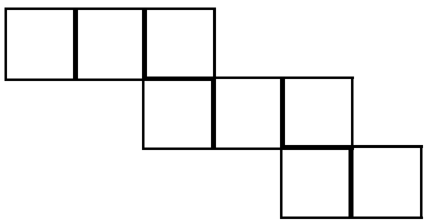
$$I \succ J$$

if $J = (i_1 + \ldots + i_{p_1}, \ldots, i_{p_{s-1}+1} + \ldots + i_{p_s}, \ldots, i_{p_{k-1}+1} + \ldots + i_n)$

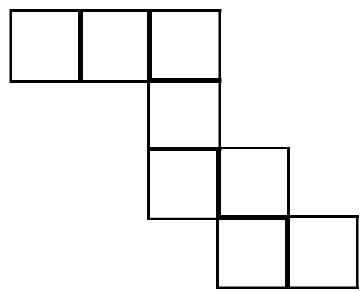
Notation: $lp(I, J) = \prod_{s=1}^{\ell(J)} i_{p_s}$.

For instance, $(3, 3, 2) \prec (3, 1, 2, 2)$

$p_1 = 1$
$p_2 = 3$
$p_3 = 4$



$\prec$



Reverse Refinement Order.

Let $I = (i_1, \ldots, i_n)$, $J = (j_1, \ldots, j_k)$, $|J| = |I|$ then I is said to be greater in the **reverse refinement order** (or, simply, **finer**) than J,

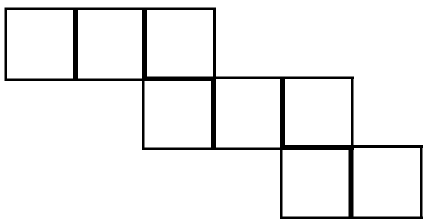
$$I \succ J$$

if $J = (i_1 + \ldots + i_{p_1}, \ldots, i_{p_{s-1}+1} + \ldots + i_{p_s}, \ldots, i_{p_{k-1}+1} + \ldots + i_n)$

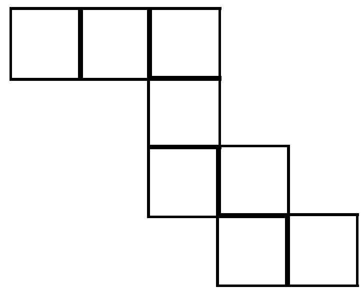
Notation: $lp(I, J) = \prod_{s=1}^{\ell(J)} i_{p_s}$.

For instance, $(3, 3, 2) \prec (3, 1, 2, 2)$

$p_1 = 1$
$p_2 = 3$
$p_3 = 4$



$\prec$



$p_1 - p_0 = 1, \quad p_2 - p_1 = 2, \quad p_3 - p_2 = 1$
$lp(3122, 332) = 3 \cdot 2 \cdot 2$

Two Multiplications.

For two compositions $I = (i_1, \ldots, i_{r-1}, i_r)$ and $J = (j_1, j_2, \ldots, j_s)$ one defines two multiplications

$$I \triangleright J = (i_1, \ldots, i_{r-1}, i_r + j_1, j_2, \ldots, j_s),$$

with $\ell(I \triangleright J) = \ell(I) + \ell(J) - 1$

and

$$I \cdot J = (i_1, \ldots, i_r, j_1, \ldots, j_s),$$

with $\ell(I \cdot J) = \ell(I) + \ell(J)$

Quasideterminants [Gelfand, Retakh (1991)].

A quasideterminant (with respect to the bottom left element) of an almost-triangular matrix with free entries a_{ij} and commutative off-diagonal entries b_j is a sum of all weighted paths starting at the bottom row, ending at the first column, taking north $\uparrow$ and east $\leftarrow$ steps and making eastward turns only at the (upper) off-diagonal entries.

$$\begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix}$$

Quasideterminants [Gelfand, Retakh (1991)].

A quasideterminant (with respect to the bottom left element) of an almost-triangular matrix with free entries a_{ij} and commutative off-diagonal entries b_j is a sum of all weighted paths starting at the bottom row, ending at the first column, taking north $\uparrow$ and east $\leftarrow$ steps and making eastward turns only at the (upper) off-diagonal entries.

$$\begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} = a_{31}$$

Quasideterminants [Gelfand, Retakh (1991)].

A quasideterminant (with respect to the bottom left element) of an almost-triangular matrix with free entries a_{ij} and commutative off-diagonal entries b_j is a sum of all weighted paths starting at the bottom row, ending at the first column, taking north $\uparrow$ and east $\leftarrow$ steps and making eastward turns only at the (upper) off-diagonal entries.

$$\begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} = a_{31} - \frac{a_{32}a_{11}}{b_1}$$

$$\begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix}$$

Quasideterminants [Gelfand, Retakh (1991)].

A quasideterminant (with respect to the bottom left element) of an almost-triangular matrix with free entries a_{ij} and commutative off-diagonal entries b_j is a sum of all weighted paths starting at the bottom row, ending at the first column, taking north $\uparrow$ and east $\leftarrow$ steps and making eastward turns only at the (upper) off-diagonal entries.

$$\begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} = a_{31} - \frac{a_{32}a_{11}}{b_1} - \frac{a_{33}a_{21}}{b_2}$$

$$\begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix}$$

Quasideterminants [Gelfand, Retakh (1991)].

A quasideterminant (with respect to the bottom left element) of an almost-triangular matrix with free entries a_{ij} and commutative off-diagonal entries b_j is a sum of all weighted paths starting at the bottom row, ending at the first column, taking north $\uparrow$ and east $\leftarrow$ steps and making eastward turns only at the (upper) off-diagonal entries.

$$\begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} = a_{31} - \frac{a_{32}a_{11}}{b_1} - \frac{a_{33}a_{21}}{b_2} + \frac{a_{33}a_{22}a_{11}}{b_1 b_2}$$

$$\begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix}$$

Recursive Formula for Quasideterminants.

$$\begin{vmatrix} a_{11} & b_1 & 0 & 0 \\ a_{21} & a_{22} & b_2 & 0 \\ a_{31} & a_{32} & a_{33} & b_4 \\ \boxed{a_{41}} & a_{42} & a_{43} & a_{44} \end{vmatrix} =$$

Recursive Formula for Quasideterminants.

$$\begin{vmatrix} a_{11} & b_1 & 0 & 0 \\ a_{21} & a_{22} & b_2 & 0 \\ a_{31} & a_{32} & a_{33} & b_4 \\ \boxed{a_{41}} & a_{42} & a_{43} & a_{44} \end{vmatrix} =$$

$$= -\frac{1}{b_4} a_{44} \begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} +$$

Recursive Formula for Quasideterminants.

$$\begin{vmatrix} a_{11} & b_1 & 0 & 0 \\ a_{21} & a_{22} & b_2 & 0 \\ a_{31} & a_{32} & a_{33} & b_4 \\ \boxed{a_{41}} & a_{42} & a_{43} & a_{44} \end{vmatrix} =$$

$$= -\frac{1}{b_4} a_{44} \begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{41}} & a_{42} & a_{43} \end{vmatrix}$$

Quasideterminants in the Commutative Limit.

If

$$A(\{b\}) = \begin{vmatrix} a_{11} & b_1 & 0 & \dots & 0 & 0 \\ a_{21} & a_{22} & b_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \dots & a_{n-1,n-1} & b_{n-1} \\ \boxed{a_{n,1}} & a_{n,2} & a_{n,3} & \dots & a_{n,n-1} & a_{n,n} \end{vmatrix}$$

then, when all a_{ij} commute,

$$A(\{b\}) = (-1)^{n-1} \frac{Det(A)}{\prod_{i=1}^{n-1} b_i}$$

Noncommutative Symmetric Functions.

- Monomial and forgotten symmetric functions.

Noncommutative Symmetric Functions.

- Monomial and forgotten symmetric functions.
- Fundamental symmetric functions.

Noncommutative Symmetric Functions.

- Monomial and forgotten symmetric functions.
- Fundamental symmetric functions.
- Elementary and complete (homogeneous) symmetric functions.

Noncommutative Symmetric Functions.

- Monomial and forgotten symmetric functions.
- Fundamental symmetric functions.
- Elementary and complete (homogeneous) symmetric functions.
- Ribbon Schur functions.

Noncommutative Symmetric Functions.

- Monomial and forgotten symmetric functions.
- Fundamental symmetric functions.
- Elementary and complete (homogeneous) symmetric functions.
- Ribbon Schur functions.
- Noncommutative Cauchy identity and a bilinear form on **NSym**.

Noncommutative Symmetric Functions: power sums.

In the original paper, [Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon (1994)], elementary symmetric functions were taken as generators of **NSym**.

Noncommutative Symmetric Functions: power sums.

In the original paper, [Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon (1994)], elementary symmetric functions were taken as generators of **NSym**. One can consider non-commutative power sums (of the first kind) $\Psi^I = \Psi_{i_1} \cdot \ldots \cdot \Psi_{i_n}$ as generators. (In particular, when all variables are declared to be commutative, $\Psi_n \to p_n$.)

Noncommutative Symmetric Functions: power sums.

In the original paper, [Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon (1994)], elementary symmetric functions were taken as generators of **NSym**. One can consider non-commutative power sums (of the first kind) $\Psi^I = \Psi_{i_1} \cdot \ldots \cdot \Psi_{i_n}$ as generators. (In particular, when all variables are declared to be commutative, $\Psi_n \to p_n$.)

Define **elementary symmetric** functions Λ_n:

$$\Lambda_n = \frac{(-1)^{n-1}}{n} \begin{vmatrix} \Psi_1 & 1 & 0 & \ldots & \ldots \\ \Psi_2 & \Psi_1 & 2 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{n-1} & \Psi_{n-2} & \ldots & \Psi_1 & n-1 \\ \boxed{\Psi_n} & \Psi_{n-1} & \ldots & \ldots & \Psi_1 \end{vmatrix}$$

NSym: Monomials.

Define **noncommutative monomial symmetric function** corresponding to a composition $I = (i_1, \ldots, i_n)$ as a quasideterminant of an n by n (where $n = \ell(I)$) matrix:

$$M^I = \frac{(-1)^{n-1}}{n} \begin{vmatrix} \Psi_{i_n} & 1 & 0 & \ldots & 0 & 0 \\ \Psi_{i_{n-1}+i_n} & \Psi_{i_{n-1}} & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{i_2+\ldots+i_n} & \ldots & \ldots & \ldots & \Psi_{i_2} & n-1 \\ \boxed{\Psi_{i_1+\ldots+i_n}} & \ldots & \ldots & \ldots & \Psi_{i_1+i_2} & \Psi_{i_1} \end{vmatrix}$$

NSym: Monomials.

Define **noncommutative monomial symmetric function** corresponding to a composition $I = (i_1, \ldots, i_n)$ as a quasideterminant of an n by n (where $n = \ell(I)$) matrix:

$$M^I = \frac{(-1)^{n-1}}{n} \begin{vmatrix} \Psi_{i_n} & 1 & 0 & \ldots & 0 & 0 \\ \Psi_{i_{n-1}+i_n} & \Psi_{i_{n-1}} & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{i_2+\ldots+i_n} & \ldots & \ldots & \ldots & \Psi_{i_2} & n-1 \\ \boxed{\Psi_{i_1+\ldots+i_n}} & \ldots & \ldots & \ldots & \Psi_{i_1+i_2} & \Psi_{i_1} \end{vmatrix}$$

In particular, $M^{1^n} = \Lambda_n$.

NSym: Monomials.

Define **noncommutative monomial symmetric function** corresponding to a composition $I = (i_1, \ldots, i_n)$ as a quasideterminant of an n by n (where $n = \ell(I)$) matrix:

$$M^I = \frac{(-1)^{n-1}}{n} \begin{vmatrix} \Psi_{i_n} & 1 & 0 & \ldots & 0 & 0 \\ \Psi_{i_{n-1}+i_n} & \Psi_{i_{n-1}} & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{i_2+\ldots+i_n} & \ldots & \ldots & \ldots & \Psi_{i_2} & n-1 \\ \boxed{\Psi_{i_1+\ldots+i_n}} & \ldots & \ldots & \ldots & \Psi_{i_1+i_2} & \Psi_{i_1} \end{vmatrix}$$

In the commutative limit $\Psi_n \to p_n$:

$$m_{\boldsymbol{\lambda}} = \sum_{I=\sigma(\boldsymbol{\lambda})} M^I,$$

where the sum is over all distinct permutations of parts of $\boldsymbol{\lambda}$.

NSym: Forgotten and Complete.

The **complete symmetric** functions S_n:

$$S_n = \frac{1}{n} \begin{vmatrix} \Psi_1 & -(n-1) & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{n-1} & \Psi_{n-2} & \dots & \dots & -1 \\ \boxed{\Psi_n} & \Psi_{n-1} & \dots & \dots & \Psi_1 \end{vmatrix}$$

NSym: Forgotten and Complete.

The **complete symmetric** functions S_n:

$$S_n = \frac{1}{n}\begin{vmatrix} \Psi_1 & -(n-1) & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{n-1} & \Psi_{n-2} & \dots & \dots & -1 \\ \boxed{\Psi_n} & \Psi_{n-1} & \dots & \dots & \Psi_1 \end{vmatrix}$$

Define **forgotten** symmetric functions

$$F^I = \frac{1}{n}\begin{vmatrix} \Psi_{i_n} & -(n-1) & 0 & \dots & 0 & 0 \\ \Psi_{i_{n-1}+i_n} & \Psi_{i_{n-1}} & -(n-2) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{i_2+\ldots+i_n} & \dots & \dots & \dots & \Psi_{i_2} & -1 \\ \boxed{\Psi_{i_1+\ldots+i_n}} & \dots & \dots & \dots & \Psi_{i_1+i_2} & \Psi_{i_1} \end{vmatrix}$$

Fundamental Noncommutative Symmetric Functions.

For every composition I one can define, by analogy with Gessel's fundamental quasi-symmetric functions, **fundamental noncommutative** symmetric function

$$L^I = \sum_{J \succeq I} M^J$$

Fundamental Noncommutative Symmetric Functions.

For every composition I one can define, by analogy with Gessel's fundamental quasi-symmetric functions, **fundamental noncommutative** symmetric function

$$L^I = \sum_{J \succeq I} M^J$$

In particular, $L^{1^n} = \Lambda_n = M^{1^n}$ and $L^n = S_n = F^{1^n}$.

NSym: ribbon Schur functions.

For every composition $I = (i_1, \ldots, i_n)$ **ribbon Schur** functions are defined as

$$R_I = (-1)^{\ell(I)-1} \begin{vmatrix} S_{i_n} & 1 & 0 & \ldots & \ldots \\ S_{i_n+i_{n-1}} & S_{i_{n-1}} & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{i_n+\ldots+i_2} & S_{i_{n-1}+\ldots+i_2} & \ldots & S_{i_2} & 1 \\ \boxed{S_{i_n+\ldots+i_1}} & S_{i_{n-1}+\ldots+i_1} & \ldots & \ldots & S_{i_1} \end{vmatrix}$$

Properties of Noncommutative Symmetric Functions.

- Behavior under the involution.

Properties of Noncommutative Symmetric Functions.

- Behavior under the involution.
- Multiplication rules.

Properties of Noncommutative Symmetric Functions.

- Behavior under the involution.
- Multiplication rules.
- Transitions between bases.

Properties of Noncommutative Symmetric Functions.

- Behavior under the involution.
- Multiplication rules.
- Transitions between bases.
- Cauchy identity.

Involution.

Recall that in the classical theory there is a map ω

Involution.

Recall that in the classical theory there is a map ω

$$\omega(p_{\boldsymbol{\lambda}}) = (-1)^{\ell(\boldsymbol{\lambda})-|\boldsymbol{\lambda}|}p_{\boldsymbol{\lambda}} \quad \text{such that}$$

Involution.

Recall that in the classical theory there is a map ω

$$\omega(p_{\boldsymbol{\lambda}}) = (-1)^{\ell(\boldsymbol{\lambda})-|\boldsymbol{\lambda}|} p_{\boldsymbol{\lambda}} \quad \text{such that}$$

$$\omega(m_{\boldsymbol{\lambda}}) = f_{\boldsymbol{\lambda}} \quad \text{and} \quad \omega(s_{\boldsymbol{\lambda}}) = s_{\boldsymbol{\lambda}'}$$

Involution.

Recall that in the classical theory there is a map ω

$$\omega(p_{\boldsymbol{\lambda}}) = (-1)^{\ell(\boldsymbol{\lambda})-|\boldsymbol{\lambda}|} p_{\boldsymbol{\lambda}} \quad \text{such that}$$

$$\omega(m_{\boldsymbol{\lambda}}) = f_{\boldsymbol{\lambda}} \quad \text{and} \quad \omega(s_{\boldsymbol{\lambda}}) = s_{\boldsymbol{\lambda}'}$$

There is a noncommutative analog of ω such that:

$$\omega\left(\Psi^{I}\right) = (-1)^{\ell(I)-|I|} \Psi^{\overline{I}} \quad \text{then}$$

Involution.

Recall that in the classical theory there is a map ω

$$\omega(p_{\boldsymbol{\lambda}}) = (-1)^{\ell(\boldsymbol{\lambda})-|\boldsymbol{\lambda}|} p_{\boldsymbol{\lambda}} \quad \text{such that}$$

$$\omega(m_{\boldsymbol{\lambda}}) = f_{\boldsymbol{\lambda}} \quad \text{and} \quad \omega(s_{\boldsymbol{\lambda}}) = s_{\boldsymbol{\lambda}'}$$

There is a noncommutative analog of ω such that:

$$\omega\left(\Psi^I\right) = (-1)^{\ell(I)-|I|}\Psi^{\overline{I}} \quad \text{then}$$

$$\omega\left(\Lambda_n\right) = S_n$$

$$\omega\left(M^I\right) = (-1)^{|I|-\ell(I)} F^{\overline{I}}$$

$$\omega\left(L^I\right) = L^{I^{\sim}}$$

$$\omega\left(R^I\right) = R^{I^{\sim}}$$

Multiplication.

Monomials:

$$M^J \cdot M^I = \sum_{K \preceq J} \binom{\ell(I) + \ell(K)}{\ell(J)} M^{K \cdot I} + \binom{\ell(I) + \ell(K) - 1}{\ell(J)} M^{K \triangleright I}$$

Multiplication.

Monomials:

$$M^J \cdot M^I = \sum_{K \preceq J} \binom{\ell(I)+\ell(K)}{\ell(J)} M^{K\cdot I} + \binom{\ell(I)+\ell(K)-1}{\ell(J)} M^{K \triangleright I}$$

Fundamental (with thanks to J.-C. Novelli for the formula in full generality):

$$L^I \cdot L^J =$$

$$\sum_{\substack{K \preceq I \\ M \succeq J}} \binom{|I|+\ell(J)-\ell(I)}{\ell(K)+\ell(M)-\ell(I)} L^{K\cdot M} + \binom{|I|+\ell(J)-\ell(I)}{\ell(K)+\ell(M)-1-\ell(I)} L^{K \triangleright M}$$

Multiplication.

Monomials:

$$M^J \cdot M^I = \sum_{K \preceq J} \binom{\ell(I)+\ell(K)}{\ell(J)} M^{K\cdot I} + \binom{\ell(I)+\ell(K)-1}{\ell(J)} M^{K \triangleright I}$$

Fundamental (with thanks to J.-C. Novelli for the formula in full generality):

$$L^I \cdot L^J =$$

$$\sum_{\substack{K \preceq I \\ M \succeq J}} \binom{|I|+\ell(J)-\ell(I)}{\ell(K)+\ell(M)-\ell(I)} L^{K\cdot M} + \binom{|I|+\ell(J)-\ell(I)}{\ell(K)+\ell(M)-1-\ell(I)} L^{K \triangleright M}$$

Ribbon Schur: $R_I \cdot R_J = R_{I.J} + R_{I \triangleright J}$

Nonnegativity of Transition Coefficients Between Bases.

$$F^I = \sum_{J \preceq I} M^J, \qquad \text{in particular,} \qquad S_n = \sum_{|I|=n} M^I$$

$$L^I = \sum_{J \succeq I} M^J$$

Nonnegativity of Transition Coefficients Between Bases.

$$F^I = \sum_{J \preceq I} M^J, \qquad \text{in particular,} \qquad S_n = \sum_{|I|=n} M^I$$

$$L^I = \sum_{J \succeq I} M^J$$

$$R^I = \sum_J K_{IJ} M^J$$

$$R^I = \sum_J G_{IJ} L^J$$

Coefficients G_{IJ} (and consequently K_{IJ}) are nonnegative.
[F. Hivert, J.-C. Novelli, L.T., J.-Y. Thibon (2007)]

Hook Ribbon Schur Functions.

$$R^{(r+1,1^s)} \equiv R^{(r|s)} = \binom{r+s}{s} \sum_{|I|=r+1} M^{I\cdot 1^s} = \binom{r+s}{s} L^{(r|s)}$$

Hook Ribbon Schur Functions.

$$R^{(r+1,1^s)} \equiv R^{(r|s)} = \binom{r+s}{s} \sum_{|I|=r+1} M^{I\cdot 1^s} = \binom{r+s}{s} L^{(r|s)}$$

$$R^{(r|s)} = \frac{(-1)^s}{r+s+1} \begin{vmatrix} \Psi_1 & a_1 & 0 & \dots & 0 & 0 \\ \Psi_2 & \Psi_1 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \Psi_{r+s} & \Psi_{s+r-1} & \Psi_{s+r-2} & \dots & \Psi_1 & a_n \\ \boxed{\Psi_{s+r+1}} & \Psi_{r+s} & \Psi_{s+r-1} & \dots & \Psi_2 & \Psi_1 \end{vmatrix}$$

with $(a_1, \dots, a_n) = (1, 2, \dots, s, -r, -(r-1), \dots, -1)$

A Noncommutative Identity.

In the Exercise 10, Ch. I, §5 of Macdonald, it is shown that

$$\sum_{|\boldsymbol{\lambda}|=n} X^{\ell(\boldsymbol{\lambda})-1} m_{\boldsymbol{\lambda}} = \sum_{k=0}^{n-1} s_{n-k,1^k} \left(X-1\right)^k$$

A Noncommutative Identity.

In the Exercise 10, Ch. I, §5 of Macdonald, it is shown that

$$\sum_{|\boldsymbol{\lambda}|=n} X^{\ell(\boldsymbol{\lambda})-1} m_{\boldsymbol{\lambda}} = \sum_{k=0}^{n-1} s_{n-k,1^k} \left(X-1\right)^k$$

There is a noncommutative version of this identity:

$$\sum_{|I|=n} X^{\ell(I)-1} M^I = \sum_{k=0}^{n-1} R_{1^k,n-k} (X-1)^k$$

A Noncommutative Identity.

In the Exercise 10, Ch. I, §5 of Macdonald, it is shown that

$$\sum_{|\boldsymbol{\lambda}|=n} X^{\ell(\boldsymbol{\lambda})-1} m_{\boldsymbol{\lambda}} = \sum_{k=0}^{n-1} s_{n-k,1^k} \,(X-1)^k$$

There is a noncommutative version of this identity:

$$\sum_{|I|=n} X^{\ell(I)-1} M^I = \sum_{k=0}^{n-1} R_{1^k,n-k}(X-1)^k$$

$\Psi_n = \sum_{k=0}^{n-1} (-1)^k R_{1^k,n-k}$ and $\sum_{|I|=n} L^I = \sum_{k=0}^{n-1} R_{1^k n-k}$

at $X=0$ (GKLLRT, 1994); at $X=2$ (B.-C.-V. Ung, 1998)

Noncommutative Cauchy identity and a bilinear form.

Proposition.
Given two noncommutative alphabets X and Y, the following identity is true:

$$\sum_I M^I(X)S^I(Y) = \sum_I L^I(X)R^I(Y)$$

Noncommutative Cauchy identity and a bilinear form.

Proposition.
Given two noncommutative alphabets X and Y, the following identity is true:

$$\sum_I M^I(X)S^I(Y) = \sum_I L^I(X)R^I(Y)$$

Then define a noncommutative bilinear form on **NSym** by requiring that M^I and S^I are dual to each other.

$$\langle M^I | S^J \rangle = \delta_{IJ},$$

it follows that

$$\langle L^I | R^J \rangle = \delta_{IJ}$$

Some properties of the bilinear form.

- The form is non-degenerate;

Some properties of the bilinear form.

- The form is non-degenerate;
- Positive: $\langle R_I | R_J \rangle = G_{IJ} \geq 0$

Some properties of the bilinear form.

- The form is non-degenerate;
- Positive: $\langle R_I | R_J \rangle = G_{IJ} \geq 0$
- Furthermore, ω is an isometry, i.e. for any two functions $H, G \in$ **NSym**

$$\langle \omega(H) | \omega(G) \rangle = \langle H | G \rangle$$

Some properties of the bilinear form.

- The form is non-degenerate;
- Positive: $\langle R_I | R_J \rangle = G_{IJ} \geq 0$
- Furthermore, ω is an isometry, i.e. for any two functions $H, G \in$ **NSym**

$$\langle \omega(H) | \omega(G) \rangle = \langle H | G \rangle$$

- Neither fundamental nor ribbon Schur basis is symmetric with respect to this form.

Some properties of the bilinear form.

- The form is non-degenerate;
- Positive: $\langle R_I | R_J \rangle = G_{IJ} \geq 0$
- Furthermore, ω is an isometry, i.e. for any two functions $H, G \in$ **NSym**

$$\langle \omega(H) | \omega(G) \rangle = \langle H | G \rangle$$

- Neither fundamental nor ribbon Schur basis is symmetric with respect to this form.

$$\langle \Psi^I | \Psi^J \rangle = \sum_{J \preceq K \preceq I} (-1)^{\ell(K) - \ell(J)} lp(K, J) \prod_{s=1}^{\ell(K)} (\ell(K) - s + 1)^{p_s - p_{s-1}},$$

In particular, $\langle \Psi^I | \Psi^I \rangle = \left(\prod_{k=1}^{\ell(I)} i_k \right) \ell(I)!$

ansion of Schur Functions in the symmetric Fundamental Basis.

Denote the commutative image of the noncommutative fundamental symmetric function L^I by l_I, then

$$s_{\boldsymbol{\lambda}} = \sum_J l_J,$$

where contributing compositions J are determined by descents in the standard fillings of $\boldsymbol{\lambda}$ exactly as in the quasi-symmetric case.

ansion of Schur Functions in the symmetric Fundamental Basis

Denote the commutative image of the noncommutative fundamental symmetric function L^I by l_I, then

$$s_{\boldsymbol{\lambda}} = \sum_J l_J,$$

where contributing compositions J are determined by descents in the standard fillings of $\boldsymbol{\lambda}$ exactly as in the quasi-symmetric case. Pull Schur functions to **NSym** by defining **noncommutative Schur functions** as

$$s^I = \sum_J L^J$$

with contributing J determined as above.

ansion of Schur Functions in the symmetric Fundamental Basis.

Denote the commutative image of the noncommutative fundamental symmetric function L^I by l_I, then

$$s_{\boldsymbol{\lambda}} = \sum_J l_J,$$

where contributing compositions J are determined by descents in the standard fillings of $\boldsymbol{\lambda}$ exactly as in the quasi-symmetric case. Pull Schur functions to **NSym** by defining **noncommutative Schur functions** as

$$s^I = \sum_J L^J$$

with contributing J determined as above.

Observation:

$$\langle s^I | s^J \rangle = \delta_{IJ}$$

Are there Noncommutative Macdonald Polynomials?

Classical Macdonald polynomials are simultaneous generalization of Hall-Littlewood and Jack polynomials, is there a noncommutative analog?

Are there Noncommutative Macdonald Polynomials?

Classical Macdonald polynomials are simultaneous generalization of Hall-Littlewood and Jack polynomials, is there a noncommutative analog? Classically, defining

$\omega_{q,t}(p_{\boldsymbol{\lambda}}) = (-1)^{|\boldsymbol{\lambda}|-\ell(\boldsymbol{\lambda})} \prod_{k=1}^{n} \frac{1-q^{\lambda_k}}{1-t^{\lambda_k}} p_{\boldsymbol{\lambda}}$,

Are there Noncommutative Macdonald Polynomials?

Classical Macdonald polynomials are simultaneous generalization of Hall-Littlewood and Jack polynomials, is there a noncommutative analog? Classically, defining

$\omega_{q,t}(p_{\boldsymbol\lambda}) = (-1)^{|\boldsymbol\lambda|-\ell(\boldsymbol\lambda)} \prod_{k=1}^{n} \frac{1-q^{\lambda_k}}{1-t^{\lambda_k}} p_{\boldsymbol\lambda}$, one has

$\omega_{q,t}(P_{\boldsymbol\lambda}(q,t)) = Q_{\boldsymbol\lambda'}(t^{-1}, q^{-1})$, where $Q_{\boldsymbol\lambda}(q,t) = b_{\boldsymbol\lambda}(q,t) P_{\boldsymbol\lambda}(q,t)$

$$\sum_{\boldsymbol\lambda} m_{\boldsymbol\lambda}(x) h_{\boldsymbol\lambda}\left(\left[\frac{1-t}{1-q}\right] y\right) = \sum_{\boldsymbol\lambda} Q_{\boldsymbol\lambda}(x;q,t) P_{\boldsymbol\lambda}(y;q,t)$$

$P_{\boldsymbol\lambda}(t,t) = s_{\boldsymbol\lambda}$

$P_{\boldsymbol\lambda}(0,t) = P_{\boldsymbol\lambda}(t)$ – Hall-Littlewood polynomials

$\lim_{t\to 1} P_{\boldsymbol\lambda}(t^\alpha, t) = P_{\boldsymbol\lambda}(\alpha)$ – Jack polynomials

$P_{\boldsymbol\lambda}(q,1) = m_{\boldsymbol\lambda} \quad P_{\boldsymbol\lambda}(1,t) = e_{\boldsymbol\lambda'}$

Deformed Noncommutative Symmetric Functions.

- Functions which Interpolate between Ribbon Schur and Monomial.

Deformed Noncommutative Symmetric Functions.

- Functions which Interpolate between Ribbon Schur and Monomial.

$$\sum_I M^I(X) S^I[(1-t)Y] = \sum_I b_I(t) L^I(X;t) R^I(Y;t)$$

Deformed Noncommutative Symmetric Functions.

- Functions which Interpolate between Ribbon Schur and Monomial.

$$\sum_I M^I(X)S^I[(1-t)Y] = \sum_I b_I(t)L^I(X;t)R^I(Y;t)$$

$$L^I(1) = M^I \quad \text{and} \quad L^I(0) = L^I$$
$$R^I(1) = M^I \quad \text{and} \quad R^I(0) = R^I$$

Deformed Noncommutative Symmetric Functions.

- Functions which Interpolate between Ribbon Schur and Monomial.

$$\sum_I M^I(X)S^I[(1-t)Y] = \sum_I b_I(t)L^I(X;t)R^I(Y;t)$$

$$L^I(1) = M^I \quad \text{and} \quad L^I(0) = L^I$$
$$R^I(1) = M^I \quad \text{and} \quad R^I(0) = R^I$$

- Noncommutative analogs of Jack polynomials.

Deformed Noncommutative Symmetric Functions.

- Functions which Interpolate between Ribbon Schur and Monomial.

$$\sum_I M^I(X)S^I[(1-t)Y] = \sum_I b_I(t)L^I(X;t)R^I(Y;t)$$

$$L^I(1) = M^I \quad \text{and} \quad L^I(0) = L^I$$
$$R^I(1) = M^I \quad \text{and} \quad R^I(0) = R^I$$

- Noncommutative analogs of Jack polynomials.

$$\omega_\alpha J^I(\alpha) \sim J^{\tilde{I}}\left(\frac{1}{\alpha}\right)$$

Functions Interpolating between Ribbon Schur and Monomial.

For every composition I define ribbon Hall-Littlewood functions as

$$R^I(t) = (-1)^{n-1}\frac{1-t}{1-t^n}\begin{vmatrix} \frac{S_{i_n}(t)}{1-t} & \frac{1-t}{1-t} & 0 & \cdots & \cdots \\ \frac{S_{i_{n-1}+i_n}(t)}{1-t} & \frac{S_{i_{n-1}}(t)}{1-t} & \frac{1-t^2}{1-t} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \boxed{\frac{S_{i_1+\ldots+i_n}(t)}{1-t}} & \cdots & \cdots & \cdots & \frac{S_{i_1}(t)}{1-t} \end{vmatrix},$$

where $S_k(t) = S_k[(1-t)X]$ and $n = \ell(I)$.

Functions Interpolating between Ribbon Schur and Monomial.

For every composition I define ribbon Hall-Littlewood functions as

$$R^I(t) = (-1)^{n-1}\frac{1-t}{1-t^n}\begin{vmatrix} \frac{S_{i_n}(t)}{1-t} & \frac{1-t}{1-t} & 0 & \cdots & \cdots \\ \frac{S_{i_{n-1}+i_n}(t)}{1-t} & \frac{S_{i_{n-1}}(t)}{1-t} & \frac{1-t^2}{1-t} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \boxed{\frac{S_{i_1+\ldots+i_n}(t)}{1-t}} & \cdots & \cdots & \cdots & \frac{S_{i_1}(t)}{1-t} \end{vmatrix},$$

where $S_k(t) = S_k[(1-t)X]$ and $n = \ell(I)$. Then

$$R^I(0) = R^I \quad \text{as } S_k(0) = S_k$$

$$R^I(1) = M^I \quad \text{as } \lim_{t\to 1}\frac{1}{1-t}S_k(t) = \Psi_n$$

Functions Interpolating between Fundamental and Monomial.

Define fundamental Hall-Littlewood polynomials:

$$L^I(t) = \sum_{J \succeq I} (-1)^{\ell(I)-\ell(J)} t^{mj(I,J)} L^J,$$

with $mj(I, J) = \sum_{k=1}^{s} (s - k + 1)(p_k - p_{k-1} - 1)$

where s is the length of the refining composition

and $L^I(1) = M^I; \quad L^I(0) = L^I$

Functions Interpolating between Fundamental and Monomial.

Define fundamental Hall-Littlewood polynomials:

$$L^I(t) = \sum_{J \succeq I} (-1)^{\ell(I)-\ell(J)} t^{mj(I,J)} L^J,$$

with $mj(I,J) = \sum_{k=1}^{s} (s-k+1)(p_k - p_{k-1} - 1)$

where s is the length of the refining composition

and $L^I(1) = M^I; \quad L^I(0) = L^I$

Then one has the following t-Cauchy identity:

$$\sum_I M^I S^I(t) = \sum_I (t;t)_n L^I(t) R^I(t), \quad \text{with} \quad n = \ell(I)$$

An Identity for Fundamental H-L Polynomials.

There is, what seems to be, a noncommutative generalization of the following identity (Macdonald, Ex. 1, §3, Ch. III):

$$\sum_{|\boldsymbol{\lambda}|=n} t^{n(\boldsymbol{\lambda})} P_{\boldsymbol{\lambda}}(t) = h_n,$$

where

$$n(\boldsymbol{\lambda}) = \sum_i \binom{\lambda_i'}{2}$$

An Identity for Fundamental H-L Polynomials.

There is, what seems to be, a noncommutative generalization of the following identity (Macdonald, Ex. 1, §3, Ch. III):

$$\sum_{|\boldsymbol{\lambda}|=n} t^{n(\boldsymbol{\lambda})} P_{\boldsymbol{\lambda}}(t) = h_n,$$

where

$$n(\boldsymbol{\lambda}) = \sum_i \binom{\lambda_i'}{2}$$

This identity reads:

$$\sum_{|I|=n} t^{\binom{\ell(I)}{2}} L^I(t) = L^n$$

Multiplication of Ribbon Hall-Littlewood Polynomials.

$$R^J(t)R^I(t) = \sum_{K \preceq J} t^{\ell(J)-\ell(K)} \begin{bmatrix} \ell(J)+\ell(I) \\ \ell(K) \end{bmatrix}_t R^{K\cdot I}(t) +$$

$$+\, t^{\ell(J)-\ell(K)} \begin{bmatrix} \ell(J)+\ell(I)-1 \\ \ell(K) \end{bmatrix}_t R^{K \triangleright I}(t)$$

$$Q^{(r)}(t) \cdot Q^I(t) = Q^{r\cdot I}(t) + (1 - t^{\ell(I)}) Q^{r \triangleright I}(t)$$

where $Q^I(t) = (t;t)_{\ell(I)} R^I(t)$

in particular, when $I = 1^s$

$$Q^{(r)}(t) \cdot Q^{1^s}(t) = Q^{r\cdot 1^s}(t) + (1 - t^s) Q^{r \triangleright 1^s}(t)$$

Noncommutative Hook Jack Polynomials.

Will use Frobenius notation and write $(r|s)$ for a composition $I = (r+1, 1^s)$. Define

$$J^{(r|s)}(\alpha) = (-1)^s \begin{vmatrix} \Psi_1 & a_1 & 0 & \dots & 0 & 0 \\ \Psi_2 & \Psi_1 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \Psi_{r+s} & \Psi_{s+r-1} & \Psi_{s+r-2} & \dots & \Psi_1 & a_n \\ \boxed{\Psi_{s+r+1}} & \Psi_{r+s} & \Psi_{s+r-1} & \dots & \Psi_2 & \Psi_1 \end{vmatrix}$$

with

$$(a_1, \dots, a_n) = (1, 2, \dots, s, -r\alpha, -(r-1)\alpha, \dots, -\alpha)$$

Properties of Noncommutative Hook Jack Polynomials.

- $$\omega_\alpha J^{(r|s)}(\alpha) = \alpha J^{(s|r)}\left(\frac{1}{\alpha}\right)$$

Properties of Noncommutative Hook Jack Polynomials.

- $$\omega_\alpha J^{(r|s)}(\alpha) = \alpha J^{(s|r)}\left(\frac{1}{\alpha}\right)$$
- Define
$$P^{(r|s)}(\alpha) = \frac{\alpha}{\alpha(s+1)+r} J^{(r|s)}(\alpha)$$

  Then
$$P^{(r|s)}(1) = R^{(r|s)} \quad P^{(r|s)}(\infty) = M^{(r|s)} \quad P^{(r|s)}(0)$$

Properties of Noncommutative Hook Jack Polynomials.

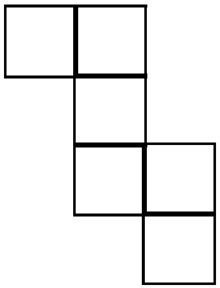
- $$\omega_\alpha J^{(r|s)}(\alpha) = \alpha J^{(s|r)}\left(\frac{1}{\alpha}\right)$$

- Define
$$P^{(r|s)}(\alpha) = \frac{\alpha}{\alpha(s+1)+r} J^{(r|s)}(\alpha)$$
Then
$$P^{(r|s)}(1) = R^{(r|s)} \quad P^{(r|s)}(\infty) = M^{(r|s)} \quad P^{(r|s)}(0)$$

- $$J^{(r|s)}(\alpha) = \sum_J c_J M^J,$$
where c_J are polynomials in α with positive integer coefficients.

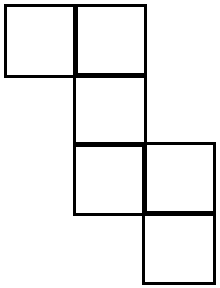
Decomposition of a Ribbon into Hooks.

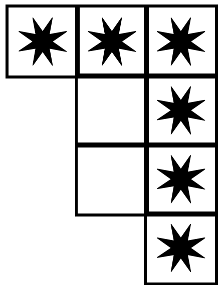
Consider $I = (2121)$, i.e.



Decomposition of a Ribbon into Hooks.

Consider $I = (2121)$, i.e.

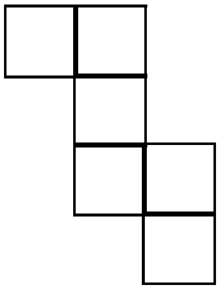


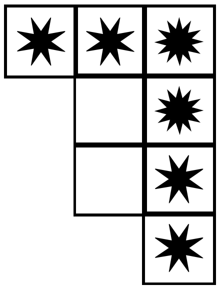


$(2|3)$

Decomposition of a Ribbon into Hooks.

Consider $I = (2121)$, i.e.

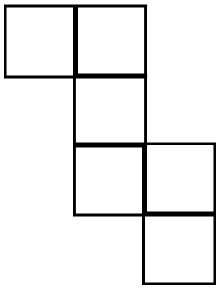


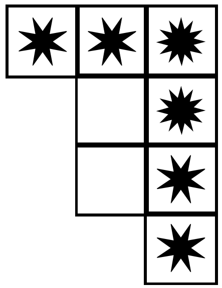


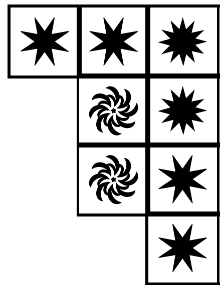
$(2|3)/(0|1)$

Decomposition of a Ribbon into Hooks.

Consider $I = (2121)$, i.e.



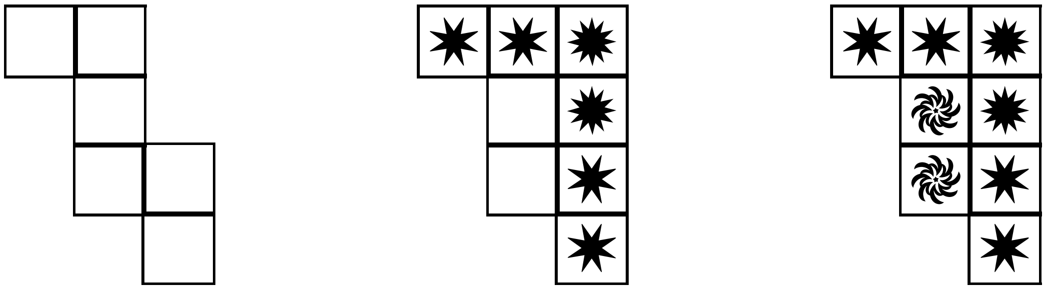




$(2|3)/(0|1)$ $\qquad$ $(20|31)/(0|1)$

Decomposition of a Ribbon into Hooks.

Consider $I = (2121)$, i.e.



$(2|3)/(0|1)$ $(20|31)/(0|1)$

This is an example of a hook decomposition of a ribbon:

$$I = J/K$$

$$2121 = (20|31)/(0|1)$$

Giambelli Type Formula for Ribbon Schur Functions.

Let $I = J/K$ with $J = (\alpha_1 \ldots \alpha_r | \beta_1 \ldots \beta_r)$ and $K = (\delta_1 \ldots \delta_{r-1} | \gamma_1 \ldots \gamma_{r-1})$, where $\{\alpha_k\}, \{\beta_k\}, \{\delta_k\},$ and $\{\gamma_k\}$ are decreasing and $\alpha_k > \delta_k \geq \alpha_{k+1},\ \beta_k > \gamma_k \geq \beta_{k+1}$ for all k:

$$R^I = \begin{vmatrix} \Lambda_{\beta_1-\gamma_1} & 1 & 0 & \ldots & 0 & \ldots & \ldots & \vdots \\ \Lambda_{\beta_1-\gamma_2} & \ldots & 1 & 0 & \ldots & \vdots & \ldots & \vdots \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ \Lambda_{\beta_1-\gamma_{r-1}} & \ldots & \ldots & \Lambda_{\beta_r-\gamma_{r-1}} & 1 & 0 & \ldots & \vdots \\ R^{(\alpha_r|\beta_1)} & \ldots & \ldots & R^{(\alpha_r|\beta_r)} & S_{\alpha_r-\delta_{r-1}} & 1 & 0 & \ldots \\ \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots & 0 \\ R^{(\alpha_2|\beta_1)} & \ldots & \ldots & R^{(\alpha_2|\beta_r)} & S_{\alpha_2-\delta_{r-1}} & \ldots & \ldots & 1 \\ \boxed{R^{(\alpha_1|\beta_1)}} & \ldots & \ldots & R^{(\alpha_1|\beta_r)} & S_{\alpha_1-\delta_{r-1}} & \ldots & \ldots & S_{\alpha_1-\delta_1} \end{vmatrix}$$

([Lascoux-Pragacz (1984)] in the commutative case)

Giambelli Type Formula for Monomials.

An arbitrary noncommutative monomial symmetric function can be similarly expressed through hook monomials:

$M^I =$

$$\left|\begin{array}{cccccccc}
\Lambda_{\beta_1-\gamma_1} & 2r-2 & 0 & \dots & 0 & \dots & \dots & \vdots \\
\Lambda_{\beta_1-\gamma_2} & \dots & 2r-3 & 0 & \dots & \vdots & \dots & \vdots \\
\vdots & \dots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\
\Lambda_{\beta_1-\gamma_{r-1}} & \dots & \dots & \Lambda_{\beta_r-\gamma_{r-1}} & r & 0 & \dots & \vdots \\
M^{(\alpha_r|\beta_1)} & \dots & \dots & M^{(\alpha_r|\beta_r)} & \Psi_{\alpha_r-\delta_{r-1}} & r-1 & 0 & \dots \\
\vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots & 0 \\
M^{(\alpha_2|\beta_1)} & \dots & \dots & M^{(\alpha_2|\beta_r)} & \Psi_{\alpha_2-\delta_{r-1}} & \dots & \dots & 1 \\
\boxed{M^{(\alpha_1|\beta_1)}} & \dots & \dots & M^{(\alpha_1|\beta_r)} & \Psi_{\alpha_1-\delta_{r-1}} & \dots & \dots & \Psi_{\alpha_1-\delta_1}
\end{array}\right|$$

Examples of Giambelli Type Formulas.

Return to $I = (2121) = (20|31)/(0|1)$. Applying the above formulas with

$$\alpha_1 = 2, \quad \alpha_2 = 0$$
$$\beta_1 = 3, \quad \beta_2 = 1$$
$$\gamma_1 = 0$$
$$\delta_1 = 1$$

one obtains

$$R^{2121} = \begin{vmatrix} \Lambda_2 & 1 & 0 \\ \Lambda_4 & \Lambda_2 & 1 \\ \boxed{R^{(2|3)}} & R^{(2|1)} & S_2 \end{vmatrix} \quad M^{2121} = \begin{vmatrix} \Lambda_2 & 2 & 0 \\ \Lambda_4 & \Lambda_2 & 1 \\ \boxed{M^{(2|3)}} & M^{(2|1)} & \Psi_2 \end{vmatrix}$$