



Quasisymmetric Schur functions

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Compositions and partitions

A **composition** $\alpha_1 \dots \alpha_k$ of n is a list of positive integers whose sum is n : **2213** \vdash 8.

A composition is a **partition** if $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k > 0$: **3221** \vdash 8.

Any composition **determines** a partition: **$\lambda(2213) = 3221$** .

$\alpha = \alpha_1 \dots \alpha_k$ is a **coarsening** of $\beta = \beta_1 \dots \beta_l$ (β is a **refinement** of α) if

$$\underbrace{\beta_1 + \dots + \beta_i}_{\alpha_1} \underbrace{\beta_{i+1} + \dots + \beta_j}_{\alpha_2} \dots \underbrace{\beta_m + \dots + \beta_l}_{\alpha_k}$$

is true: **53** \geq **2213**.

Macdonald polynomials ...

1988: (Macdonald) Introduced $\{P_\lambda(X; q, t)\}_\lambda$

$$(T) \quad P_\lambda(X; q, t) = s_\lambda(X) + \sum_{\mu < \lambda} s_\mu(X) \varepsilon_{\lambda, \mu}(q, t)$$

$$(\varepsilon_{\lambda, \mu}(q, t) \in \mathbb{Q}(q, t))$$

$$(O) \quad \langle P_\lambda(X; q, t), P_\mu(X; q, t) \rangle_{q, t} = 0 \quad \text{if } \lambda \neq \mu$$

... go nonsymmetric

$$P_\lambda(X; q, t) = \prod_{u \in \text{dg}'(\lambda^\circ)} (1 - q^{l(u)+1} t^{a(u)}) \sum_{\lambda(\mu)=\lambda} \frac{E_\gamma(X; q^{-1}, t^{-1})}{\prod (1 - q^{l(u)+1} t^{a(u)})}$$

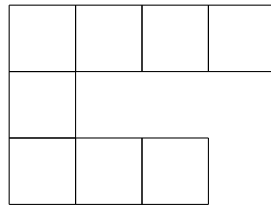
letting $q, t \rightarrow 0$

$$s_\lambda(X) = \sum_{\lambda(\mu)=\lambda} E_\gamma(X; \infty, \infty).$$

(Lascoux-Schützenberger (90); Reiner-Shimozono (95); Haglund, Haiman, Loehr (05); Mason (06))

Composition diagrams and tableaux

The **composition diagram** $\alpha = \alpha_1 \dots \alpha_k > 0$ is the array of **boxes** with α_i boxes in row i from the **top**.



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A **(standard) composition tableau** of **shape** α is a filling of α with (each first n) $1, 2, 3, \dots$ such that

Rules for composition tableaux I

- First column entries **strictly increase** top to bottom.
- Rows **weakly decrease** left to right.

Example

5	4	3	1
6			
8	7	2	

Rules for composition tableaux II

- When the filling is **extended** for $1 \leq i < j, 2 \leq k$

$$(j, k) \neq 0, (j, k) \geq (i, k) \Rightarrow (j, k) > (i, k - 1).$$

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6	0	0	0
8	7	2	0

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Weights and weakness

Given a composition tableau T we have

$$x^T := x_1^{\#1s} x_2^{\#2s} x_3^{\#3s} \dots$$

Example

$$T = \begin{array}{|c|c|c|c|} \hline 5 & 4 & 3 & 1 \\ \hline 6 & & & \\ \hline 8 & 7 & 2 & \\ \hline \end{array} \quad x^T = x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8$$

A **weak composition** $\gamma_1 \dots \gamma_k$ of n is a list of **non-negative** integers whose sum is n : **2021003** \vDash 8.

Demazure atoms

The **collapse** of a weak composition γ , $\alpha(\gamma)$, is γ with 0's removed: $\alpha(2021003) = 2213$.

The **foundation** of a weak composition γ , $\mathcal{F}_o(\gamma)$, is all i so $\gamma_i > 0$: $\mathcal{F}_o(2021003) = \{1, 3, 4, 7\}$.

Then

$$NS_\gamma(X) = E_{\gamma_n, \dots, \gamma_1}(x_n, \dots, x_1; \infty, \infty) = \sum_{\gamma} x^T$$

over composition tableaux shape $\alpha(\gamma)$, entries $1, 2, 3, \dots, n$, first column $\mathcal{F}_o(\gamma)$.

NS_γ for $\gamma = 1030$

1		
3	3	3

1		
3	3	2

1		
3	3	1

1		
3	2	2

1		
3	2	1

1		
3	1	1

$$NS_\gamma = x_1x_3^3 + x_1x_2x_3^2 + x_1^2x_3^2 + x_1x_2^2x_3 + x_1^2x_2x_3 + x_1^3x_3.$$

NS_γ for $\gamma = 1030$

1	0	0
3	3	3

1		
3	3	2

1		
3	3	1

1		
3	2	2

1		
3	2	1

1		
3	1	1

$$NS_\gamma = x_1x_3^3 + x_1x_2x_3^2 + x_1^2x_3^2 + x_1x_2^2x_3 + x_1^2x_2x_3 + x_1^3x_3.$$

NS_γ for $\gamma = 1030$

1		
3	3	3

1		
3	3	2

1		
3	3	1

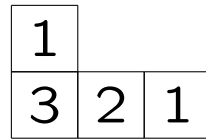
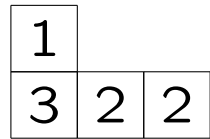
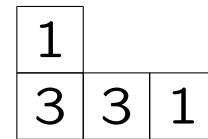
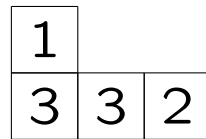
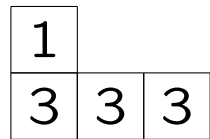
1		
3	2	2

1		
3	2	1

1	0	0
3	1	1

$$NS_\gamma = x_1x_3^3 + x_1x_2x_3^2 + x_1^2x_3^2 + x_1x_2^2x_3 + x_1^2x_2x_3 + x_1^3x_3.$$

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$$NS_\gamma = x_1x_3^3 + x_1x_2x_3^2 + x_1^2x_3^2 + x_1x_2^2x_3 + x_1^2x_2x_3 + \dots$$

Quasisymmetric functions I

Let $\mathcal{Q} \subset \mathbb{Q}[[x_1, x_2, \dots]]$ be the algebra of all **quasisymmetric functions**

$$\mathcal{Q} := \mathcal{Q}_0 \oplus \mathcal{Q}_1 \oplus \dots$$

where

$$\mathcal{Q}_n := \text{span}_{\mathbb{Q}}\{M_\alpha \mid \alpha = \alpha_1 \dots \alpha_k \models n\}$$

$$M_\alpha := \sum_{i_1 < i_2 < \dots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_k}^{\alpha_k}.$$

Example $M_{121} = \sum_{i_1 < i_2 < i_3} x_{i_1}^1 x_{i_2}^2 x_{i_3}^1 = x_1 x_2^2 x_3 + \dots$

Quasisymmetric functions II

For $\alpha \vDash n$ define

$$F_\alpha = \sum_{\beta \leq \alpha} M_\beta.$$

The F_α are the **fundamental** quasisymmetric functions.

Example

$$F_{121} = M_{121} + M_{1111}$$

Symmetric functions I

Let $\Lambda \subset \mathbb{Q}[[x_1, x_2, \dots]]$ be the algebra of all **symmetric functions**

$$\Lambda := \Lambda_0 \oplus \Lambda_1 \oplus \dots$$

where

$$\Lambda_n := \text{span}_{\mathbb{Q}}\{m_\lambda \mid \lambda \vdash n\}$$

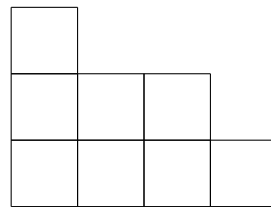
$$m_\lambda := \sum_{\lambda(\alpha)=\lambda} M_\alpha.$$

Example

$$m_{211} = M_{211} + M_{121} + M_{112}$$

Contretableaux

The **diagram** $\lambda = \lambda_1 \geq \dots \geq \lambda_k > 0$ is the array of **boxes** with λ_i boxes in row i from the **bottom**.



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A **(standard) contretableau** T of **shape** λ is a filling of λ with (each first n) $1, 2, 3, \dots$ so rows **weakly decrease** and columns **strictly increase**.

Contretableaux

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5			
6	4	2	
8	7	3	1

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A **(standard) contretableau** T of **shape** λ is a filling of λ with (each first n) $1, 2, 3, \dots$ so rows **weakly decrease** and columns **strictly increase**.

Descents and subsets

$D(T)$ is i where $i + 1$ is strictly south and weakly east:

5			
6	4	2	
8	7	3	1

composition $\alpha_1 \dots \alpha_k \vDash n \leftrightarrow$ subset $\{i_1, \dots, i_{k-1}\} \subseteq [n - 1]$

β $2312 \vDash 8 \leftrightarrow \{2, 5, 6\} \subseteq [7]$ $S(\beta)$

Symmetric functions II

For $\lambda \vdash n$ define

$$s_\lambda = \sum_{\beta} d_{\lambda\beta} F_\beta$$

where $d_{\lambda\beta}$ = number of standard tableaux T of shape λ and $D(T) = S(\beta)$.

The s_λ are the Schur functions.

Example

$$s_{21} = F_{21} + F_{12}$$

from

2	
3	1

1	
3	2

Quasisymmetric Schur functions

If $m_\lambda = \sum_{\lambda(\alpha)=\lambda} M_\alpha$ then $s_\lambda = \sum_{\lambda(\alpha)=\lambda} \mathcal{S}_\alpha$

but what is \mathcal{S}_α ?

The quasisymmetric Schur function is given by ...

Quasisymmetric Schur functions

If $m_\lambda = \sum_{\lambda(\alpha)=\lambda} M_\alpha$ then $s_\lambda = \sum_{\lambda(\alpha)=\lambda} \mathcal{S}_\alpha$

but what is \mathcal{S}_α ?

The quasisymmetric Schur function is given by

$$\mathcal{S}_\alpha = \sum_{\alpha(\gamma)=\alpha} N S_\gamma.$$

S_α for $\alpha = 12$

1	
2	2

1	
3	2

1	
3	3

2	
3	3

$$\begin{aligned} S_{12} &= NS_{120} + NS_{102} + NS_{012} \\ &= x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 + x_2x_3^2 \\ &= M_{12} + M_{111} \end{aligned}$$

and

$$s_{21} = F_{21} + F_{12} = M_{21} + M_{12} + 2M_{111} = s_{21} + S_{12}.$$

What properties should \mathcal{S}_α have?

- \mathbb{Z} -basis for \mathcal{Q} .
- Expression in F_β .
- Quasisymmetric Kostka numbers.
- Quasisymmetric Pieri rules.
- Quasisymmetric LR rule.

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- \mathbb{Z} -basis for \mathcal{Q} .
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Expression in F_β

$\mathcal{D}(T)$ is i where $i + 1$ is north in column (immediately south if both in first column), or strictly east:

4	3	2
5	1	

For $\lambda \vdash n$

$$s_\lambda = \sum_{\beta} d_{\lambda\beta} F_\beta$$

where $d_{\lambda\beta}$ = number of standard tableaux T of shape λ and $D(T) = S(\beta)$.

Expression in F_β

$\mathcal{D}(T)$ is i where $i + 1$ is north in column (immediately south if both in first column), or strictly east:

4	3	2
5	1	

For $\alpha \vDash n$

$$S_\alpha = \sum_{\beta} d_{\alpha\beta} F_\beta$$

where $d_{\alpha\beta}$ = number of standard tableaux T of shape α and $\mathcal{D}(T) = S(\beta)$.

S_α for $\alpha = 32$

3	2	1
5	4	

4	3	1
5	2	

4	3	2
5	1	

and

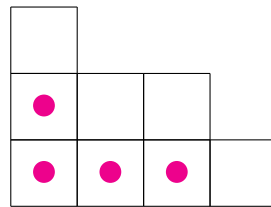
$$S_{32} = F_{32} + F_{221} + F_{131}.$$

Skew diagrams and strips

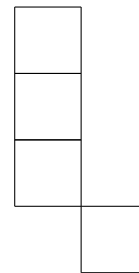
For λ, μ the skew diagram λ/μ of size $|\lambda/\mu|$ is the array of boxes contained in λ but not in μ .

A skew diagram λ/μ is a row (column) strip if

no 2 boxes in same column (row).



row strip



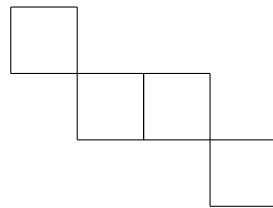
col strip

Skew diagrams and strips

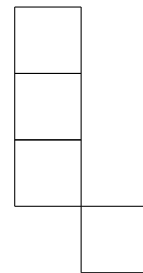
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row strip



col strip

Pieri rules for partitions λ, μ

$$s_n s_\lambda = \sum_{\mu} s_\mu \quad s_{1^n} s_\lambda = \sum_{\mu} s_\mu$$

- $\delta = \mu/\lambda$ row strip, $|\delta| = n$.
- $\varepsilon = \mu/\lambda$ column strip, $|\varepsilon| = n$.

Removal from compositions

Let $\alpha = \alpha_1 \dots \alpha_k$ have largest part m , and $s \in [m]$

$$rem_s(\alpha) = \alpha_1 \dots \alpha_{i-1} (s-1) \alpha_{i+1} \dots \alpha_k$$

for largest i .

If $S = \{s_1 < \dots < s_j\}$ then

$$row_S(\alpha) = rem_{s_1}(\dots (rem_{s_{j-1}}(rem_{s_j}(\alpha))) \dots).$$

If $M = \{m_1 \leq \dots \leq m_j\}$ then

$$col_M(\alpha) = rem_{m_j}(\dots (rem_{m_2}(rem_{m_1}(\alpha))) \dots).$$

rem_s in action

rem_s removes rightmost box from lowest row length s .

Example

$$rem_1(113) = \begin{array}{|c|} \hline \square \\ \hline \bullet \\ \hline \square & \square & \square \\ \hline \end{array} = \begin{array}{|c|} \hline \square \\ \hline \square & \square & \square \\ \hline \end{array} = 13$$

Quasisymmetric Pieri rules for compositions α, β

$$\mathcal{S}_n \mathcal{S}_\alpha = \sum_{\beta} \mathcal{S}_\beta \quad \mathcal{S}_{1^n} \mathcal{S}_\alpha = \sum_{\beta} \mathcal{S}_\beta$$

- $\delta = \lambda(\beta)/\lambda(\alpha)$ row strip, $|\delta| = n$.
- $row_{\mathcal{S}(\delta)}(\beta) = \alpha$.
- $\varepsilon = \lambda(\beta)/\lambda(\alpha)$ column strip, $|\varepsilon| = n$.
- $col_{M(\varepsilon)}(\beta) = \alpha$.

$S_1 S_{13}$ in action

41/31, 32/31, 311/31, 311/31

with row strips containing a box in column 4, 2, 1, 1 respectively.

$$\begin{aligned} \text{row}_{\{4\}}(14) &= \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \bullet \\ \hline \end{array} & \text{row}_{\{2\}}(23) &= \begin{array}{|c|c|c|} \hline & \bullet & \\ \hline & & \\ \hline \end{array} \\ \text{row}_{\{1\}}(131) &= \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \bullet & & \\ \hline \end{array} & \text{row}_{\{1\}}(113) &= \begin{array}{|c|c|c|} \hline & & \\ \hline \bullet & & \\ \hline & & \\ \hline \end{array} \end{aligned}$$

and

$$S_1 S_{13} = S_{14} + S_{23} + S_{131} + S_{113}.$$