Diagonal Harmonics (early 90's)

$$DH_n = \mathbb{Q}[X_n, Y_n] / \left\langle Sym^+ \right\rangle$$

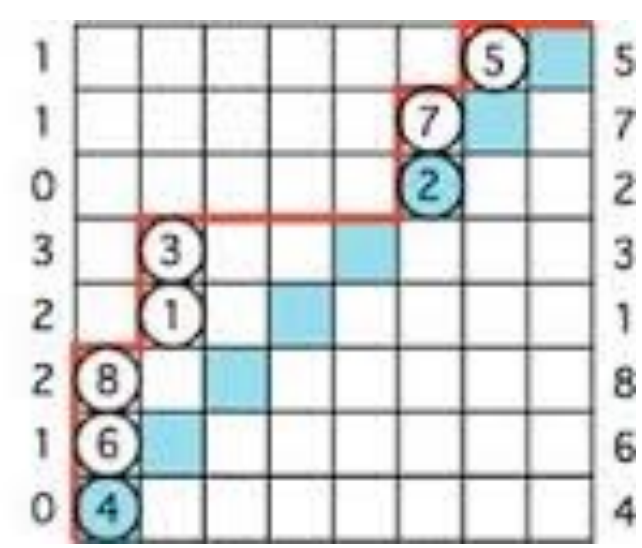
Symmetric function expression (late 90's)

$$\mathcal{F}_{qt}(DH_n) = \nabla(e_n)$$

Shuffle Conjecture (early 2000's)

$$\nabla(e_n) = \sum_{PF} q^{dinv(PF)} t^{area(PF)} wt(PF)$$

$$PF = \begin{bmatrix} 4 & 6 & 8 & 1 & 3 & 2 & 7 & 5 \\ 0 & 1 & 2 & 2 & 3 & 0 & 1 & 1 \end{bmatrix} \iff$$



Diagonal Harmonics ⟶ Shuffle conjecture

1994 - Haiman: (Conjectures on the quotient ring by diagonal invariants)
Definition of the diagonal harmonics

1996 - Garsia, Haiman: (A remarkable *q,t*-Catalan sequence and *q*-Lagrange inversion)
conjectured symmetric function expression for the Frobenius character, alternant multiplicity = q,t-Catalan

1999 - F. Bergeron, Garsia, Haiman, Tesler: (Identities and Positivity Conjectures for Some Remarkable Operators in the Theory of Symmetric Functions)
introduction of the operators ∇ and Δ_f

2002 - Garsia, Haglund: (A positivity result in the theory of Macdonald polynomials)
combinatorial formula for the q,t-Catalan (q,t-dimension of alternants in diagonal harmonics)

2002 - Haiman: (Vanishing theorems and character formulas for the Hilbert scheme of points in the plane)
proof of the conjectured symmetric function expression for the Frobenius character - $\dim DH_n = (n+1)^{n-1}$

2005 - Haglund, Haiman, Loehr, Remmel, Ulyanov: (A combinatorial formula for the character of the diagonal coinvariants)
combinatorial formula for the monomial expansion of the Frobenius character - “the shuffle conjecture”

2018 - Carsson and Mellit: (A proof of the shuffle conjecture)

???? - Schur expansion of Frobenius image
indexed by hook shapes is the “q,t-Schröder” (Haglund 2003)

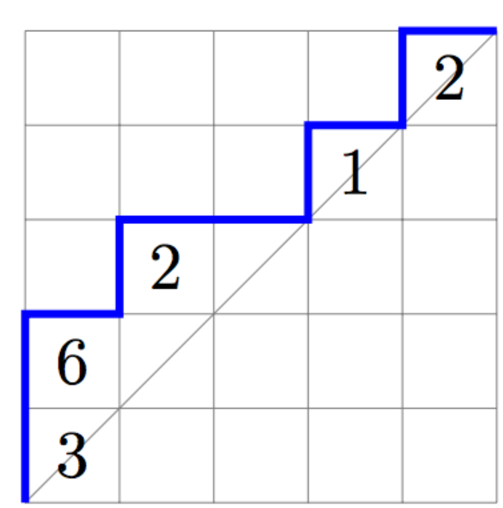
The Delta conjecture

Conjecture 1.1 (Delta Conjecture)**.** *For any integers* $n > k \geq 0$,

$$\Delta'_{e_k} e_n = \sum_{P \in \mathcal{LD}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \prod_{i:\, a_i(P) > a_{i-1}(P)} \left(1 + z/t^{a_i(P)}\right) x^P \Bigg|_{z^{n-k-1}} \tag{7}$$

$$= \sum_{P \in \mathcal{LD}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \prod_{i \in \mathrm{Val}(P)} \left(1 + z/q^{d_i(P)+1}\right) x^P \Bigg|_{z^{n-k-1}}. \tag{8}$$

Equivalently, we can replace the left-hand side with $\Delta_{e_k} e_n$ *for integers* $n \geq k \geq 0$, *multiply both right-hand sides by* $(1 + z)$, *and then take the coefficient of* z^{n-k}.



a_i	d_i
0	0
0	1
1	1
1	2
0	0

FIGURE 2. A sample labeled Dyck path $P \in \mathcal{LD}_5$ with $\mathrm{area}(P) = 2$, $\mathrm{dinv}(P) = 4$, $\mathrm{comp}(P) = \{1, 2, 1, 1\}$, and $\mathrm{Val}(P) = \{4, 5\}$.

Tri-agonal Harmonics ⟵ Delta conjecture

2015 - Haglund, Remmel, Wilson : (The Delta Conjecture)
A conjectured combinatorial formula for $\Delta'_{e_k}(e_n)$
the case k=n-1 is "the shuffle conjecture"

2016 - Romero : (The Delta Conjecture at q=1)

2017 - Garsia, Haglund, Remmel, Yoo : (A proof of the Delta conjecture when q=0)

2016 - Haglund, Rhodes, Shimozono : (Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture)
A module whose graded Frobenius characteristic is the Delta conjecture at q=0 (up to application of \omega and rev_q

2017/2018 - we started working to understand the super-harmonics as a module and it also has graded Frobenius characteristic (exactly matching) the symmetric function expression in the Delta conjecture

Since ordered set partitions is similar to the harmonics in super space that we've been working on. What would happen if we added an extra set of commuting variables? Conjecture:::

$$\mathcal{F}_{qz}(\mathbb{Q}[X_n, Y_n; \Theta_n]/\left\langle Sym^+\right\rangle) = \Delta'_{e_{n-1}+ze_{n-2}+\cdots+z^{n-1}}(e_n)$$

In [1]:
```
s = SymmetricFunctions(QQ['q','t'].fraction_field()).s()
Ht = s.symmetric_function_ring().macdonald().Ht()
```

In [2]:
```
s(Ht[2,2])
```

Out[2]:
```
q^2*t^2*s[1, 1, 1, 1] + (q^2*t+q*t^2+q*t)*s[2, 1, 1] + (q^2+t^2)*s[2, 2] + (q*t+q+t)*s[3, 1] + s[4]
```

$$\nabla(\tilde{H}_\mu[X;q,t]) = t^{n(\mu)} q^{n(\mu')} \tilde{H}_\mu[X;q,t]$$

In [4]:
```
Ht[4,2,1].nabla()
```

Out[4]:
```
q^7*t^4*McdHt[4, 2, 1]
```

$$\mathcal{F}_{qt}(DH_n) = \nabla(e_n)$$

In [3]:
```
s[1,1,1].nabla()
```

Out[3]:
```
(q^3+q^2*t+q*t^2+t^3+q*t)*s[1, 1, 1] + (q^2+q*t+t^2+q+t)*s[2, 1] + s[3]
```

$$B_\mu = \sum_{c\in\mu} t^{c_y} q^{c_x}$$

$$\Delta_f(\tilde{H}_\mu[X;q,t]) = f[B_\mu]\tilde{H}_\mu[X;q,t]$$

$$\Delta'_f(\tilde{H}_\mu[X;q,t]) = f[B_\mu - 1]\tilde{H}_\mu[X;q,t]$$

In [1]:
```
# Formula with respect to the Delta conjecture
SymmetricFunctions(QQ['q','t','z'].fraction_field()).inject_shorthands(verbose=False)
(q,t,z) = s.base_ring().gens()
Ht = s.symmetric_function_ring().macdonald().Ht()
Bmu = lambda mu: sum(t**c1*q**c2 for (c1,c2) in Partition(mu).cells())
def Deltap(f,g):
    """
    Computes \Delta_f'(g) where \Delta_f'(Ht(lambda)) = f[B_\mu-1] Ht(lambda)
    and Ht is the Macdonald symmetric function basis
    """
    return g.parent()(sum(c*f((Bmu(la)-1)*s[[]]).coefficient([])*Ht(la) for (la,c) in Ht(g)))
```

In [2]:
```
Deltap(e[2],s[1,1,1])
```

Out[2]:
```
(q^3+q^2*t+q*t^2+t^3+q*t)*s[1, 1, 1] + (q^2+q*t+t^2+q+t)*s[2, 1] + s[3]
```

$$\mathcal{F}_{qz}(\mathbb{Q}[X_n;\Theta_n]/\langle Sym^+\rangle) = \Delta'_{e_{n-1}+ze_{n-2}+\cdots+z^{n-1}}(e_n)|_{t=0}$$

In [6]:
```
s(Deltap(e[2]+z*e[1]+z^2,e[3])).map_coefficients(lambda c: c.subs(t=0))
```

Out[6]:
```
(q^3+q^2*z+q*z+z^2)*s[1, 1, 1] + (q^2+q*z+q+z)*s[2, 1] + s[3]
```

In [7]:
```
s(Deltap(e[3]+z*e[2]+z^2*e[1]+z^3,e[4])).map_coefficients(lambda c: c.subs(t=0))
```

Out[7]:
```
(q^6+q^5*z+q^4*z+q^3*z^2+q^3*z+q^2*z^2+q*z^2+z^3)*s[1, 1, 1, 1] + (q^5+q^4*z+q^4+2*q^3*z+q^2*z^2+q^3+2*q^2*z+q*
z^2+q*z+z^2)*s[2, 1, 1] + (q^4+q^3*z+q^2*z+q*z^2+q^2+q*z)*s[2, 2] + (q^3+q^2*z+q^2+q*z+q+z)*s[3, 1] + s[4]
```

$$Coinv_n^{k,k'} := \mathbb{Q}[X_n^{(1)}, \ldots, X_n^{(k)}; \Theta_n^{(1)}, \ldots, \Theta_n^{(k')}] / \left\langle Sym^+ \right\rangle$$

Master conjecture:

There are symmetric functions in two sets of variables $\mathcal{E}_n[Z; X]$ such that

$$\mathcal{F}_{Q_k, T_{k'}}(Coinv_n^{k,k'}) = \mathcal{E}_n[Q_k - \epsilon T_{k'}; X]$$

tables of these symmetric functions up to n=5 are in `arXiv:1105.4358v4`

Special cases of the conjecture

quotient	dimension	dim of alts	status
$\mathbb{Q}[X_n]/ < Sym^+ >$	$n!$	1	'classical'
$\mathbb{Q}[X_n, Y_n]/ < Sym^+ >$	$(n+1)^{n-1}$	$\frac{1}{n+1}\binom{2n}{n}$	proven ~2000
$\mathbb{Q}[X_n, Y_n, Z_n]/ < Sym^+ >$	$2^n(n+1)^{(n-2)}$ http://oeis.org/A127670	$\binom{4n+1}{n+1} - 9\binom{4n+1}{n-1}$ http://oeis.org/A000260	not known
$\mathbb{Q}[\Theta_n]/ < Sym^+ >$	2^{n-1}	1	'easy'
$\mathbb{Q}[\Theta_n, \mathcal{T}_n]/ < Sym^+ >$	$\binom{2n+1}{n+1}$ http://oeis.org/A001700	n	seems like an interesting one to try
$\mathbb{Q}[X_n, \Theta_n]/ < Sym^+ >$	$\sum_{k=1}^{n} k!S(n,k)$	2^{n-1}	working on it
$\mathbb{Q}[X_n, Y_n, \Theta_n]/ < Sym^+ >$	$\frac{1}{2n+2}\sum_{k=0}^{n+1}\binom{n+1}{k}k^n$ http://oeis.org/A201595	http://oeis.org/A001003	conjectured comb formula

How to verify the conjectures:

The dimension tells you if two modules are isomorphic as a vector space
The character tells you if two modules are isomorphic as S_n modules

1. **find a basis for M**

for an algebra modulo an idea F/I

$\mathcal{B} = \{w : w \text{ is not divisible by a leading term of the Gröbner basis}\}$

2. **define the S_n action on this basis**

3. **compute the character for each permutation (of each cycle structure)**

$$\chi_M(\sigma_\mu) = \sum_{b \in \mathcal{B}} \sigma_\mu(b)|_{\text{coeff } b}$$

4. **compute the Frobenius image**

$$\mathcal{F}(M) = \sum_{\mu \vdash n} \chi_M(\sigma_\mu) \frac{p_\mu}{z_\mu}$$

two modules are isomorphic iff their Frobenius images are equal