Notes 1

Creation operators

Creation operators (or non-commutative Bernstein operators) were introduced in [1]. They give rise to a new basis (the so-called immaculate basis) of NSym. Our goal is to study the subalgebra of endomorphism of NSym spanned by these operators.

After analyzing the relationship between the creation and dual (perp) operators, some straightening relations on the creation operators will be given. As an application, we will derive some formulas for the expansion of compositions of these elements when the index of the operator is 0.

1.1. Relations between $\mathbb{B}_i$ and $F_i^\perp$

Let us denote by H_i the complete homogeneous non-commutative symmetric function, by F_α the fundamental quasi-symmetric function indexed by the composition α; and by $F_\alpha^\perp$ the linear transformation of NSym that is adjoint to multiplication by F_α in QSym.

Definition 1.1. (See [1, Definition 3.1]) Let $m \in \mathbb{Z}$, the **non-commutative Bernstein operator** $\mathbb{B}_m$ is defined as follows

$$\mathbb{B}_m = \sum_{i\geq 0} (-1)^i H_{m+i} F_{1^i}^\perp$$

Lemma 1.2. *The following relations hold*

(i) $\sum_{i\geq 0} \mathbb{B}_i F_i^\perp = \mathrm{Id}$

(ii) $\sum_{i\geq -s} \mathbb{B}_i F_{i+s}^{\perp} = 0$ *for all* $s \geq 1$.

Proof. (i). Recall that by definition $H_0 = \mathrm{Id}$, and $\sum_{i+j=r} (-1)^j F_i^{\perp} F_{1^j}^{\perp} = 0$ for all $r \geq 1$. Then

$$\begin{aligned}\sum_{i\geq 0} \mathbb{B}_i F_i^{\perp} &= \sum_{i\geq 0} \left(\sum_{j\geq 0} (-1)^j H_{i+j} F_{1^j}^{\perp} \right) F_i^{\perp} = \sum_{r\geq 0} H_r \left(\sum_{i+j=r} (-1)^j F_{1^j}^{\perp} F_i^{\perp} \right) \\ &= H_0 = \mathrm{Id}\end{aligned}$$

Taking into account that $H_{-m} = 0$ for all $m < 0$, (ii) can be shown in a similar way. More precisely

$$\begin{aligned}\sum_{i\geq -s} \mathbb{B}_i F_{i+s}^{\perp} &= \sum_{i\geq -s} \left(\sum_{j\geq 0} (-1)^j H_{i+j} F_{1^j}^{\perp} \right) F_{i+s}^{\perp} = \sum_{k\geq 0} \left(\sum_{j\geq 0} (-1)^j H_{k-s+j} F_{1^j}^{\perp} \right) F_k^{\perp} \\ &= \sum_{r\geq 0} H_{r-s} \left(\sum_{k+j=r} (-1)^j F_{1^j}^{\perp} F_k^{\perp} \right) = 0\end{aligned}$$

□

Corollary 1.3. *Let* $s \geq 1$

$$\mathbb{B}_0 = \mathrm{Id} - \sum_{i\geq 1} \mathbb{B}_i F_i^{\perp}, \qquad \mathbb{B}_{-s} = -\sum_{i>-s} \mathbb{B}_i F_{i+s}^{\perp}$$

Corollary 1.4. *Let* $m \in \mathbb{Z}$ *and* $r \geq 0$.

(i) $F_r^{\perp} \mathbb{B}_m = \sum_{i=0}^{r} \mathbb{B}_{m-i} F_{r-i}^{\perp}$ *if* $m > r$.

(ii) $F_m^{\perp} \mathbb{B}_m = \mathrm{Id} - \sum_{i\geq 1} \mathbb{B}_{m+i} F_{m+i}^{\perp}$

(ii) $F_r^{\perp} \mathbb{B}_m = -\sum_{i\geq 1} \mathbb{B}_{m+i} F_{r+i}^{\perp}$ *if* $m < r$.

Proof. For $m \in \mathbb{Z}$ and $r \geq 0$, we have $F_r^{\perp} \mathbb{B}_m = \sum_{i=0}^{r} \mathbb{B}_{m-i} F_{r-i}^{\perp}$ (see [1]). The result now follows from Corollary 1.3. □

1.2. Algebra spanned by $\{\mathbb{B}_m\}_{m\in\mathbb{Z}}$

Straightening relation for $\mathbb{B}_0 \mathbb{B}_m$

Given $m > 0$, from Corollary 1.3 we obtain

$$\mathbb{B}_0 \mathbb{B}_m = \left(\mathrm{Id} - \sum_{i\geq 1} \mathbb{B}_i F_i^{\perp} \right) \mathbb{B}_m = \mathbb{B}_m - \sum_{i\geq 1} \mathbb{B}_i F_i^{\perp} \mathbb{B}_m.$$

Next, divide the latest sum into three summands and apply Corollary 1.4.

$$\begin{aligned}\mathbb{B}_0\mathbb{B}_m &= \mathbb{B}_m - \sum_{1\le i<m} \mathbb{B}_i F_i^\perp \mathbb{B}_m - \mathbb{B}_m F_m^\perp \mathbb{B}_m - \sum_{i>m} \mathbb{B}_i F_i^\perp \mathbb{B}_m \\ &= \mathbb{B}_m - \sum_{1\le i<m} \mathbb{B}_i \left(\sum_{j=0}^{i} \mathbb{B}_{m-j} F_{i-j}^\perp\right) - \mathbb{B}_m \left(\mathrm{Id} - \sum_{i\ge 1} \mathbb{B}_{m+i} F_{m+i}^\perp\right) \\ &\quad + \sum_{i>m} \mathbb{B}_i \left(\sum_{j\ge 1} \mathbb{B}_{m+j} F_{i+j}^\perp\right) = \sum_{1\le i<m} \sum_{j=0}^{i} \mathbb{B}_i \mathbb{B}_{m-j} F_{i-j}^\perp + \sum_{i\ge 1} \mathbb{B}_m \mathbb{B}_{m+i} F_{m+i}^\perp \\ &\quad + \sum_{i>m} \sum_{j\ge 1} \mathbb{B}_i \mathbb{B}_{m+j} F_{i+j}^\perp\end{aligned}$$

Thus

$$\mathbb{B}_0\mathbb{B}_m = -\sum_{1\le i<m} \sum_{j=0}^{i} \mathbb{B}_i \mathbb{B}_{m-j} F_{i-j}^\perp + \sum_{i\ge 1} \mathbb{B}_m \mathbb{B}_{m+i} F_{m+i}^\perp + \sum_{i>m} \sum_{j\ge 1} \mathbb{B}_i \mathbb{B}_{m+j} F_{i+j}^\perp \tag{1.1}$$

Straightening relation for $\mathbb{B}_{-s}\mathbb{B}_m$

In this case, one soon realizes that the negative index makes things messier!!! More formulas are needed! Let us pick $m > 0$, $s \ge 1$ and start, like before, applying Corollary 1.3.

$$\mathbb{B}_{-s}\mathbb{B}_m = -\sum_{i>-s} \mathbb{B}_i F_{i+s}^\perp \mathbb{B}_m$$

Now apart from dealing with $F_{i+s}^\perp \mathbb{B}_m$, we also have to take care with the indices of the $\mathbb{B}_i$'s since they can be negative or zero! Thus, we will consider the three following sums:

$$\mathbb{B}_{-s}\mathbb{B}_m = -\sum_{-s<i<0} \mathbb{B}_i F_{i+s}^\perp \mathbb{B}_m - \mathbb{B}_0 F_s^\perp \mathbb{B}_m - \sum_{i\ge 1} \mathbb{B}_i F_{i+s}^\perp \mathbb{B}_m \tag{1.2}$$

In what follows, we will compute $\mathbb{B}_i F_r^\perp \mathbb{B}_m$ attending on whether either $i > 0$, $i = 0$ or $i < 0$.

- $i > 0$ This is the easiest case! Leave $\mathbb{B}_i$ like it is, and apply Corollary 1.4 to obtain the following formulas:

- $m > r$

$$\mathbb{B}_i F_r^{\perp} \mathbb{B}_m = \sum_{j=0}^{r} \mathbb{B}_i \mathbb{B}_{m-j} F_{r-j}^{\perp}$$

- $m = r$

$$\mathbb{B}_i F_m^{\perp} \mathbb{B}_m = \mathbb{B}_i - \sum_{j\geq 1} \mathbb{B}_i \mathbb{B}_{m+j} F_{m+j}^{\perp}$$

- $m < r$

$$\mathbb{B}_i F_r^{\perp} \mathbb{B}_m = -\sum_{j\geq 1} \mathbb{B}_i \mathbb{B}_{m+j} F_{r+j}^{\perp}$$

- $i = 0$ In order to get a formula for $\mathbb{B}_0 F_r^{\perp} \mathbb{B}_m$ consider as before, three subcases:

  - $m > r$ By Corollary 1.4 (i) we get

$$\mathbb{B}_0 F_r^{\perp} \mathbb{B}_m = \mathbb{B}_0 \left(\sum_{k=0}^{r} \mathbb{B}_{m-k} F_{r-k}^{\perp} \right) = \sum_{k=0}^{r} \mathbb{B}_0 \mathbb{B}_{m-k} F_{r-k}^{\perp}$$

  Our hypothesis implies that $m - k > 0$ for every $0 \leq k \leq r$, which allows us to use (1.1). Therefore:

$$\begin{aligned} \mathbb{B}_0 F_r^{\perp} \mathbb{B}_m &= -\sum_{k=0}^{r} \sum_{1\leq j<m-k} \sum_{\ell=0}^{j} \mathbb{B}_j \mathbb{B}_{m-k-\ell} F_{j-\ell}^{\perp} F_{r-k}^{\perp} \\ &+ \sum_{k=0}^{r} \sum_{j\geq 1} \mathbb{B}_{m-k} \mathbb{B}_{m-k+j} F_{m-k+j}^{\perp} F_{r-k}^{\perp} + \sum_{k=0}^{r} \sum_{j>m-k} \sum_{\ell\geq 1} \mathbb{B}_j \mathbb{B}_{m-k+\ell} F_{j+\ell}^{\perp} F_{r-k}^{\perp} \end{aligned}$$

  - $m = r$ From Corollary 1.4 (ii) we obtain

$$\mathbb{B}_0 F_m^{\perp} \mathbb{B}_m = \mathbb{B}_0 \left(\mathrm{Id} - \sum_{k\geq 1} \mathbb{B}_{m+k} F_{m+k}^{\perp} \right) = \mathbb{B}_0 - \sum_{k\geq 1} \mathbb{B}_0 \mathbb{B}_{m+k} F_{m+k}^{\perp}$$

  Next, applying Corollary 1.3 to the first summand and (1.1) to the second one, we get

$$\begin{aligned} \mathbb{B}_0 F_m^{\perp} \mathbb{B}_m &= \mathrm{Id} - \sum_{j\geq 1} \mathbb{B}_j F_j^{\perp} + \sum_{k\geq 1} \sum_{1\leq j<m+k-1} \sum_{\ell=0}^{j} \mathbb{B}_j \mathbb{B}_{m+k-\ell} F_{j-\ell}^{\perp} F_{m+k}^{\perp} \\ &- \sum_{k,\, j\geq 1} \mathbb{B}_{m+k} \mathbb{B}_{m+k+j} F_{m+k+j}^{\perp} F_{m+k}^{\perp} - \sum_{k\geq 1} \sum_{j>m+k} \sum_{\ell\geq 1} \mathbb{B}_j \mathbb{B}_{m+k+\ell} F_{j+\ell}^{\perp} F_{m+k}^{\perp} \end{aligned}$$

- $m < r$ Apply first Corollary 1.4 (iii) and secondly Corollary 1.3 we have

$$\begin{aligned}\mathbb{B}_0 F_r^\perp \mathbb{B}_m &= -\mathbb{B}_0 \sum_{j\geq 1} \mathbb{B}_{m+j} F_{r+j}^\perp = -\left(\mathrm{Id} - \sum_{k\geq 1} \mathbb{B}_k F_k^\perp\right) \sum_{j\geq 1} \mathbb{B}_{m+j} F_{r+j}^\perp \\ &= -\sum_{j\geq 1} \mathbb{B}_{m+j} F_{r+j}^\perp + \sum_{j,\,k\geq 1} \mathbb{B}_k F_k^\perp \mathbb{B}_{m+j} F_{r+j}^\perp\end{aligned}$$

This case is now trickier, since the expression of $F_k^\perp \mathbb{B}_{m+j}$ depends on whether either $k < m+j$, $k = m+j$ or $k > m+j$. Keeping this fact in mind from Corollary 1.4 we get

$$\begin{aligned}\mathbb{B}_0 F_r^\perp \mathbb{B}_m = &-\sum_{j\geq 1} \mathbb{B}_{m+j} F_{r+j}^\perp + \sum_{j\geq 1} \sum_{1\leq k<m+j} \sum_{\ell=0}^{k} \mathbb{B}_k \mathbb{B}_{m+j-\ell} F_{k-\ell}^\perp F_{r+j}^\perp \\ &+ \sum_{j\geq 1} \mathbb{B}_{m+j} \left(\mathrm{Id} - \sum_{\ell\geq 1} \mathbb{B}_{m+j+\ell} F_{m+j+\ell}^\perp\right) F_{r+j}^\perp \\ &- \sum_{j\geq 1} \sum_{k>m+j} \sum_{\ell\geq 1} \mathbb{B}_k \mathbb{B}_{m+j+\ell} F_{k+\ell}^\perp F_{r+j}^\perp\end{aligned}$$

Thus:

$$\begin{aligned}\mathbb{B}_0 F_r^\perp \mathbb{B}_m = &\sum_{j\geq 1} \sum_{1\leq k<m+j} \sum_{\ell=0}^{k} \mathbb{B}_k \mathbb{B}_{m+j-\ell} F_{k-\ell}^\perp F_{r+j}^\perp \\ &- \sum_{j,\,\ell\geq 1} \mathbb{B}_{m+j} \mathbb{B}_{m+j+\ell} F_{m+j+\ell}^\perp F_{r+j}^\perp - \sum_{j\geq 1} \sum_{k>m+j} \sum_{\ell\geq 1} \mathbb{B}_k \mathbb{B}_{m+j+\ell} F_{k+\ell}^\perp F_{r+j}^\perp\end{aligned}$$

- $i := -s < 0$ Definitively, this is the worst case! The most tedious one!!! Again, the cases $m > r$, $m = r$ and $m < r$ need to be considered. Well, ignore this "little" detail for a moment, and substitute the expression of $\mathbb{B}_{-s}$ obtained in Corollary 1.3.

$$\mathbb{B}_{-s} F_r^\perp \mathbb{B}_m = \left(-\sum_{j>-s} \mathbb{B}_j F_{j+s}^\perp\right) F_r^\perp \mathbb{B}_m$$

Since the $\mathbb{B}_j$'s appearing in the expression above can be either negative or zero, we need to split this sum into three new ones. More concretely:

$$\mathbb{B}_{-s} F_r^\perp \mathbb{B}_m = -\sum_{-s<j<0} \mathbb{B}_j F_{j+s}^\perp F_r^\perp \mathbb{B}_m - \mathbb{B}_0 F_s^\perp F_r^\perp \mathbb{B}_m - \sum_{j>0} \mathbb{B}_j F_{j+s}^\perp F_r^\perp \mathbb{B}_m$$

The general formulas will become really messy. So let us just try to give more details about how to proceed to straighten $\mathbb{B}_{-s}F_r^\perp\mathbb{B}_m$. For example, for $m > r$ we will have

$$\mathbb{B}_{-s}F_r^\perp\mathbb{B}_m = -\sum_{-s<j<0}\sum_{k=0}^{r}\mathbb{B}_jF_{j+s}^\perp\mathbb{B}_{m-k}F_{r-k}^\perp - \sum_{k=0}^{r}\mathbb{B}_0F_s^\perp\mathbb{B}_{m-k}F_{r-k}^\perp - \sum_{j>0}\sum_{k=0}^{r}\mathbb{B}_jF_{j+s}^\perp\mathbb{B}_{m-k}F_{r-k}^\perp$$

Now, we need to look at carefully the expressions of $F_{j+s}^\perp\mathbb{B}_{m-k}$ and $F_s^\perp\mathbb{B}_{m-k}$ and apply Corollary 1.4. Next, after that Corollary 1.3 needs to applied in the first and second sums. Note that it will bring new $F_\ell^\perp$'s, which will have to be straightened to the right of the corresponding $\mathbb{B}_h$'s.

At this point, I think a concrete example might (or not?) help us to understand exactly what is going on. As you will see, even dealing with a simple guy like $\mathbb{B}_{-2}\mathbb{B}_1$ is a pain!

Example 1.5. Let us compute for example $\mathbb{B}_{-2}\mathbb{B}_1$

$$\mathbb{B}_{-2}\mathbb{B}_1 = -\sum_{i>-2}\mathbb{B}_iF_{i+2}^\perp\mathbb{B}_1 = -\mathbb{B}_{-1}F_1^\perp\mathbb{B}_1 - \mathbb{B}_0F_2^\perp\mathbb{B}_1 - \sum_{i>0}\mathbb{B}_iF_{i+2}^\perp\mathbb{B}_1$$

We have already formulas for $\mathbb{B}_0F_2^\perp\mathbb{B}_1$ and $\mathbb{B}_iF_{i+2}^\perp\mathbb{B}_1$ where $i > 0$. Here they are:

$$\mathbb{B}_0F_2^\perp\mathbb{B}_1 = \sum_{j\geq 1}\sum_{1\geq k<j+1}\sum_{\ell=0}^{k}\mathbb{B}_k\mathbb{B}_{j-\ell+1}F_{k-\ell}^\perp F_{j+2}^\perp + \sum_{j,\ell\geq 1}\mathbb{B}_{j+1}\mathbb{B}_{j+\ell+1}F_{j+\ell+1}^\perp F_{j+2}^\perp - \sum_{j\geq 1}\sum_{k>j+1}\sum_{\ell\geq 1}\mathbb{B}_k\mathbb{B}_{j+\ell+1}F_{k+\ell}^\perp F_{j+2}^\perp \tag{1.3}$$

$$\mathbb{B}_iF_{i+2}^\perp\mathbb{B}_1 = -\sum_{j\geq 1}\mathbb{B}_i\mathbb{B}_{j+1}F_{i+j+2}^\perp \tag{1.4}$$

It remains to compute $\mathbb{B}_{-1}F_1^\perp\mathbb{B}_1$. By Corollary 1.4 (ii) we have

$$\mathbb{B}_{-1}F_1^\perp\mathbb{B}_1 = \mathbb{B}_{-1}\left(\mathrm{Id} - \sum_{j\geq 1}\mathbb{B}_{j+1}F_{j+1}^\perp\right) = \mathbb{B}_{-1} - \sum_{j\geq 1}\mathbb{B}_{-1}\mathbb{B}_{j+1}F_{j+1}^\perp = -\sum_{k>-1}\mathbb{B}_kF_{k+1}^\perp - \sum_{k>-1}\sum_{j\geq 1}\mathbb{B}_kF_{k+1}^\perp\mathbb{B}_{j+1}F_{j+1}^\perp$$

Let us compute the above two sums separately. For the fist one, just note above the presence of $\mathbb{B}_0$, and use Corollary 1.3 to kill it!

$$\sum_{k>-1} \mathbb{B}_k F_{k+1}^{\perp} = \mathbb{B}_0 F_1^{\perp} + \sum_{k>0} \mathbb{B}_k F_{k+1}^{\perp} = F_1^{\perp} - \sum_{j\geq 1} \mathbb{B}_j F_j^{\perp} F_1^{\perp} + \sum_{k>0} \mathbb{B}_k F_{k+1}^{\perp}$$

Regarding to the second one, $\mathbb{B}_0$ forces us split it into two new sums:

$$\sum_{k>-1} \sum_{j\geq 1} \mathbb{B}_k F_{k+1}^{\perp} \mathbb{B}_{j+1} F_{j+1}^{\perp} = \sum_{j\geq 1} \mathbb{B}_0 F_1^{\perp} \mathbb{B}_{j+1} F_{j+1}^{\perp} + \sum_{k,\, j\geq 1} \mathbb{B}_k F_{k+1}^{\perp} \mathbb{B}_{j+1} F_{j+1}^{\perp}$$

Note that

$$\begin{aligned} \sum_{j\geq 1} \mathbb{B}_0 F_1^{\perp} \mathbb{B}_{j+1} F_{j+1}^{\perp} &= \sum_{j\geq 1} \mathbb{B}_0 \left(\mathbb{B}_{j+1} F_1^{\perp} + \mathbb{B}_j \right) F_{j+1}^{\perp} = \sum_{j\geq 1} \mathbb{B}_0 \left(\mathbb{B}_{j+1} F_1^{\perp} F_{j+1}^{\perp} + \mathbb{B}_j F_{j+1}^{\perp} \right) \\ &= \sum_{j\geq 1} \left(\mathbb{B}_{j+1} F_1^{\perp} F_{j+1}^{\perp} + \mathbb{B}_j F_{j+1}^{\perp} \right) - \sum_{k,\, j\geq 1} \mathbb{B}_k F_k^{\perp} \mathbb{B}_{j+1} F_1^{\perp} F_{j+1}^{\perp} - \sum_{k,\, j\geq 1} \mathbb{B}_k F_k^{\perp} \mathbb{B}_j F_{j+1}^{\perp} \end{aligned}$$

Thus to finish, given $k,\, j \geq 1$ we should compute $F_{k+1}^{\perp} \mathbb{B}_{j+1}$, $F_k^{\perp} \mathbb{B}_{j+1}$ and $F_k^{\perp} \mathbb{B}_j$. Consider the corresponding cases and apply Corollary 1.4.

1.3. Some applications: formulas for $\mathfrak{S}_{0a}$ and $\mathfrak{S}_{0ab}$

Given $a, b \geq 1$. The goal is to rewrite $\mathfrak{S}_{0a}$ and $\mathfrak{S}_{0ab}$ in terms of basis elements. Taking into account the following facts:

- $\mathfrak{S}_{0a} = \mathbb{B}_0 \mathbb{B}_a(1)$, $\mathfrak{S}_{0ab} = \mathbb{B}_0 \mathbb{B}_a \mathbb{B}_b(1)$ (See [?, Definition 3.2])
- $\mathbb{B}_m = \sum_{i\geq 0} (-1)^i H_{m+i} F_{1^i}^{\perp}$
- $F_0^{\perp} = \mathrm{Id}$ and $F_i^{\perp}(1) = 0$ for all $i \neq 0$.

One can expect that the results we have already obtained might help us. In fact, an application of (1.1) gives

$$\mathbb{B}_0 \mathbb{B}_a = - \sum_{1\leq i<a} \sum_{j=0}^{i} \mathbb{B}_i \mathbb{B}_{a-j} F_{i-j}^{\perp} + \sum_{i\geq 1} \mathbb{B}_a \mathbb{B}_{a+i} F_{a+i}^{\perp} + \sum_{i>a} \sum_{j\geq 1} \mathbb{B}_i \mathbb{B}_{a+j} F_{i+j}^{\perp}$$

Notice that the only term which will contribute to the expression of $\mathfrak{S}_{0a}$ appears in the first sum; just by making $j = i$. All the remaining summands vanish when we evaluate them at 1! Therefore:

$$\boxed{\mathfrak{S}_{0a} = -\sum_{1\leq i<a} \mathfrak{S}_{i\,a-i}}$$

For $\mathfrak{S}_{0ab}$, since

$$\mathbb{B}_0\mathbb{B}_a\mathbb{B}_b = -\sum_{1\leq i<a}\sum_{j=0}^{i} \mathbb{B}_i\mathbb{B}_{a-j}F^{\perp}_{i-j}\mathbb{B}_b + \sum_{i\geq 1} \mathbb{B}_a\mathbb{B}_{a+i}F^{\perp}_{a+i}\mathbb{B}_b + \sum_{i>a}\sum_{j\geq 1} \mathbb{B}_i\mathbb{B}_{a+j}F^{\perp}_{i+j}\mathbb{B}_b,$$

we have to compute $F^{\perp}_{i-j}\mathbb{B}_b$, $F^{\perp}_{a+i}\mathbb{B}_b$ and $F^{\perp}_{i+j}\mathbb{B}_b$. Since we would like to apply Corollary 1.4, we will need to consider these two cases: $\boxed{a\leq b}$, $\boxed{a>b}$.

Case $\boxed{a\leq b}$

In order to be as much clear as possible, we will deal with any of the sums above separately.

$\star$ Given $1\leq i<a$ and $0\leq j\leq i$ we have that $b>i-j$, and Corollary 1.4 (i) applies to get

$$\begin{aligned}
-\sum_{1\leq i<a}\sum_{j=0}^{i} \mathbb{B}_i\mathbb{B}_{a-j}F^{\perp}_{i-j}\mathbb{B}_b &= -\sum_{1\leq i<a} \mathbb{B}_i\mathbb{B}_{a-i}\mathbb{B}_b - \sum_{1\leq i<a}\sum_{j=0}^{i-1} \mathbb{B}_i\mathbb{B}_{a-j}F^{\perp}_{i-j}\mathbb{B}_b \\
&= -\sum_{1\leq i<a} \mathbb{B}_i\mathbb{B}_{a-i}\mathbb{B}_b - \sum_{1\leq i<a}\sum_{j=0}^{i-1}\sum_{k=0}^{i-j} \mathbb{B}_i\mathbb{B}_{a-j}\mathbb{B}_{b-k}F^{\perp}_{i-j-k} = -\sum_{1\leq i<a} \mathbb{B}_i\mathbb{B}_{a-i}\mathbb{B}_b \\
&\quad - \sum_{1\leq i<a}\sum_{j=0}^{i-1} \mathbb{B}_i\mathbb{B}_{a-j}\mathbb{B}_{b-i+j} - \sum_{1\leq i<a}\sum_{j=0}^{i-1}\sum_{k=0}^{i-j-1} \mathbb{B}_i\mathbb{B}_{a-j}\mathbb{B}_{b-k}F^{\perp}_{i-j-k}
\end{aligned}$$

$\star$ For $i\geq 1$, we need to split the sum

$$\sum_{i\geq 1} \mathbb{B}_a\mathbb{B}_{a+i}F^{\perp}_{a+i}\mathbb{B}_b,$$

into three new ones, depending on whether either $i<b-a$, $i=b-a$ or $i>b-a$. Note that in case $b=a$, we are always in the latest situation, namely, $i>b-a$.

$$\begin{aligned}
\sum_{i\geq 1} \mathbb{B}_a\mathbb{B}_{a+i}F^{\perp}_{a+i}\mathbb{B}_b &= \sum_{1\leq i<b-a} \mathbb{B}_a\mathbb{B}_{a+i}F^{\perp}_{a+i}\mathbb{B}_b + \mathbb{B}_a\mathbb{B}_bF^{\perp}_b\mathbb{B}_b + \sum_{i>b-a} \mathbb{B}_a\mathbb{B}_{a+i}F^{\perp}_{a+i}\mathbb{B}_b \\
&= \sum_{1\leq i<b-a}\sum_{j=0}^{a+i} \mathbb{B}_a\mathbb{B}_{a+i}\mathbb{B}_{b-j}F^{\perp}_{a+i-j} + \mathbb{B}_a\mathbb{B}_b - \sum_{i\geq 1} \mathbb{B}_a\mathbb{B}_b\mathbb{B}_{b+i}F^{\perp}_{b+i}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i>b-a}\sum_{j\geq 1} \mathbb{B}_a\mathbb{B}_{a+i}\mathbb{B}_{b+j}F^{\perp}_{a+i+j} = \sum_{1\leq i<b-a} \mathbb{B}_a\mathbb{B}_{a+i}\mathbb{B}_{b-a-i} \\
&+ \sum_{1\leq i<b-a}\sum_{j=0}^{a+i-1} \mathbb{B}_a\mathbb{B}_{a+i}\mathbb{B}_{b-j}F^{\perp}_{a+i-j} + \mathbb{B}_a\mathbb{B}_b - \sum_{i\geq 1}\mathbb{B}_a\mathbb{B}_b\mathbb{B}_{b+i}F^{\perp}_{b+i} \\
&- \sum_{i>b-a}\sum_{j\geq 1} \mathbb{B}_a\mathbb{B}_{a+i}\mathbb{B}_{b+j}F^{\perp}_{a+i+j}
\end{aligned}
$$

⋆ Take now $i > a$ and $j \geq 1$. In this situation, we need to distinguish more cases:

- $a < i < b$ and $1 \leq j < b - i$.
- $a < i < b$ and $j = b - i$.
- $a < i < b$ and $j > b - i$.
- $i \geq b$ and $j \geq 1$.

Note that when $a = b$, none of these cases need to be considered! Corollary 1.4 (iii) can be applied directly!

$$
\begin{aligned}
&\sum_{i>a}\sum_{j\geq 1} \mathbb{B}_i\mathbb{B}_{a+j}F^{\perp}_{i+j}\mathbb{B}_b = \sum_{a<i<b}\sum_{1\leq j<b-i} \mathbb{B}_i\mathbb{B}_{a+j}F^{\perp}_{i+j}\mathbb{B}_b + \sum_{a<i<b}\mathbb{B}_i\mathbb{B}_{a+b-i}F^{\perp}_b\mathbb{B}_b \\
&\quad + \sum_{a<i<b}\sum_{j>b-i} \mathbb{B}_i\mathbb{B}_{a+j}F^{\perp}_{i+j}\mathbb{B}_b + \sum_{i\geq b}\sum_{j\geq 1} \mathbb{B}_i\mathbb{B}_{a+j}F^{\perp}_{i+j}\mathbb{B}_b \\
&= \sum_{a<i<b}\sum_{1\leq j<b-i}\sum_{k=0}^{i+j} \mathbb{B}_i\mathbb{B}_{a+j}\mathbb{B}_{b-k}F^{\perp}_{i+j-k} + \sum_{a<i<b}\mathbb{B}_i\mathbb{B}_{a+b-i} \\
&\quad - \sum_{a<i<b}\sum_{j\geq 1}\mathbb{B}_i\mathbb{B}_{a+b-i}\mathbb{B}_{b+j}F^{\perp}_{b+j} - \sum_{a<i<b}\sum_{j>b-i}\sum_{k\geq 1}\mathbb{B}_i\mathbb{B}_{a+j}\mathbb{B}_{b+k}F^{\perp}_{i+j+k} \\
&\quad - \sum_{i\geq b}\sum_{j\geq 1}\sum_{k\geq 1}\mathbb{B}_i\mathbb{B}_{a+j}\mathbb{B}_{b+k}F^{\perp}_{i+j+k} = \sum_{a<i<b}\sum_{1\leq j<b-i}\mathbb{B}_i\mathbb{B}_{a+j}\mathbb{B}_{b-i-j} \\
&\quad + \sum_{a<i<b}\sum_{1\leq j<b-i}\sum_{k=0}^{i+j-1} \mathbb{B}_i\mathbb{B}_{a+j}\mathbb{B}_{b-k}F^{\perp}_{i+j-k} + \sum_{a<i<b}\mathbb{B}_i\mathbb{B}_{a+b-i} \\
&\quad - \sum_{a<i<b}\sum_{j\geq 1}\mathbb{B}_i\mathbb{B}_{a+b-i}\mathbb{B}_{b+j}F^{\perp}_{b+j} - \sum_{a<i<b}\sum_{j>b-i}\sum_{k\geq 1}\mathbb{B}_i\mathbb{B}_{a+j}\mathbb{B}_{b+k}F^{\perp}_{i+j+k} \\
&\quad - \sum_{i\geq b}\sum_{j\geq 1}\sum_{k\geq 1}\mathbb{B}_i\mathbb{B}_{a+j}\mathbb{B}_{b+k}F^{\perp}_{i+j+k}
\end{aligned}
$$

Putting all together, we get

$$\boxed{\begin{aligned}\mathfrak{S}_{0ab} = &-\sum_{1\le i<a}\mathfrak{S}_{i\ a-i\ b} - \sum_{1\le i<a}\sum_{j=0}^{i-1}\mathfrak{S}_{i\ a-j\ b-i+j} + \sum_{1\le i<b-a}\mathfrak{S}_{a\ a+i\ b-a-i} + \mathfrak{S}_{ab}\\ &+\sum_{a<i<b}\sum_{1\le j<b-i}\mathfrak{S}_{i\ a+j\ b-i-j} + \sum_{a<i<b}\mathfrak{S}_{i\ a+b-i}\end{aligned}}$$

Note that for $\mathfrak{S}_{0aa}$ we have the following nicer expression:

$$\boxed{\mathfrak{S}_{0aa} = -\sum_{1\le i<a}\mathfrak{S}_{i\ a-i\ a} - \sum_{1\le i<a}\sum_{j=0}^{i-1}\mathfrak{S}_{i\ a-j\ a-i+j}}$$

Case $a > b$

Recall that

$$\mathbb{B}_0\mathbb{B}_a\mathbb{B}_b = -\sum_{1\le i<a}\sum_{j=0}^{i}\mathbb{B}_i\mathbb{B}_{a-j}F^{\perp}_{i-j}\mathbb{B}_b + \sum_{i\ge 1}\mathbb{B}_a\mathbb{B}_{a+i}F^{\perp}_{a+i}\mathbb{B}_b + \sum_{i>a}\sum_{j\ge 1}\mathbb{B}_i\mathbb{B}_{a+j}F^{\perp}_{i+j}\mathbb{B}_b$$

We will proceed, as before, studying separately each sum. Let us keep the first one to the end, since as you will see this is the most tedious one!

$\star$ Given $i \ge 1$, we always have that $a + i > b$. So Corollary 1.4 (iii) applies to get

$$\sum_{i\ge 1}\mathbb{B}_a\mathbb{B}_{a+i}F^{\perp}_{a+i}\mathbb{B}_b = -\sum_{i,j\ge 1}\mathbb{B}_a\mathbb{B}_{a+i}\mathbb{B}_{b+j}F^{\perp}_{a+i+j}$$

$\star$ For $i > a$ and $j \ge 1$, again we can always assert that $i + j > b$; so a second use of Corollary 1.4 (iii) gives

$$\sum_{i>a}\sum_{j\ge 1}\mathbb{B}_i\mathbb{B}_{a+j}F^{\perp}_{i+j}\mathbb{B}_b = -\sum_{i>a}\sum_{j,k\ge 1}\mathbb{B}_i\mathbb{B}_{a+j}\mathbb{B}_{b+k}F^{\perp}_{i+j+k}$$

To finish, let us consider the sum

$$-\sum_{1\le i<a}\sum_{j=0}^{i}\mathbb{B}_i\mathbb{B}_{a-j}F^{\perp}_{i-j}\mathbb{B}_b$$

Note that

$$-\sum_{1\le i<a}\sum_{j=0}^{i}\mathbb{B}_i\mathbb{B}_{a-j}F^{\perp}_{i-j}\mathbb{B}_b = -\sum_{1\le i<a}\mathbb{B}_i\mathbb{B}_{a-i}\mathbb{B}_b - \sum_{1\le i<a}\sum_{j=0}^{i-1}\mathbb{B}_i\mathbb{B}_{a-j}F^{\perp}_{i-j}\mathbb{B}_b \tag{1.5}$$

Thus, we only have to deal with the second sum in (1.5). Again, we need to distinguish some cases, namely:

- $1 \le i < b$ and $0 \le j \le i-1$.
- $i = b$.
- $b < i < a$ and $0 \le j \le i-1$.

Hence

$$-\sum_{1\le i<a}\sum_{j=0}^{i-1}\mathbb{B}_i\mathbb{B}_{a-j}F_{i-j}^{\perp}\mathbb{B}_b = -\sum_{1\le i<b}\sum_{j=0}^{i-1}\mathbb{B}_i\mathbb{B}_{a-j}F_{i-j}^{\perp}\mathbb{B}_b - \sum_{j=0}^{b-1}\mathbb{B}_b\mathbb{B}_{a-j}F_{b-j}^{\perp}\mathbb{B}_b$$
$$-\sum_{b<i<a}\sum_{j=0}^{i-1}\mathbb{B}_i\mathbb{B}_{a-j}F_{i-j}^{\perp}\mathbb{B}_b \tag{1.6}$$

Let us now compute the sums appearing in (1.6) separately. (I hope nobody will be confused! Sorry for the mess!)

$\star\star$ For $1 \le i < b$ and $0 \le j \le i-1$ we have that $b > i-j$. So, by Corollary 1.4 (i) we get

$$-\sum_{1\le i<b}\sum_{j=0}^{i-1}\mathbb{B}_i\mathbb{B}_{a-j}F_{i-j}^{\perp}\mathbb{B}_b = -\sum_{1\le i<b}\sum_{j=0}^{i-1}\sum_{k=0}^{i-j}\mathbb{B}_i\mathbb{B}_{a-j}\mathbb{B}_{b-k}F_{i-j-k}^{\perp}$$
$$= -\sum_{1\le i<b}\sum_{j=0}^{i-1}\mathbb{B}_i\mathbb{B}_{a-j}\mathbb{B}_{b-i+j} - \sum_{1\le i<b}\sum_{j=0}^{i-1}\sum_{k=0}^{i-j-1}\mathbb{B}_i\mathbb{B}_{a-j}\mathbb{B}_{b-k}F_{i-j-k}^{\perp}$$

$\star\star$ Splitting

$$-\sum_{j=0}^{b-1}\mathbb{B}_b\mathbb{B}_{a-j}F_{b-j}^{\perp}\mathbb{B}_b$$

into two terms, and applying Corollary 1.4 (i) and (ii) we obtain

$$-\sum_{j=0}^{b-1}\mathbb{B}_b\mathbb{B}_{a-j}F_{b-j}^{\perp}\mathbb{B}_b = -\mathbb{B}_b\mathbb{B}_aF_b^{\perp}\mathbb{B}_b - \sum_{j=1}^{b-1}\mathbb{B}_b\mathbb{B}_{a-j}F_{b-j}^{\perp}\mathbb{B}_b = -\mathbb{B}_b\mathbb{B}_a$$
$$+\sum_{k\ge1}\mathbb{B}_b\mathbb{B}_a\mathbb{B}_{b+k}F_{b+k}^{\perp} - \sum_{j=1}^{b-1}\sum_{k=0}^{b-j}\mathbb{B}_b\mathbb{B}_{a-j}\mathbb{B}_{b-k}F_{b-j-k}^{\perp} = -\mathbb{B}_b\mathbb{B}_a$$
$$+\sum_{k\ge1}\mathbb{B}_b\mathbb{B}_a\mathbb{B}_{b+k}F_{b+k}^{\perp} - \sum_{j=1}^{b-1}\mathbb{B}_b\mathbb{B}_{a-j}\mathbb{B}_j - \sum_{j=1}^{b-1}\sum_{k=0}^{b-j-1}\mathbb{B}_b\mathbb{B}_{a-j}\mathbb{B}_{b-k}F_{b-j-k}^{\perp}$$

$\star\star$ To finish, let us consider the latest sum in (1.6). Split it into two new ones, and apply Corollary 1.4 appropriately to obtain

$$
\begin{aligned}
-\sum_{b<i<a}\sum_{j=0}^{i-1}\mathbb{B}_i\mathbb{B}_{a-j}F_{i-j}^{\perp}\mathbb{B}_b &= -\sum_{b<i<a}\mathbb{B}_i\mathbb{B}_aF_i^{\perp}\mathbb{B}_b-\sum_{b<i<a}\sum_{j=1}^{i-1}\mathbb{B}_i\mathbb{B}_{a-j}F_{i-j}^{\perp}\mathbb{B}_b\\
&=\sum_{b<i<a}\sum_{k\geq 1}\mathbb{B}_i\mathbb{B}_a\mathbb{B}_{b+k}F_{i+k}^{\perp}-\sum_{b<i<a}\sum_{1\leq j<i-b}\mathbb{B}_i\mathbb{B}_{a-j}F_{i-j}^{\perp}\mathbb{B}_b-\sum_{b<i<a}\mathbb{B}_i\mathbb{B}_{a-i+b}F_b^{\perp}\mathbb{B}_b\\
&\quad-\sum_{b<i<a}\sum_{i-b<j\leq i-1}\mathbb{B}_i\mathbb{B}_{a-j}F_{i-j}^{\perp}\mathbb{B}_b
\end{aligned}
\tag{1.7}
$$

Below, we straighten the new factors arising from (1.7).

- For $b<i<a$ and $1\leq j<i-b$, we have that $i-j>b$; so Corollary 1.4 (iii) allows us to conclude that

$$
-\sum_{b<i<a}\sum_{1\leq j<i-b}\mathbb{B}_i\mathbb{B}_{a-j}F_{i-j}^{\perp}\mathbb{B}_b=\sum_{b<i<a}\sum_{1\leq j<i-b}\sum_{k\geq 1}\mathbb{B}_i\mathbb{B}_{a-j}\mathbb{B}_{b+k}F_{i-j+k}^{\perp}
$$

- By Corollary 1.4 (ii) we get

$$
-\sum_{b<i<a}\mathbb{B}_i\mathbb{B}_{a-i+b}F_b^{\perp}\mathbb{B}_b=-\sum_{b<i<a}\mathbb{B}_i\mathbb{B}_{a-i+b}+\sum_{b<i<a}\sum_{k\geq 1}\mathbb{B}_i\mathbb{B}_{a-i+b}\mathbb{B}_{b+k}F_{b+k}^{\perp}
$$

- Given $b<i<a$ and $i-b<j\leq i-1$, we always have that $b>i-j$. Thus, we can apply Corollary 1.4 (i) and conclude that

$$
\begin{aligned}
-\sum_{b<i<a}\sum_{i-b<j\leq i-1}\mathbb{B}_i\mathbb{B}_{a-j}F_{i-j}^{\perp}\mathbb{B}_b &= -\sum_{b<i<a}\sum_{i-b<j\leq i-1}\sum_{k=0}^{i-j}\mathbb{B}_i\mathbb{B}_{a-j}\mathbb{B}_{b-k}F_{i-j-k}^{\perp}\\
&=-\sum_{b<i<a}\sum_{i-b<j\leq i-1}\mathbb{B}_i\mathbb{B}_{a-j}\mathbb{B}_{b-i+j}-\sum_{b<i<a}\sum_{i-b<j\leq i-1}\sum_{k=0}^{i-j-1}\mathbb{B}_i\mathbb{B}_{a-j}\mathbb{B}_{b-k}F_{i-j-k}^{\perp}
\end{aligned}
$$

At this point, if one is not lost in this "sum jungle", one realizes that

$$
\boxed{
\begin{aligned}
\mathfrak{S}_{0ab} &= -\sum_{1\leq i<a}\mathfrak{S}_{i\ a-i\ b}-\sum_{1\leq i<b}\sum_{j=0}^{i-1}\mathfrak{S}_{i\ a-j\ b-i+j}-\mathfrak{S}_{b\,a}-\sum_{j=0}^{b-1}\mathfrak{S}_{b\ a-j\ j}\\
&\quad-\sum_{b<i<a}\mathfrak{S}_{i\ a-i+b}-\sum_{b<i<a}\sum_{i-b<j\leq i-1}\mathfrak{S}_{i\ a-j\ b-i+j}
\end{aligned}
}
$$

Remark 1.6. Note that in order to get a formula for $\mathfrak{S}_{0ab}$, we could have stopped our calculations before!!! Actually, as soon as we get an operator $F_i^{\perp}$ appearing to the right most position of a term, we can already forget about this term, since it will vanish after evaluating it at 1. Just remember that $F_i^{\perp}(1) = 0$.

I went through the whole calculation of $\mathbb{B}_0\mathbb{B}_a\mathbb{B}_b$, since I am not sure whether it could be useful to get an expression of $\mathfrak{S}_{0abc}$ (?)

[1] C. Berg, N. Bergeron, F. Saliola, L. Serrano, M. Zabrocki:
A lift of the Shur and Hall-Littlewood bases to Non-commutative symmetric functions.
`arXiv:1208.5191v1 [math.CO]`