Consider 2 directions of generalizing the Hopf algebra of binary trees.

\( \text{deg} = d = \# \text{ leaf} - 1 \)

\( \text{ideg} = \# \text{ of internal nodes} \)


\( A \quad \Delta(T) = \sum_{\text{all leaf}} \frac{\delta}{\delta} \)

preserves \( \text{deg} \), not \( \text{ideg} \)

\( \Delta(\Psi) = \Psi \otimes \Psi \)

\( \Psi \)

\( B \quad \Delta'(T) = \sum_{\text{Allowable Leaf}} \frac{\delta}{\delta} \)

Preserves both \( \text{deg} \), \( \text{ideg} \)

\( \text{Mult} \): only using allowable cuts

\( \text{Mult} \): Same as binary tree, over all \( \frac{\delta}{\delta} \)

Q1. \( A \cong B \) ?

Hint: consider the dimension of primitive elements of each degree

Q2. For \( B \), define

\( m\text{deg}(T) = z_1^{b_1} \cdots z_k^{b_k} \)

\( b_i = \# \text{ of internal node} \)

with degree \( i-1 \)

\( \Delta(T) = \sum T_1 \otimes T_2 \)

\( m\text{deg} T = m\text{deg} T_1 \cdot m\text{deg} T_2 \)

Find Hilbert Series

Q2' For \( B \), define

Increasing (upward)

Non-degenerate, Stirling permutation:

Labelled, multiset permutation

Planar trees, avoiding \( X \cdots Y \cdots X \)

\( \Psi \)

\( 441255163 \)
Q2': \[ \hat{H}(z_1, z_2, \ldots) = \sum_{\text{INC TREES}} \text{mdeg}(T) \] 

Other Hilbert Series are specialization of this.

Q3: Find a "good" M basis.

Define an "interesting" partial order

(on planar trees / Labelled planar trees)

Q4: Quasi-sym in Non-com Variable
indexed by set composition

\[
\{1,2,3,4\} \rightarrow (\{2\}, \{1,4\}, \{3\}) \leftrightarrow \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 1 & 3 & 2
\end{array}
\]

labelled \rightarrow \text{stirling} \rightarrow \text{permutation} \rightarrow \text{Word} \rightarrow \text{set composition}

Planar \rightarrow \text{tree}

Find a "natural" injection

Hopf INC

Non-deg Labelled \rightarrow \text{NC-QSYM}

Planar trees

Q5: Lift the Novelli-Thibon Hopf algebra on parking functions to left INC BIN trees

Planar binary tree

Inc on left \leftrightarrow Parking

labelled tree \leftrightarrow Function

Our \[ \frac{1}{q} \] no longer works!