Consider 2 directions of generalizing the Hopf algebra of binary trees.

deg = d = # leaf − 1

(1) − Generalizing Trees.

ideg = # of internal nodes

Ⓐ $\Delta(T) = \sum_{\text{all leaf}} \otimes$

preserves deg, not ideg

$\Delta(\ ) = 1 \otimes \ + \underbrace{\ \otimes\ }\ + \ \otimes 1$

Mult: Same as binary tree, over all

Ⓑ $\Delta'(T) = \sum_{\substack{\text{Allowoble}\\ \text{Leaf}}} \otimes$

Preserves both deg, ideg

Mult′: only using allowoble cuts

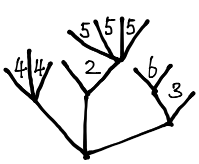
Q1 Ⓐ $\cong$ Ⓑ ?

Hint: consider the dimension of primitive elements of each degree

(2) − Generalizing labelling

Increasing (upward)
Non-degenerate
Labelled
planar trees

stirling permutation:
multiset permutation
avoiding $X \dots Y \dots X$, $X > Y$



$\longrightarrow$ 4412555163

Q1′: Ⓐ $\cong$ Ⓑ ?

Q2: For Ⓑ, define

$\text{mdeg}(T) = z_1^{b_1} \cdots z_k^{b_k}$.

b_i = # of internal node with degree $i-1$

$\Delta(T) = \sum T_1 \otimes T_2$

$\text{mdeg}\, T = \text{mdeg}\, T_1 \cdot \text{mdeg}\, T_2$

Find Hilbert Series

Q2' : $\hat{H}(z_1, z_2, \dots) = \sum_{\substack{\text{INC LABELLED} \\ \text{TREES}}} \operatorname{mdeg}(T)$ "Other Hilbert Series are specialization of this"

$H(z_1, z_2, \dots) = \sum_{\substack{\text{INC PLANAR} \\ \text{TREES}}} \operatorname{mdeg}(T)$

Q3 : Find a "good" M basis.

(define an "interesting" partial order on planar trees / Labelled planar trees)

Q4 : Quasi-sym in Non-com Variable indexed by set composition

$\{1,2,3,4\} \longrightarrow (\{2\}, \{1,4\}, \{3\}) \longleftrightarrow \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 2 \end{matrix}$

labelled planar tree $\longrightarrow$ stirling permutation $\sim$ word $\longrightarrow$ set composition

Find a "natural" injection

Hopf INC Non-deg Labelled planar trees $\hookrightarrow$ NC-QSYM

Q5 : Lift the Novelli-Thibon Hopf algebra on parking functions to left INC BIN trees

Planar binary tree INC ON left labelled tree $\longleftrightarrow$ Parking Function

our ↯ no longer works!