Crystal Bases for $\mathcal{U}_q(osp(1,2n))$

$B(0,n) = osp(1,2n)$ is the only simple Lie superalgebra (which is not a Lie algebra) for which all finite dimensional representations are completely reducible.

$$osp(1,2n) = \left\{ \begin{array}{cl} \begin{pmatrix} 0 & a & b \\ -b & A & B \\ a & C & -A^T \end{pmatrix} & \begin{array}{l} \}1 \\ \}n \\ \}n \end{array} \\ \begin{array}{ccc} \underbrace{\quad}_{1} & \underbrace{\quad}_{n} & \underbrace{\quad}_{n} \end{array} & \end{array} \;\middle|\; B^T = B,\ C^T = C \right\}$$

$$osp(1,2) = \left\{ \begin{pmatrix} 0 & a & b \\ -b & c & d \\ a & e & -c \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{C} \right\}$$

Let $e = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Then $[h,e] = e$, $[h,f] = -f$, $[e,f] = h$ $\qquad (*)$

$osp(1,2) = \langle e, h, f \mid (*) \rangle$

$$osp(1,2n) = \langle \underbrace{e_1, h_1, f_1, \ldots, e_{n-1}, h_{n-1}, f_{n-1}}_{n-1 \text{ copies of } \mathfrak{sl}_2}, \underbrace{e_n, h_n, f_n}_{osp(1,2)} \mid (*) \rangle$$

$$(*) \quad [h_i, h_j] = 0 \quad , \quad [e_i, f_j] = \delta_{ij} h_i \qquad \forall\, i,j$$

$$[h_i, e_j] = \mathbf{a_{ij}}\, e_j \quad , \quad [h_i, f_j] = -\mathbf{a_{ij}}\, f_j \qquad \forall\, i,j$$

$$[e_i,[e_i, e_j]] = 0 = [f_i,[f_i, f_j]] \qquad i \neq j \text{ and if } i = n,\ j \neq n-1$$

$$[e_n[e_n[e_n, e_{n-1}]]] = [f_n[f_n[f_n, f_{n-1}]]] = 0$$

where

$$A = (\mathbf{a_{ij}})_{1 \leq i,j \leq n} = \begin{pmatrix} 2 & -1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & & & \vdots \\ 0 & -1 & 2 & & & \\ & & & \ddots & & 0 \\ \vdots & & & & 2 & -1 \\ 0 & \cdots & & 0 & -1 & 1 \end{pmatrix}$$

Let $\mathfrak{h} = \text{span}\{h_1, \ldots, h_{n-1}, h_n\}$ and define $\alpha_i \in \mathfrak{h}^*$

by $\alpha_i(h_j) = \mathbf{a_{ji}}$

$$\mathcal{U} = \mathcal{U}(\mathbf{osp(1,2n)}) = \langle e_1, h_1, f_1, \ldots, e_{n-1}, h_{n-1}, f_{n-1}, e_n, h_n, f_n \mid (*) \rangle$$

$\mathcal{U} = \mathcal{U}^- \mathcal{U}^\circ \mathcal{U}^+$ where $\mathcal{U}^-$ = subalgebra generated by f's
$\mathcal{U}^\circ$ = " " " h's
$\mathcal{U}^+$ = " " " e's

Let $\lambda \in \mathfrak{h}^*$.

Define $V(\lambda) = \mathcal{U} \big/ \langle e_i, h_i - \lambda(h_i)1 \rangle$ and

$M(\lambda)$ be the simple module $V(\lambda) \big/ I(\lambda)$

where $I(\lambda)$ is the unique maximal submodule of $V(\lambda)$

Denote the image of $\bar{1}$ in $M(\lambda)$ by u_λ.

Facts:
- Every irreducible f.d. representation of $\mathcal{U}$ is of the form $M(\lambda)$ for $\lambda \in \{ \mu \in \mathfrak{h}^* \mid \langle \mu, h_i \rangle \in \mathbb{N} \ \forall i \}$
- Every finite dimensional representation of $\mathcal{U}$ is completely reducible.

In $\mathcal{U}(osp(1,2)) = \langle e, h, f \mid [h,e]=e, [h,f] = -f, [e,f] = h \rangle$,

$$ef^r = \begin{cases} -\frac{r}{2} f^{r-1} + f^r e & \text{if } r \text{ is even} \\ f^{r-1}\left(h - \frac{r-1}{2}\right) - f^r e & \text{if } r \text{ is odd} \end{cases}$$

For $\lambda \in \mathfrak{h}^*$, $M(\lambda)$ is finite dimensional iff

$\exists \ r \in \mathbb{Z}_{\geq 1}$ such that $f^r u_\lambda = 0$ and $f^{r-1} u_\lambda \neq 0$ iff

$\lambda(h) - \frac{r-1}{2} = 0$ and r is odd iff $r = 2\lambda(h) + 1$.

Hence the dimension of $M(\lambda)$ is odd

The quantized universal enveloping algebra of $osp(1,2n)$, $\mathcal{U}_q = \mathcal{U}_q(osp(1,2n))$ is the associative superalgebra (with 1) over $\mathbb{C}(\sqrt{q})$ generated by

$E_1, K_1, K_1^{-1}, F_1, \ldots, E_{n-1}, K_{n-1}, K_{n-1}^{-1}, F_{n-1}, E_n, K_n, K_n^{-1}, F_n$ subject to

$$K_i K_j = K_j K_i \ , \quad K_i K_i^{-1} = K_i^{-1} K_i = 1$$

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j \ , \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j$$

$$E_i F_j - (-1)^{|E_i||F_j|} F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$$

$$\left.\begin{array}{l} E_i^2 E_j E_i - [2]_q E_i E_j E_i + E_j E_i^2 = 0 \\ F_i^2 F_j F_i - [2]_q F_i F_j F_i + F_j F_i^2 = 0 \end{array}\right\} \begin{array}{l} \text{if } i \neq j \text{ and} \\ \text{if } i = n, \ j \neq n-1 \end{array}$$

$$E_n^3 E_{n-1} - (q - 1 + q^{-1}) E_n^2 E_{n-1} E_n - (q - 1 + q^{-1}) E_n E_{n-1} E_n^2 + E_{n-1} E_n^3 = 0$$

$$F_n^3 F_{n-1} - (q - 1 + q^{-1}) F_n^2 F_{n-1} F_n - (q - 1 + q^{-1}) F_n F_{n-1} F_n^2 + F_{n-1} F_n^3 = 0$$

where the degree of all generators is 0 except for E_n and F_n whose degree is 1.

$\mathcal{U}_q$ has a Hopf algebra structure

$$\mathcal{U}_q = \mathcal{U}_q^- \mathcal{U}_q^0 \mathcal{U}_q^+$$

Let M be a $\mathcal{U}_q$-module.

Let $\omega = (\omega_1, \dots, \omega_n)$ where $\omega_i \in \mathbb{C}(\sqrt{q})^*$

The weight space of M of weight ω is

$$M_\omega = \{ x \in M \mid K_i x = \omega_i x \ \forall \ i \}$$

- $M_\omega \cap \sum_{\mu \neq \omega} M_\mu = \{0\}$
- $x \in M_\omega \Rightarrow E_i x \in M_{(\omega_1 q^{a_{1i}}, \dots, \omega_n q^{a_{ni}})}$ and

  $F_i x \in M_{(\omega_1 q^{-a_{1i}}, \dots, \omega_n q^{-a_{ni}})}$

A $\mathcal{U}_q$-module M is called a highest weight module of highest weight ω if $\exists\ 0 \neq v \in M$ such that

$E_i v = 0 \ \forall i$
$K_i v = \omega_i v \ \forall i$
$M = \mathcal{U}_q v = \mathcal{U}_q^- v$

Let $\omega = (\omega_1, \dots, \omega_n)$ and $\omega' = (\omega_1', \dots, \omega_n')$ where $\omega_i, \omega_i' \in \mathbb{C}(\sqrt{q})^*$

Define $\omega \leq \omega' \Leftrightarrow \forall i \quad \omega_i' \omega_i^{-1} = q^{\sum_j r_j a_{ij}}$ for $r_j \in \mathbb{N}$.

If M is a highest weight module of highest weight

ω, $M = \bigoplus_{\omega' \leq \omega} M_{\omega'}$ and $\dim M_\omega = 1$.

Let $\omega = (\omega_1, \ldots, \omega_n) \in (\mathbb{C}(\sqrt{q})^*)^n$.

Define $\mathbf{V(\omega)} = \mathcal{U}_q / \langle E_i, K_i - \omega_i 1 \mid 1 \le i \le n \rangle$

$V(\omega)$ is a highest weight module of highest weight ω with highest weight vector $\bar{1}$.

$V(\omega)$ contains a unique maximal submodule $\mathbf{I(\omega)}$

Define $\mathbf{M(\omega)} = \mathbf{V(\omega)} / \mathbf{I(\omega)}$.

$M(\omega)$ is an irreducible highest weight module with highest weight ω. Denote a highest ~~weight~~ w vector by v_ω.

A $\mathcal{U}_q$-module M is called integrable if

$M = \bigoplus_{\omega \in \mathfrak{h}^*} M_\omega$ and E_i, F_i act locally nilpotently on M.

Theorem: $M(\omega)$ is integrable iff

$$\omega_i = \pm q^{m_i} \quad \text{with } m_i \in \mathbb{N} \quad \text{and}$$

$$\omega_n = \begin{cases} \pm q^m & , m \in \mathbb{N} \\ \pm i q^m & , m \in \mathbb{N} + \frac{1}{2} \end{cases}$$

Proof: Let $r \geqslant 1$ be such that $F_n^r v_w = 0$ and $F_n^{r-1} v_w \neq 0$.

We have

$$E_n F_n^r = F_n^{r-1}\left(K_n \frac{q^{-r} + (-1)^{r-1}}{q^{-1}+1} - K_n^{-1} \frac{q^r + (-1)^{r-1}}{q+1}\right) + (-1)^r F_n^r E_n$$

So $\quad \omega_n \dfrac{q^{-r} + (-1)^{r-1}}{q^{-1}+1} - \omega_n^{-1} \dfrac{q^r + (-1)^{r-1}}{q+1} = 0$.

$$\omega_n^2 = (-1)^{n-1} q^{r-1}.$$

$$\omega_n = \begin{cases} \pm q^{(r-1)/2} & \text{if } r \text{ is odd} \\ \pm i\, q^{(r-1)/2} & \text{if } r \text{ is even} \end{cases}$$

If $i \neq n$,

$$E_i F_i^r = F_i^{r-1} \frac{\left((q^{-2(r-1)}+1) K_i - (q^{2(r-1)}+1) K_i^{-1}\right)}{q - q^{-1}} + F_i^r E_i$$

So $\;(q^{-2(r-1)}+1)\,\omega_i - (q^{2(r-1)}+1)\,\omega_i^{-1} = 0$.

$$\omega_i^2 = \frac{q^{2(r-1)}+1}{q^{-2(r-1)}+1}, \quad \omega_i = \pm q^{(r-1)}$$

($\Leftarrow$) Let $y = F_{i_1} \cdots F_{i_t} v_\omega$.

$E_i^k y = 0$ for $k >> 0$ since its weight $\not\leq \omega$.

We now show that $F_i^k y = 0$ for $k >> 0$.

First assume $t = 0$.

If $i = n$, consider $v = F_n^{2m+1} v_\omega$.

For $j \neq n$, $E_j F_n^{2m+1} v_\omega = F_n^{2m+1} E_j v_\omega = 0$

$$E_n F_n^{2m+1} v_\omega = F_n^{2m} \left(\pm q^m \frac{q^{-(2m+1)} + (-1)^{2m}}{q^{-1}+1} \mp q^{-m} \frac{q^{2m+1} + (-1)^{2m}}{q+1} \right) v_\omega$$

$$= 0$$

i $\quad -i$
if $m \in \mathbb{N} + \frac{1}{2}$

So if $v \neq 0$, it generates a proper submodule of $M(\omega)$. Since $M(\omega)$ is simple, $F_n^{2m+1} v_\omega = 0$.

Similarly, if $i \neq n$, we get $F_i^{m_i} v_\omega = 0$.

Now assume $t \geq 1$. If $i = i_1$, by induction we are done. If $i \neq i_1$, use defining relations and induction.

$\square$

Corollary: $\mathcal{U}_q(osp(1,2))$ has irreducible modules of all dimensions.

Let $\sigma : \{\alpha_1, \dots, \alpha_n\} \to \{\pm 1\}$ and

$$\lambda \in \Sigma := \left\{ \lambda \in \mathfrak{h}^* \ : \ \lambda(h_i) \in \mathbb{Z} \ , \ 1 \leq i \leq n-1, \ \lambda(h_n) \in \tfrac{1}{2}\mathbb{Z} \right\}.$$

Let $\omega_{\lambda,\sigma} = \begin{cases} (\sigma(\alpha_1) q^{\lambda(h_1)}, \dots, \sigma(\alpha_n) q^{\lambda(h_n)}) & \text{if } \lambda(h_n) \in \mathbb{Z} \\ (\sigma(\alpha_1) q^{\lambda(h_1)}, \dots, \sigma(\alpha_n)\, i\, q^{\lambda(h_n)}) & \text{if } \lambda(h_n) \in \mathbb{Z} + \frac{1}{2} \end{cases}$

Let M be a finite dimensional integrable $\mathcal{U}_q$-module

- $M = \bigoplus_{\substack{\lambda \in \Sigma \\ \sigma}} M_{\omega_{\lambda,\sigma}}$
- $M = \bigoplus_{\sigma} M_\sigma$ where $M_\sigma = \bigoplus_{\lambda \in \Sigma} M_{\lambda,\sigma}$.
  - If $M = M_\sigma$, M is said to be of type σ.
- M is the direct sum of different types of modules.
- M of type $\sigma \Rightarrow M \cong M_1 \otimes Y$ where

  M_1 is of type 1 $(1(\alpha_i) = 1 \ \forall i)$ and Y is a

  1-dimensional module of type σ.

We denote $\omega_{\lambda,1}$ by ω_λ, and M_{ω_λ} by M_λ.