

- Let $\lambda \in \Sigma^+ = \left\{ \mu \in \mathfrak{h}^* \mid \begin{array}{l} \mu(h_i) \in \mathbb{N} \quad \forall 1 \leq i \leq n-1 \\ \text{and } \mu(h_n) \in \frac{1}{2}\mathbb{N} \end{array} \right\}$

$M(\omega_\lambda)$ is finite dimensional.

- Every finite dimensional integrable irreducible $U_q(\mathfrak{osp}(1, 2n))$ -module (of type 1) is isomorphic to $M(\omega_\lambda)$ for some $\lambda \in \Sigma^+$.
- Every finite dimensional integrable $U_q(\mathfrak{osp}(1, 2n))$ -module is completely reducible.

Lemma: Let M be an integrable $U_q(\mathfrak{osp}(1, 2r))$ -mod.

Denote M_λ by $M_{\lambda(h)}$.

If $\dim M_m < \infty$, M is finite dimensional

and $\dim M_m = \dim M_{-m}$

Proof:

Assume $M_m \neq 0$ for some $m \in \frac{1}{2}\mathbb{Z}$.

Let $0 \neq x \in M_m$.

Let r be such that $E^{r+1}x = 0$ and $u_0 = E^r x \neq 0$

$u_0 \in M_{m+r}$

$U_q u_0$ is irreducible

$r+m \in \mathbb{Z} \Rightarrow \dim (U_q u_0)_0 = 1$

$r+m \in \mathbb{Z} + \frac{1}{2} \Rightarrow \dim (U_q u_0)_{1/2} = 1$

If $M = U_q u_0$, we are done.

If not, $\dim (M/\langle u_0 \rangle)_0 + \dim (M/\langle u_0 \rangle)_{1/2} < \dim M_0 + \dim M_{1/2}$

and we can use induction.

□

Let W be the Weyl group associated to

$\pi = \{\alpha_1, \dots, \alpha_n\}$. i.e W is the group generated by

r_1, \dots, r_n where for $\lambda \in \mathfrak{h}^*$,

$$r_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i, \quad 1 \leq i \leq n-1$$

$$r_n(\lambda) = \lambda - 2\langle \lambda, h_n \rangle \alpha_n$$

- W is finite
- λ is conjugate under W to exactly one element of $\Sigma^+ = \{\lambda \in \Sigma \mid \langle \lambda, h_i \rangle \geq 0\}$
- Let $\lambda \in \Sigma^+$. Then the number of $\mu \in \Sigma^+$ such that $\lambda - \mu \in \sum \mathbb{N}\alpha_i$ is finite.
- Denote by w_0 the longest word in W .

Lemma: Let M be an integrable \mathcal{U}_q -module with $\dim M_w < \infty$.

Then $\dim M_\lambda = \dim M_{w\lambda}$, where $w = w_0$ and $w \in W$

Proof:

$$\text{Let } N = \sum_{m \in \mathbb{Z}} M_{\lambda + m\alpha_n}.$$

N is f.d. integrable $\mathcal{U}_q(\mathfrak{osp}(1,2))$ -module.

By the previous lemma,

$$M_\lambda = N_{\lambda(h_n)} \quad N_{-\lambda(h_n)} = M_{r_n \lambda} \quad (r_n \lambda = \lambda - 2\lambda(h_n)\alpha_n)$$

By Lemma $\dim M_\lambda = \dim M_{r_n \lambda}$.

By \mathfrak{sl}_2 -theory, $M_\lambda = M_{r_i \lambda} \quad \forall \quad 1 \leq i \leq n-1$

□

Theorem: If $\lambda \in \Sigma^+$, $M(\lambda)$ is finite dimensional.

Every finite dimensional integrable irreducible U_q -module is $\cong M(\lambda)$ for some $\lambda \in \Sigma$.

Proof:

If $M(\lambda)_\mu \neq 0$, $\mu \in \lambda - \sum \mathbb{N}\alpha_i$.

\exists finitely many $\mu \in \Sigma^+$ such that $\mu \in \lambda - \sum \mathbb{N}\alpha_i$.
 Every weight of $M(\lambda)$ is conjugate to exactly one dominant weight in $\lambda - \sum \mathbb{N}\alpha_i$.
 Since W is finite, \exists finitely many weights of $M(\lambda)$. So $M(\lambda)$ is finite.

Let M be a f.d. integrable U_q -module.
 Let $0 \neq x \in M_\omega$ and $E_i x = 0 \ \forall i$.

$$V(\omega) \xrightarrow{\text{Irreducible}} U_q x = M \quad \Rightarrow \quad M \cong M(\omega)$$

□

Lemma: Let $\lambda \in \Sigma^+$ and $J_\lambda =$ submodule of $V(\lambda)$ generated by $F_i^{m_i+1} \mathbf{1}$ $1 \leq i \leq n-1$ and $F_n^{2m_n+1} \mathbf{1}$.

$$\text{Then } M(\lambda) \cong V(\lambda) / J_\lambda \quad m_i = \lambda(h_i)$$

Define the antiautomorphism

$S: \mathcal{U}_q \rightarrow \mathcal{U}_q$ by

$$SE_i = -K_i^{-1} E_i$$

$$SF_i = -F_i K_i$$

$$SK_i = K_i^{-1}.$$

Let M be a \mathcal{U}_q -module.

Define a \mathcal{U}_q -module structure on M^* by

$$(uf)(x) = f(S(u)x) \text{ for } f \in M^*, u \in \mathcal{U}_q \\ \text{and } x \in M.$$

We can also define a \mathcal{U}_q -module structure on the dual of M by using S^{-1} . We denote this module by M'

Lemma: $M \cong (M^*)'$

Proof: $M \ni x \mapsto \hat{x}$ where $\hat{x}(f) = f(x)$ for $f \in M^*$ \square

Lemma: Let $\lambda \in \Sigma^+$ and $w_0 \in W$ be the longest element of W .

$$\text{Then } M(\lambda)^* \cong M(-w_0\lambda)$$

Proof:

$M(\lambda)^*$ is irreducible and finite dimensional since $M(\lambda)$ is.

Let u_λ be a highest weight vector of $M(\lambda)$ and $f_\lambda \in M(\lambda)^*$ be define by

$$f_\lambda(u_\lambda) = 1 \quad \text{and} \quad f_\lambda(M(\lambda)_\mu) = 0.$$

$$K_i f_\lambda = \begin{cases} q^{-\lambda(h_i)} f_\lambda \\ -i q^{-\lambda(h_i)} f_\lambda \end{cases} \quad \text{if } i=n \text{ and } \lambda(h_n) \in \mathbb{N} + \frac{1}{2}$$

$$F_i f_\lambda = 0 \quad \forall i.$$

So $(q^{-\lambda(h_1)}, \dots, q^{-\lambda(h_n)}, -i \text{ if } \lambda(h_n) \in \mathbb{N} + \frac{1}{2})$ is the lowest weight of $M(\lambda)^*$.

Hence $-w_0\lambda$ is the highest weight and

$$M(\lambda)^* \cong M(-w_0\lambda)$$

□

Theorem: If M is a finite dimensional integrable U_q -module, then M is completely reducible.

Proof:

Let $L = M(\lambda)$ be an irreducible submodule of M .

We need to show that $0 \rightarrow L \xrightarrow{\iota} M \rightarrow M/L \rightarrow 0$

splits since then $M \cong L \oplus M/L$ and the

result follows by induction on $\dim M$.

Let y_λ be a highest weight vector of L .

Let f_λ be as before. Extend f_λ to all of

M by $f_\lambda(M_\mu) = 0 \forall \mu \neq \lambda$.

$U_q f_\lambda \hookrightarrow M^*$ gives us $(M^*)' \rightarrow (U_q f_\lambda)'$

$$\begin{array}{ccc}
 & \xrightarrow{\phi} & \\
 L \xrightarrow{\iota} M \cong (M^*)' & \longrightarrow & (U_q f_\lambda)' \cong M(\lambda) \\
 y_\lambda \longmapsto y_\lambda = \hat{y}_\lambda & \longmapsto & \hat{y}_\lambda |_{(U_q f_\lambda)} \neq 0
 \end{array}$$

$\phi \circ \iota$ is an isomorphism.

$(\phi \circ \iota)^{-1}$ is a left inverse of ι

□