- Let $\lambda \in \Sigma^+ = \left\{ \mu \in \mathfrak{h}^* \mid \begin{array}{l} \mu(h_i) \in \mathbb{N} \quad \forall\, 1 \le i \le n-1 \\ \text{and } \mu(h_n) \in \frac{1}{2}\mathbb{N} \end{array} \right\}$

  $M(\omega_\lambda)$ is finite dimensional.

- Every finite dimensional integrable irreducible $\mathcal{U}_q(osp(1,2n))$-module (of type 1) is isomorphic to $M(\omega_\lambda)$ for some $\lambda \in \Sigma^+$.

- Every finite dimensional integrable $\mathcal{U}_q(osp(1,2n))$-module is completely reducible.

Lemma: Let M be an integrable $\mathcal{U}_q(osp(1,2))$-mod.

Denote M_λ by $M_{\lambda(h)}$.

If $\dim M_m < \infty \ \forall m$, M is finite dimensional

and $\dim M_m = \dim M_{-m}$

Proof:

Assume $M_m \neq 0$ for some $m \in \frac{1}{2}\mathbb{Z}$.
Let $0 \neq x \in M_m$.
Let r be such that $E^{r+1}x = 0$ and $v_0 = E^r x \neq 0$

$v_0 \in M_{m+r}$
$\mathcal{U}_q v_0$ is irreducible

$r+m \in \mathbb{Z} \Rightarrow \dim(\mathcal{U}_q v_0)_0 = 1$
$r+m \in \mathbb{Z}+\frac{1}{2} \Rightarrow \dim(\mathcal{U}_q v_0)_{1/2} = 1$

If $M = \mathcal{U}_q v_0$, we are done.
If not, $\dim\left((M/\langle v_0\rangle)_0 + (M/\langle v_0\rangle)_{1/2} \right) < \dim M_0 + \dim M_{1/2}$

and we can use induction.

$\square$

Let W be the Weyl group associated to

$\Pi = \{\alpha_1, \ldots, \alpha_n\}$. i.e W is the group generated by

$r_1, \ldots, r_n$ where for $\lambda \in \mathfrak{h}^*$,

$$r_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i \quad , 1 \le i \le n-1$$
$$r_n(\lambda) = \lambda - 2\langle \lambda, h_n \rangle \alpha_n$$

- W is finite
- λ is conjugate under W to exactly one element of $\Sigma^+ = \{ \lambda \in \Sigma \mid \langle \lambda, h_i \rangle \ge 0 \}$
- Let $\lambda \in \Sigma^+$. Then the number of $\mu \in \Sigma^+$ such that $\lambda - \mu \in \sum \mathbb{N}\alpha_i$ is finite.
- Denote by w_0 the longest word in W.

Lemma: Let M be an integrable $\mathcal{U}_q$-module with $\dim M_\omega < \infty$.
Then $\dim M_\lambda = \dim M_{w\lambda}$, where $w = w_\lambda$ and $w \in W$

Proof:
Let $N = \sum_{m \in \mathbb{Z}} M_{\lambda + m\alpha_n}$.

N is f.d. integrable $\mathcal{U}_q(osp(1,2))$-module.

By the previous lemma,

$M_\lambda = N_{\lambda(h_n)} \quad N_{-\lambda(h_n)} = M_{r_n\lambda} \qquad (r_n\lambda = \lambda - 2\lambda(h_n)\alpha_n)$.
By Lemma $\dim M_\lambda = \dim M_{r_n\lambda}$.
By $\mathfrak{sl}_2$-theory, $M_\lambda = M_{r_i\lambda} \quad \forall \quad 1 \le i \le n-1$

$\square$

Theorem: If $\lambda \in \Sigma^+$, $M(\lambda)$ is finite dimensional.

Every finite dimensional integrable irreducible $\mathcal{U}_q$-module is $\cong$ $M(\lambda)$ for some $\lambda \in \Sigma$.

Proof:
If $M(\lambda)_\mu \neq 0$, $\mu \in \lambda - \sum \mathbb{N}\alpha_i$.

$\nexists$ finitely many $\mu \in \Sigma^+$ such that $\mu \in \lambda - \sum \mathbb{N}\alpha_i$.
Every weight of $M(\lambda)$ is conjugate to exactly one dominant weight in $\lambda - \sum \mathbb{N}\alpha_i$.
Since W is finite, $\nexists$ finitely many weights of $M(\lambda)$. So $M(\lambda)$ is finite.

Let M be a f.d. integrable $\mathcal{U}_q$-module.
Let $0 \neq x \in M_\omega$ and $E_i x = 0$ $\forall i$.

$$V(\omega) \longrightarrow \underbrace{\mathcal{U}_q x = M}_{\text{irreducible}} \qquad \Rightarrow \quad M \cong M(\omega)$$

$\square$

Lemma: Let $\lambda \in \Sigma^+$ and J_λ = submodule of $V(\lambda)$ generated by $F_i^{m_i+1}\mathbb{1}$ $1 \leq i \leq n-1$ and $F_n^{2m_n+1}\mathbb{1}$.

Then $M(\lambda) \cong V(\lambda)/J_\lambda$ $\qquad m_i = \lambda(h_i)$

Define the antiautomorphism

$$S: \mathcal{U}_q \longrightarrow \mathcal{U}_q \quad \text{by}$$

$$\begin{aligned} SE_i &= -K_i^{-1} E_i \\ SF_i &= -F_i K_i \\ SK_i &= K_i^{-1}. \end{aligned}$$

Let M be a $\mathcal{U}_q$-module.

Define a $\mathcal{U}_q$-module structure on M^* by

$$(uf)(x) = f(S(u)x) \quad \text{for } f \in M^*, u \in \mathcal{U}_q \text{ and } x \in M.$$

We can also define a $\mathcal{U}_q$-module structure on the dual of M by using S^{-1}. We denote this module by M'

Lemma: $M \cong (M^*)'$

Proof: $M \ni x \longmapsto \hat{x}$ where $\hat{x}(f) = f(x)$ for $f \in M^*$ $\square$

Lemma: Let $\lambda \in \mathbb{Z}^+$ and $w_0 \in W$ be the longest element of W.

Then $M(\lambda)^* \cong M(-w_0(\lambda))$

Proof:

$M(\lambda)^*$ is irreducible and finite dimensional since $M(\lambda)$ is.

Let v_λ be a highest weight vector of $M(\lambda)$ and $f_\lambda \in M(\lambda)^*$ be define by

$$f_\lambda(v_\lambda) = 1 \quad \text{and} \quad f_\lambda(M(\lambda)_\mu) = 0 .$$

$$K_i f_\lambda = \begin{cases} q^{-\lambda(h_i)} f_\lambda \\ -i\, q^{-\lambda(h_i)} f_\lambda & \text{if } i=n \text{ and } \lambda(h_n) \in \mathbb{N} + \frac{1}{2} \end{cases}$$

$F_i f_\lambda = 0 \quad \forall i.$

So $(q^{-\lambda(h_1)}, \ldots, \underset{-i \text{ if } \lambda(h_n) \in \mathbb{N}+\frac{1}{2}}{\underbrace{\,}} q^{-\lambda(h_n)})$ is the lowest weight of $M(\lambda)^*$

Hence $-w_0\lambda$ is the highest weight and

$M(\lambda)^* \cong M(-w_0\lambda)$

$\square$

Theorem: If M is a finite dimensional integrable $\mathcal{U}_q$-module, then M is completely reducible.

Proof:

Let $L = M(\lambda)$ be an irreducible submodule of M.

We need to show that
$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\iota"'] & M \arrow[l, dashed, bend right, "k"'] \arrow[r] & M/L \arrow[r] & 0 \end{tikzcd}$$
splits since then $M \cong L \oplus M/L$ and the result follows by induction on $\dim M$.

Let u_λ be a highest weight vector of L.

Let f_λ be as before. Extend f_λ to all of M by $f_\lambda(M_\mu) = 0$ $\forall \mu \neq \lambda$.

$\mathcal{U}_q f_\lambda \hookrightarrow M^*$ gives us $(M^*)' \longrightarrow (\mathcal{U}_q f_\lambda)'$

$$\begin{tikzcd} L \arrow[r, hook, "\iota"] & M \cong (M^*)' \arrow[r] \arrow[rr, bend left, "\phi"] & (\mathcal{U}_q f_\lambda)' \cong M(\lambda) \\ u_\lambda \arrow[r, mapsto] & u_\lambda = \hat{u}_\lambda \arrow[r, mapsto] & \hat{u}_\lambda|_{(\mathcal{U}_q f_\lambda)} \neq 0 \end{tikzcd}$$

$\phi \circ \iota$ is an isomorphism.

$(\phi \circ \iota)^{-1}$ is a left inverse of ι

$\square$