Part I:

What is so great about perfect crystals?

Part II:

Why little Miss perfect crystal is not so great after all... she is high maintence.

Part III:

Other perfect crystals.

Theorem: Why P.C. are special.

Let $\mathcal{B}$ be a perfect crystal of level $l>0$. Then for any classical dominant integral weight $\lambda \in \bar{P}_l^+$ there is a (strict) crystal isomorphism

$$\Psi : \mathcal{B}(\lambda) \longrightarrow \mathcal{B}(\varepsilon(b_\lambda)) \otimes \mathcal{B}$$

with $\qquad u_\lambda \longmapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda$

u_λ and $u_{\varepsilon(b_\lambda)}$ are the highest weight vectors of $\mathcal{B}(\lambda)$ and $\mathcal{B}(\varepsilon(b_\lambda))$ and b_λ is the element of $\mathcal{B}$ s.t. $\phi(b_\lambda) = \lambda$

Remarks: $\mathcal{B}(\lambda)$ is a $U_q(g)$ crystal with g affine and hence ∞

$\mathcal{B}$ is a $U_q'(g)$ crystal. It is perfect and finite.

$A_1^{(1)}$ Perfect crystal level 1

$$\mathcal{B} = \begin{tikzcd} - \arrow[r, bend left, "0"] & + \arrow[l, bend left, "1"] \end{tikzcd} \qquad \begin{array}{ll} b_{\Lambda_0} = - & b^{\Lambda_0} = + \\ b_{\Lambda_1} = + & b^{\Lambda_1} = - \end{array}$$

Theorem says $\exists$ an isomorphism

$$\mathcal{B}(\Lambda_0) \xrightarrow{\sim} \mathcal{B}(\varepsilon(b_{\Lambda_0})) \otimes \mathcal{B}$$
$$\| $$
$$\mathcal{B}(\Lambda_1) \otimes \mathcal{B}$$

$$u_0 \longmapsto u_1 \otimes b_{\Lambda_0} = u_1 \otimes -$$

AND $\exists$ an isomorphism

$$\mathcal{B}(\Lambda_1) \xrightarrow{\sim} \mathcal{B}(\varepsilon(b_{\Lambda_1})) \otimes \mathcal{B} = \mathcal{B}(\Lambda_0) \otimes \mathcal{B}$$
$$u_1 \longmapsto u_0 \otimes b_{\Lambda_1} = u_0 \otimes +$$

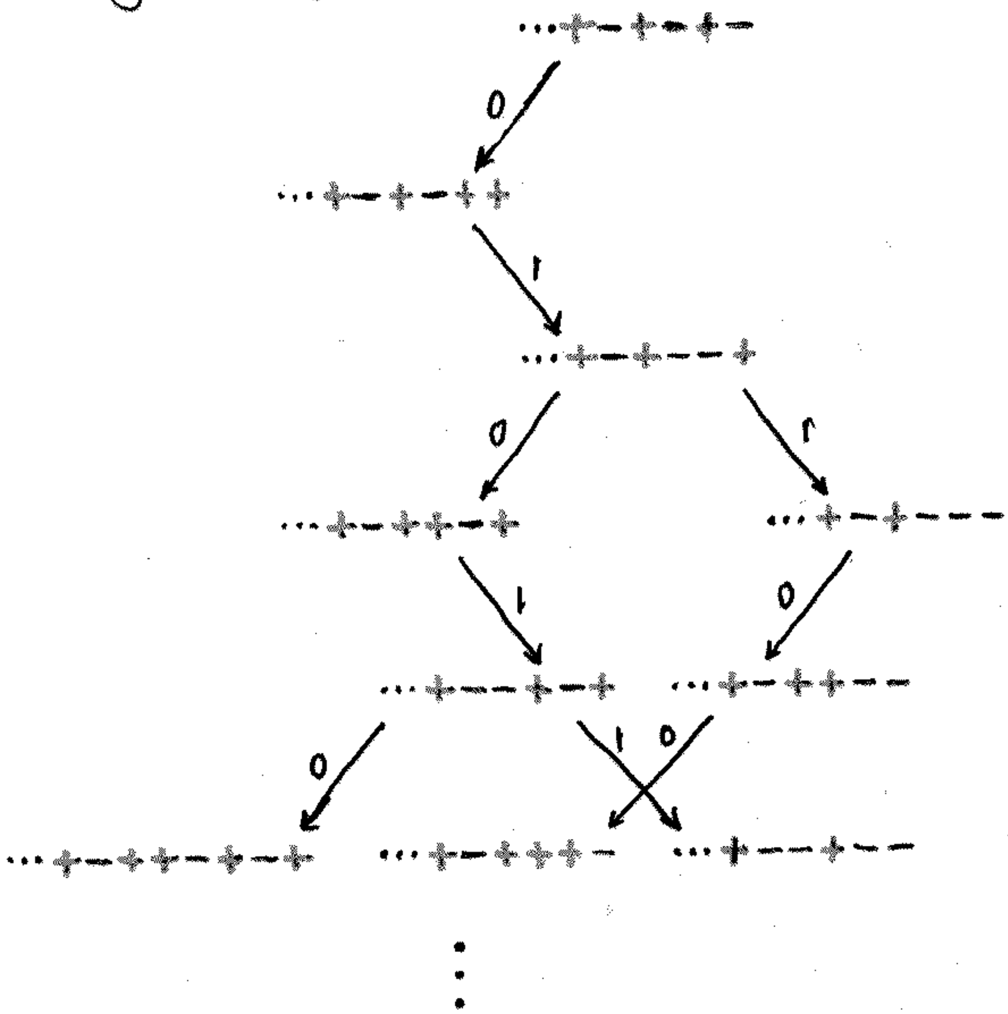
So if we want to obtain structure of $\mathcal{B}(\Lambda_0)$ we iterate these isomorphisms

$$\mathcal{B}(\Lambda_0) \cong \mathcal{B}(\Lambda_1) \otimes \mathcal{B} \cong \mathcal{B}(\Lambda_0) \otimes \mathcal{B} \otimes \mathcal{B} \cong \cdots$$
$$\cong \cdots \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}$$
$$u_0 \mapsto \cdots \otimes + \otimes - \otimes + \otimes -$$

$$\mathcal{B}(\Lambda_0) \cong \cdots \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}$$

$$u_0 \longmapsto \cdots \otimes + \otimes - \otimes + \otimes -$$

Crystal graph of $\mathcal{B}(\Lambda_0)$ of $A_1^{(1)}$



"path realization of $\mathcal{B}(\Lambda_0)$"

Note: a path is in the graph iff $\exists N_0$ s.t. $\forall k > N_0$
$2k-1$ entry is $-$ and $2k$ entry is $+$

affine $A_n^{(1)}$ $(n \geq 1)$

Perfect crystal of level 1:

$$\begin{tikzcd} \boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \cdots \arrow[r, "n-1"] & \boxed{n} \arrow[r, "n"] & \boxed{0} \arrow[llll, bend left, "0"] \end{tikzcd}$$

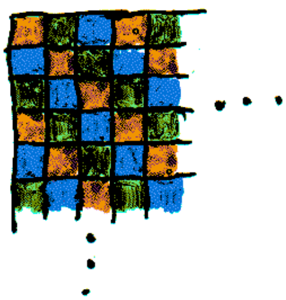
$$b_{\Lambda_i} = \boxed{i} \qquad b^{\Lambda_i} = \boxed{i+1} \qquad i = 0, 1, \ldots, n$$

Ground-state paths:

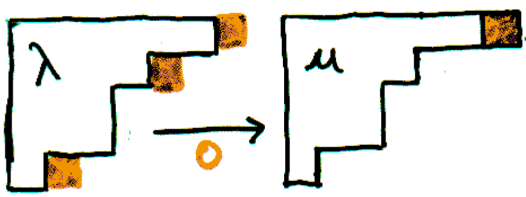
$$p_{\Lambda_i} = (\cdots 1\,2 \cdots n\,0\,1\,2 \cdots i)$$

Crystal graph $B(\Lambda_0)$ for $A_2^{(1)}$

$$\begin{tikzcd}
 & 120120 \arrow[d, "0"] & & \\
 & \cdots 120121 \arrow[dl, "1"'] \arrow[dr, "2"] & & \\
\cdots 120122 \arrow[dr, "2"] & & \cdots 120101 \arrow[d, "1"] & \\
 & \cdots 120102 \arrow[d, "0"] \arrow[drr, "2"] & \cdots 120201 \arrow[dll, "1"'] \arrow[d, "0"] & \\
\cdots 120202 & \cdots 120112 & \cdots 121201 & \cdots 120100
\end{tikzcd}$$



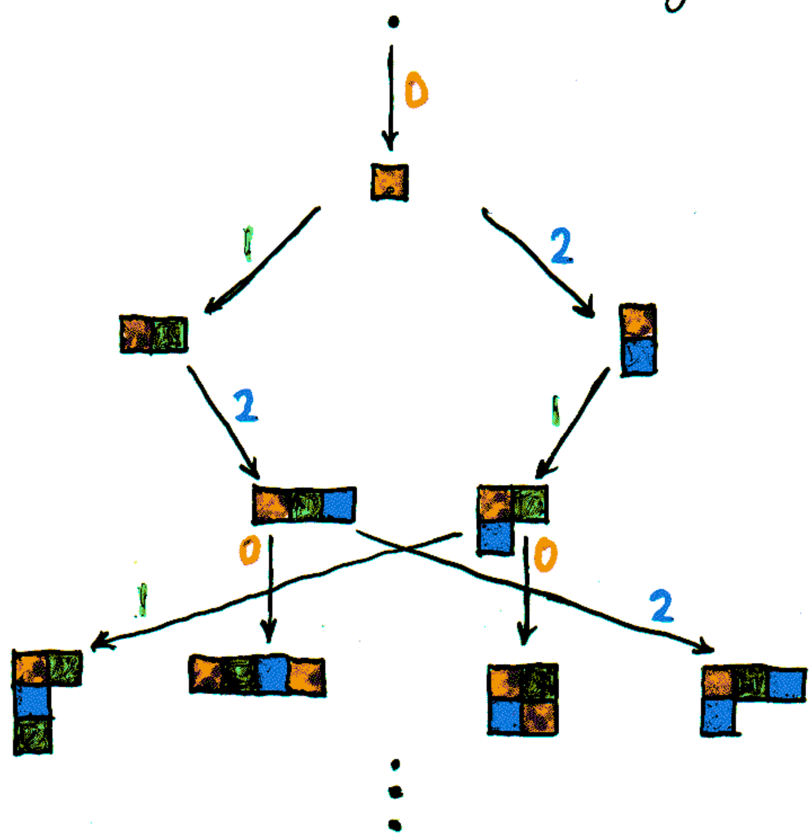
Partitions with 3 colors and no more than two parts same length



there will be an i colored edge if μ differs from λ by the rightmost unmatched i colored cell

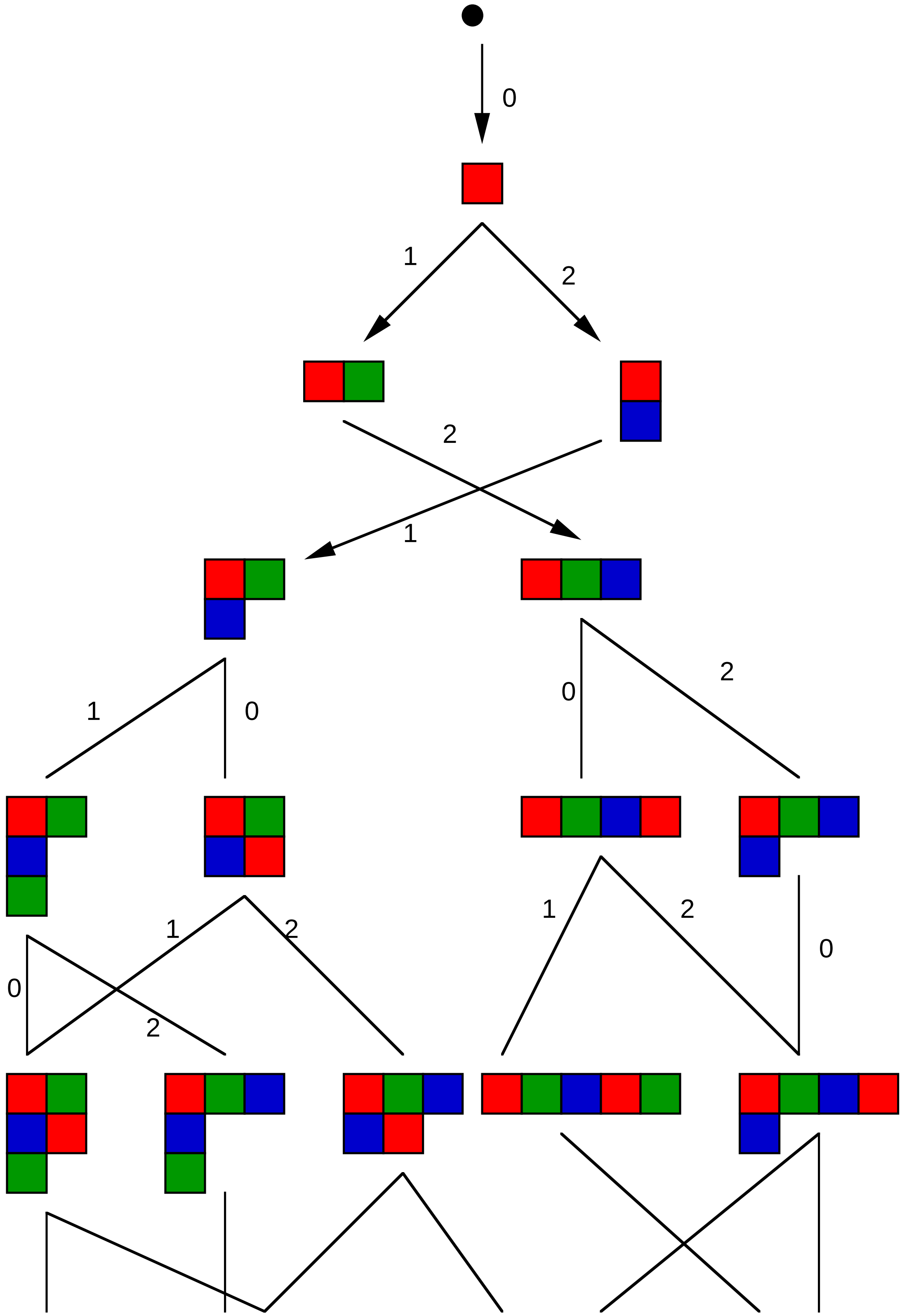
See def of matched/unmatched next slide

notice   has no 2 edge in example.

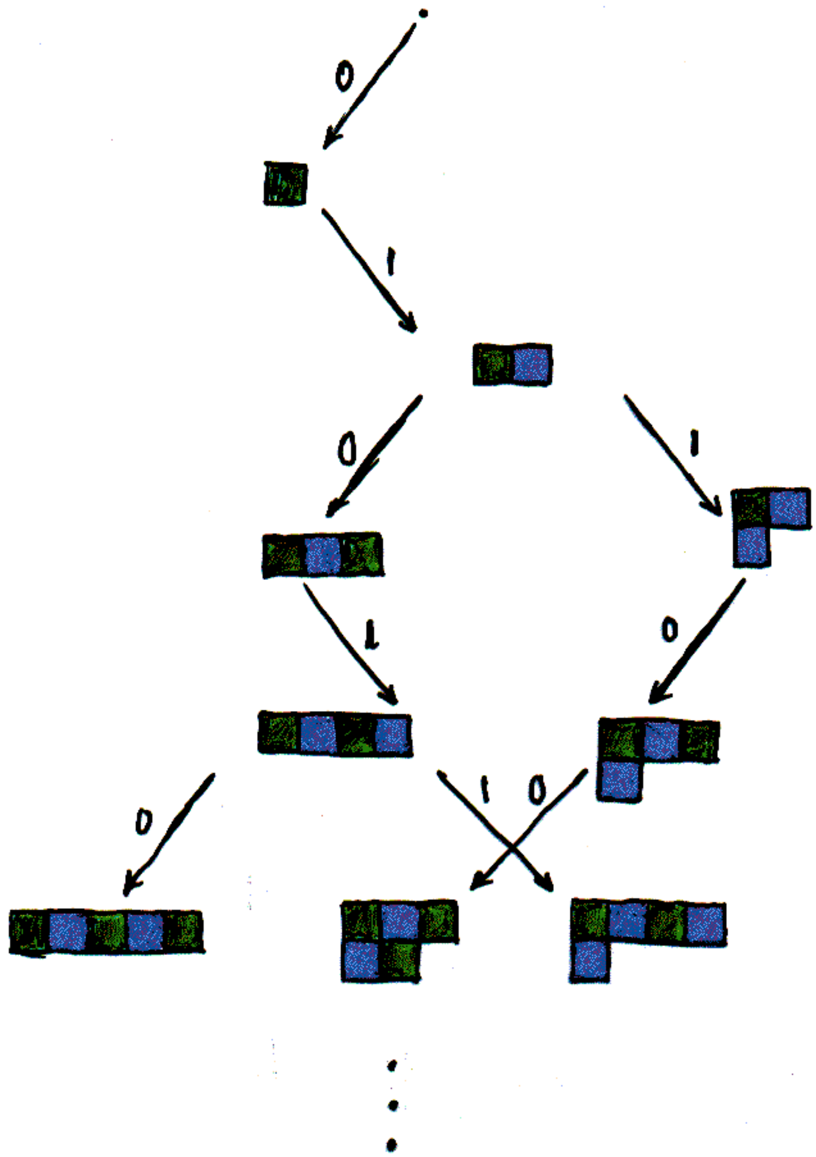


isomorphic to $\mathcal{B}(\Lambda_0)$ for $A_3^{(1)}$

The graph of partitions with no more than two parts of the same length and edges following the colored lattice rule. Isomorphic to $B(\Lambda_0)$ of affine type $A_3^{(1)}$



Graph of partitions with distinct parts edges following 2-color rule



Isomorphic to $\mathcal{B}(\Lambda_0)$ for $U_q(\widehat{sl_2})$

For $\widehat{sl}_n$ consider the graph of partitions with no more than $n-1$ parts having same length.

Put an edge of color i between partitions λ and μ if

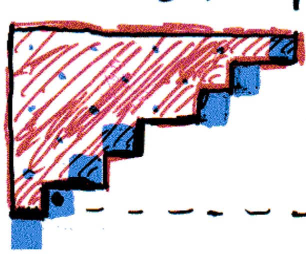
(1) μ differs from λ by a cell with coordinates (x,y) s.t. $x-y \equiv i \mod n$

(2) the cell μ/λ is the rightmost unmatched cell of λ

Fix a color i and a partition λ.
Make a word of (and) by reading the cells (x_j, y_j) with $x_j - y_j \equiv i \mod n$ that are at the inner and outer corners of λ from right to left. Put an (in the word for an outer corner and) for an inner corner.

We say the cell (x_j, y_j) is matched if the parenthesis corresponding to the cell is matched.

Ex: $n=2$
$i=0$



corresponds to word)(())((

rightmost unmatched

Theorem (Misra, Miwa):

This graph is isomorphic to $\mathcal{B}(\Lambda_0)$ for $U_q(\widehat{sl}_n)$.

Generalization for $\mathcal{B}(\Lambda_i)$ (just shift colors). This shows that

$\mathcal{B}(\Lambda_0) \cong \mathcal{B}(\Lambda_1) \cong \dots \cong \mathcal{B}(\Lambda_{n-1})$

and there are no loops in these graphs.

Generalizations exist for $\mathcal{B}(\lambda)$ for $\lambda \in P^+$

Jimbo, Misra, Miwa, Okado

& Alejandra Premat

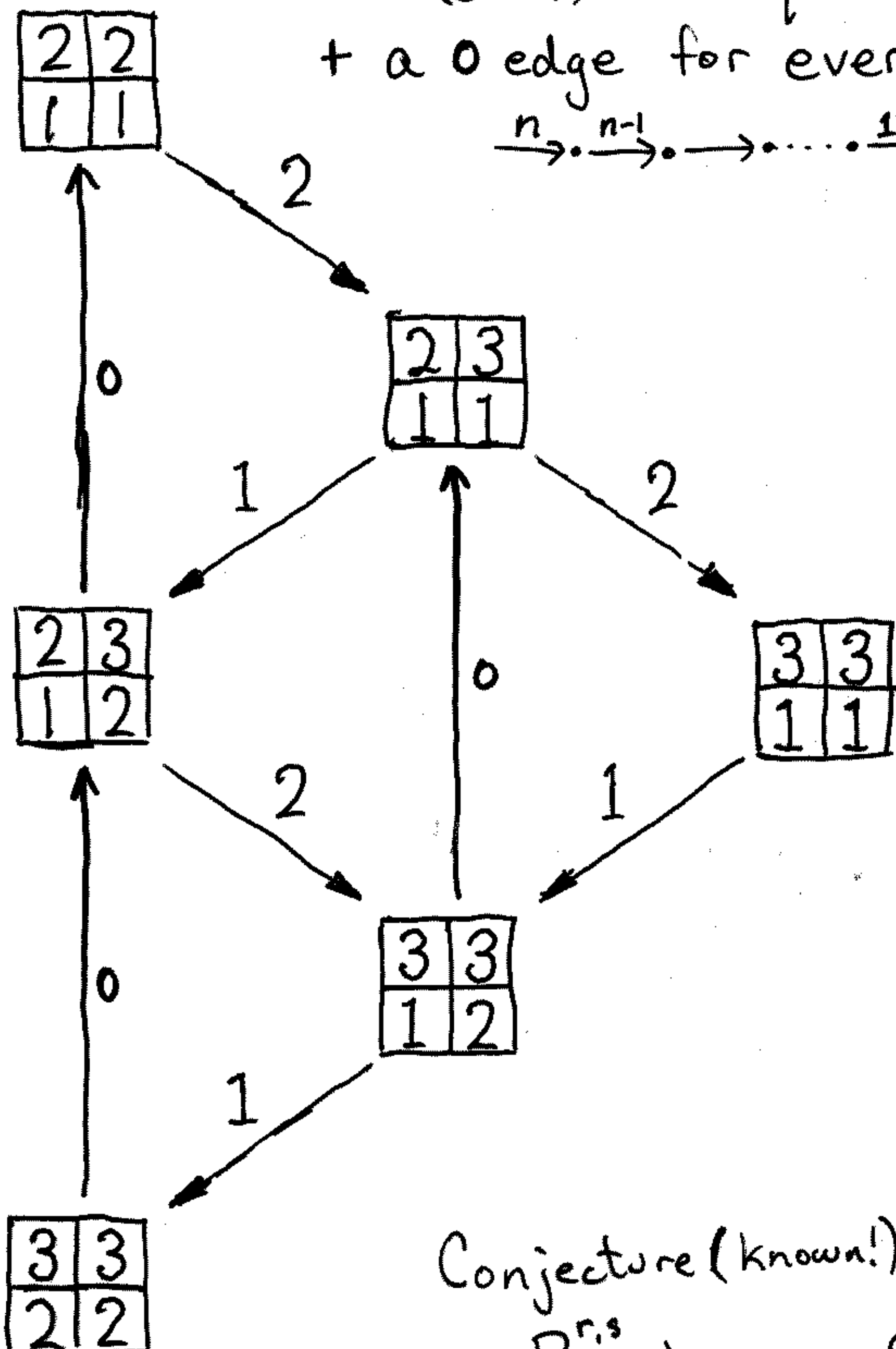
Open question: other types?

For type A_{n-1} take crystal

$B^{r,s} := \mathcal{B}(s\Lambda_r)$ of $U_q(sl_n)$ r s

+ a 0 edge for every path

$\xrightarrow{n}\cdot\xrightarrow{n-1}\cdot\longrightarrow\cdot\cdots\cdot\xrightarrow{1}$

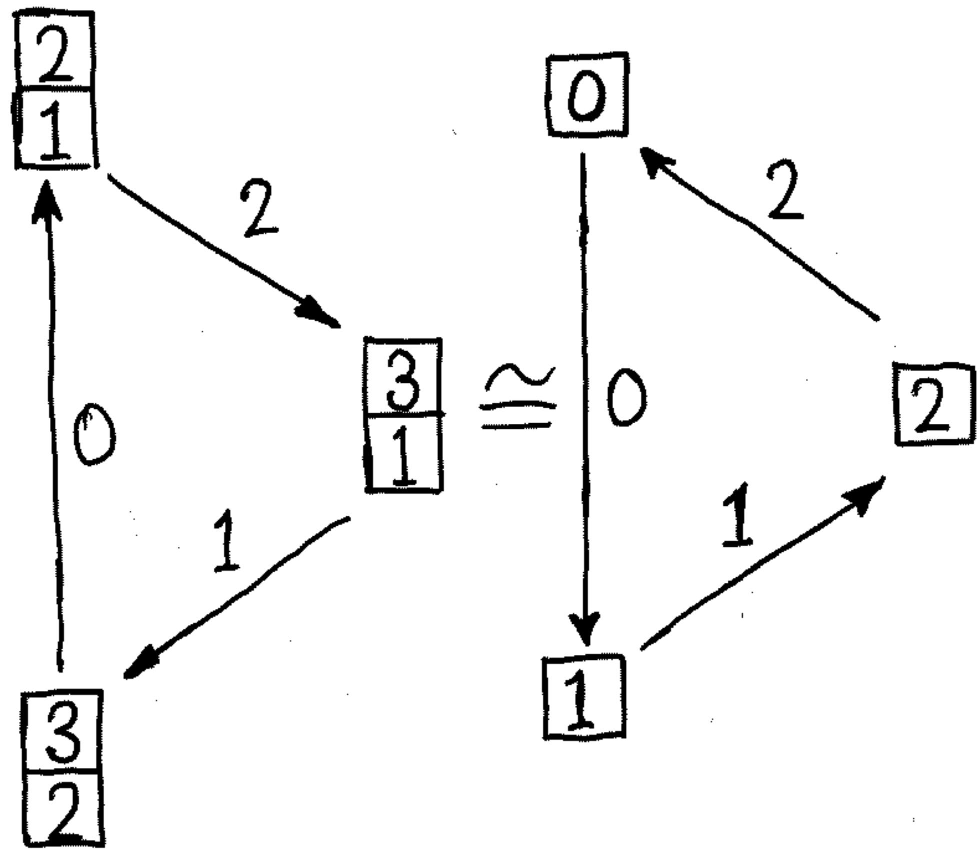


Conjecture (known!):

$B^{r,s}$ is a perfect crystal of level s for $U_q'(\widehat{sl}_n)$

Isomorphism for $B^{2,1}$ and p.c. of level 1.

reverse the arrows!



I suspect that same idea does not work for non-rectangular shapes. (e.g. )