$$\mathrm{osp}(1,2n) = \langle \underbrace{e_1, h_1, f_1, \ldots, e_{n-1}, h_{n-1}, f_{n-1}}_{n-1 \text{ copies of } \mathfrak{sl}_2}, \underbrace{e_n, h_n, f_n}_{\mathrm{osp}(1,2)} \mid (*) \rangle$$

$$(*) \quad [h_i, h_j] = 0 \quad , \quad [e_i, f_j] = \delta_{ij} h_i \qquad \forall\, i, j$$

$$[h_i, e_j] = \mathbf{a_{ij}}\, e_j \quad , \quad [h_i, f_j] = -\mathbf{a_{ij}}\, f_j \qquad \forall\, i\, j$$

$$\underbrace{[e_i,[e_i,\ldots[e_i}_{1-a_{ij}}, e_j]]] = \underbrace{[f_i,[f_i,\ldots[f_i}_{1-a_{ij}}, f_j]] = 0 \quad i \neq j \text{ and if } i = n,\ j \neq n-1$$

$$[e_n[e_n[e_n, e_{n-1}]]] = [f_n[f_n[f_n, f_{n-1}]]] = 0$$

where

$$A = (\mathbf{a_{ij}})_{1 \le i,j \le n} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & & \\ & & & \ddots & 0 \\ \vdots & & & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

Let $\mathfrak{h} = \mathrm{span}\{h_1, \ldots, h_{n-1}, h_n\}$ and define $\alpha_i \in \mathfrak{h}^*$

by $\alpha_i(h_j) = \mathbf{a_{ji}}$

$$\mathcal{U} = \mathcal{U}(\mathrm{osp}(1,2n)) = \langle e_1, h_1, f_1, \ldots, e_{n-1}, h_{n-1}, f_{n-1}, e_n, h_n, f_n \mid (*) \rangle$$

$\mathcal{U} = \mathcal{U}^- \mathcal{U}^\circ \mathcal{U}^+$ where $\mathcal{U}^-$ = subalgebra generated by f's
$\mathcal{U}^\circ$ = " " " h's
$\mathcal{U}^+$ = " " " e's

The quantized universal enveloping algebra of $osp(1,2n)$

$\mathcal{U}_q = \mathcal{U}_q(osp(1,2n))$ is the associative superalgebra

(with 1) over $\mathbb{C}(\sqrt{q})$ generated by

$E_1, K_1, K_1^{-1}, F_1, \ldots, E_{n-1}, K_{n-1}, K_{n-1}^{-1}, F_{n-1}, E_n, K_n, K_n^{-1}, F_n$ subject to

$$K_i K_j = K_j K_i \ , \quad K_i K_i^{-1} = K_i^{-1} K_i = 1$$

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j \ , \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j$$

$$E_i F_j - (-1)^{|E_i||F_j|} F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$$

$$E_i E_j = E_j E_i \quad \text{if } i \neq j-1 \text{ or } j+1$$

$$\left.\begin{aligned} &E_i^2 E_j E_i - [2]_q E_i E_j E_i + E_j E_i^2 = 0 \\ &F_i^2 F_j F_i - [2]_q F_i F_j F_i + F_j F_i^2 = 0 \end{aligned}\right\} \begin{aligned} &\text{if } i = j-1 \text{ or } j+1 \\ &\text{if } i = n, \ j \neq n-1 \end{aligned}$$

$$E_n^3 E_{n-1} - (q-1+q^{-1}) E_n^2 E_{n-1} E_n - (q-1+q^{-1}) E_n E_{n-1} E_n^2 + E_{n-1} E_n^3 = 0$$

$$F_n^3 F_{n-1} - (q-1+q^{-1}) F_n^2 F_{n-1} F_n - (q-1+q^{-1}) F_n F_{n-1} F_n^2 + F_{n-1} F_n^3 = 0$$

where the degree of all generators is 0 except for E_n and F_n whose degree is 1.

$\mathcal{U}_q$ has a Hopf algebra structure

$$\mathcal{U}_q = \mathcal{U}_q^- \mathcal{U}_q^0 \mathcal{U}_q^+$$

Let $M(m)$ denote the $\mathcal{U}_q(osp(1,2))$-module of highest weight $\begin{cases} q^m, & \text{if } m \in \mathbb{N} \\ iq^m & \text{if } m \in \mathbb{N} + \frac{1}{2} \end{cases}$

$M(m)$

$$\begin{array}{l} u_m \\ \big\downarrow \tilde{F} \\ u_{m-1} = F^{(1)} u_m \\ \big\downarrow \tilde{F} \\ \vdots \\ \big\downarrow \tilde{F} \\ u_{-m} = F^{(2m)} u_m \end{array}$$

$$F^{(k)} = F^k / [k]_{iq^{1/2}}!$$

$$E^{(k)} = E^k / [k]_{iq^{1/2}}!$$

Let $A_0 = \left\{ f/g \mid f, g \in \mathbb{C}[q^{1/2}] \text{ and } g(0) \neq 0 \right\}$

Define $L(m) = \bigoplus_{k=m}^{-m} A_0 u_k$ and $B(m) = \left\{ u_k + qL(m) \mid -m \le k \le m \right\}$

$(L(m), B(m))$ is called a crystal base of $M(m)$.

Crystal graph: $\begin{array}{l} \bullet\ \bar{u}_m \\ \downarrow \\ \bullet\ \bar{u}_{m-1} \\ \vdots \\ \downarrow \\ \bullet\ \bar{u}_{-m} \end{array}$

Let M be a finite dimensional integrable $\mathcal{U}_q(osp(1,2))$ - module.

A crystal base of M is a pair (L, B) where L is a free A_0-module and

B is a basis of $L/q^{1/2}L$

such that $\exists$ $M \overset{\phi}{\cong} \bigoplus_{\substack{\text{some} \\ m's}} M(m)$

$$L \overset{\phi}{\cong} \bigoplus_{m's} L(m)$$

$$B \cong \bigsqcup_{m's} B(m)$$

(L, B) is a crystal base of M iff

- L is a finitely generated A_0-module s.t. $M \cong \mathbb{C}(\sqrt{q}) \otimes_{A_0} L$
- $L = \oplus L_\lambda$ where $L_\lambda = M_\lambda \cap L$
- $\breve{F} L \subseteq L$, $\breve{E} L \subseteq L$
- $B = \sqcup B_\lambda$ where $B_\lambda = B \cap L_\lambda / q^{1/2} L_\lambda$
- B is a basis of $L/q^{1/2}L$
- $\breve{F}(B) \subseteq B \cup \{0\}$, $\breve{E}(B) \subseteq B \cup \{0\}$
- For $b, b' \in B$, $b = \breve{F} b' \Leftrightarrow \breve{E} b = b'$

$M(1/2)$

$$\begin{tikzcd} \bullet\, u_{1/2} \arrow[d] \\ \bullet\, u_{-1/2} \end{tikzcd}$$

Comultiplication: $\Delta E = E \otimes K^{-1} + 1 \otimes E$

$$\Delta F = F \otimes 1 + K \otimes F$$

$$\Delta K^{\pm 1} = K^{\pm 1} \otimes K^{\pm 1}$$

$M(1/2) \otimes M(1/2)$

$$\begin{tikzcd} v_1 = u_{1/2} \otimes u_{1/2} \arrow[d, "\tilde{F}"] & \\ v_0 = u_{-1/2} \otimes u_{1/2} + i q^{1/2}\, u_{1/2} \otimes u_{-1/2} \arrow[d, "\tilde{F}"] & w_0 = u_{1/2} \otimes u_{-1/2} - i q^{1/2}\, u_{-1/2} \otimes u_{1/2} \\ v_{-1} = u_{-1/2} \otimes u_{-1/2} & \end{tikzcd}$$

$$\cong \quad M(1) \quad \oplus \quad M(0)$$

$$L = A_0 v_1 \oplus A_0 v_0 \oplus A_0 v_{-1} \oplus A_0 w_0 \cong L(1) \oplus L(0)$$

$$B = \left\{ \overline{v_1}, \overline{v_0}, \overline{v_{-1}}, \overline{w_0} \right\} \cong B(1) \sqcup B(0)$$

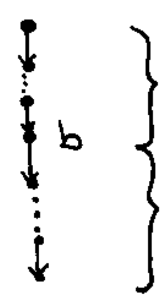
$$(\, \bar{x} := x + qL \,)$$

Theorem: Let M_1 and M_2 be finite dimensional $\mathcal{U}_q(osp(1,2))$-modules and (L_i, B_i) be a crystal base of M_i. Then $(L_1 \otimes L_2, B_1 \otimes B_2)$ is a crystal base of $M_1 \otimes M_2$ and

$$\tilde{F}(b \otimes b') = \begin{cases} \tilde{F}b \otimes b' & , \text{ if } \ \phi(b) > \varepsilon(b') \\ b \otimes \tilde{F}b' & , \text{ if } \ \phi(b) \leq \varepsilon(b') \end{cases}$$

$$\tilde{E}(b \otimes b') = \begin{cases} \tilde{E}b \otimes b' & , \text{ if } \ \phi(b) \geq \varepsilon(b') \\ b \otimes \tilde{E}b' & , \text{ if } \ \phi(b) > \varepsilon(b') \end{cases}$$

where



example $M(2) \otimes M(3/2)$