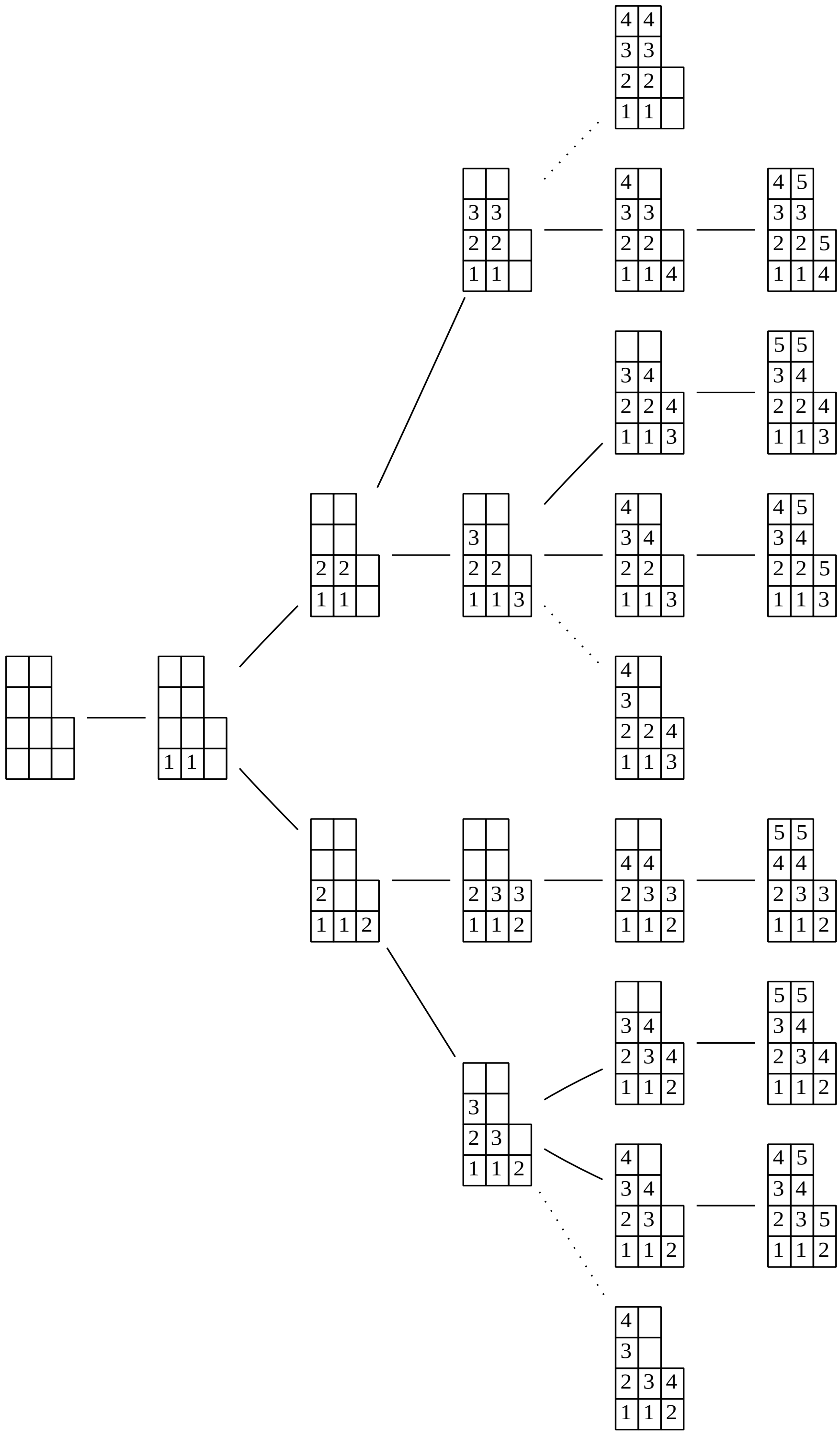

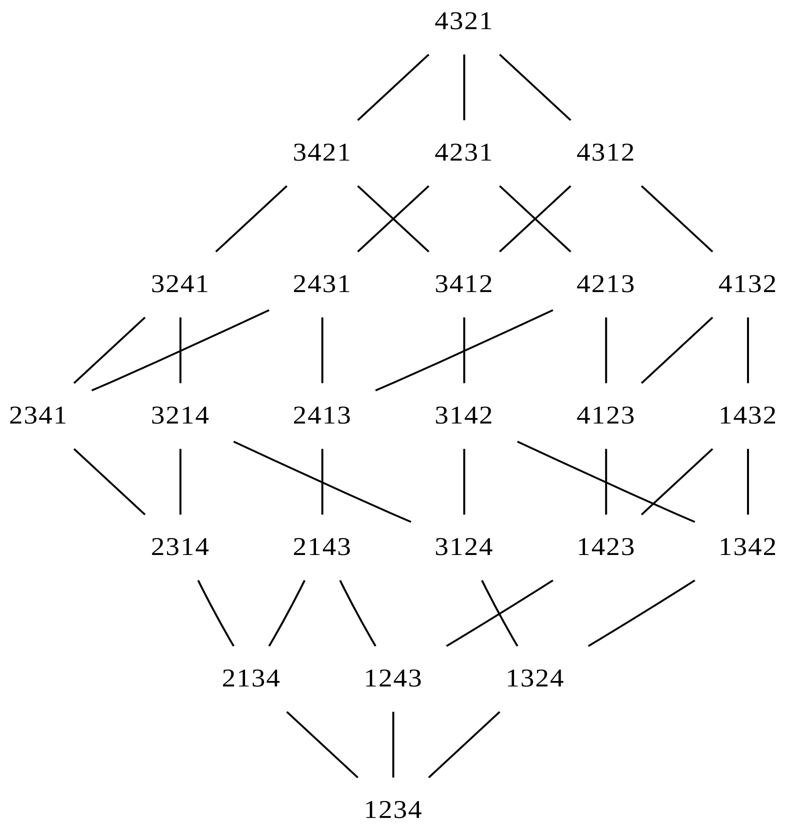
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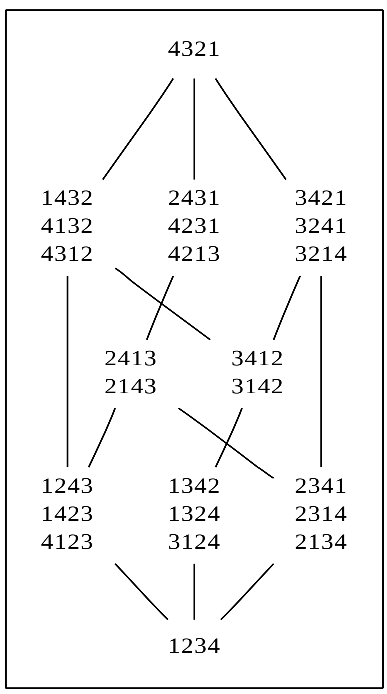
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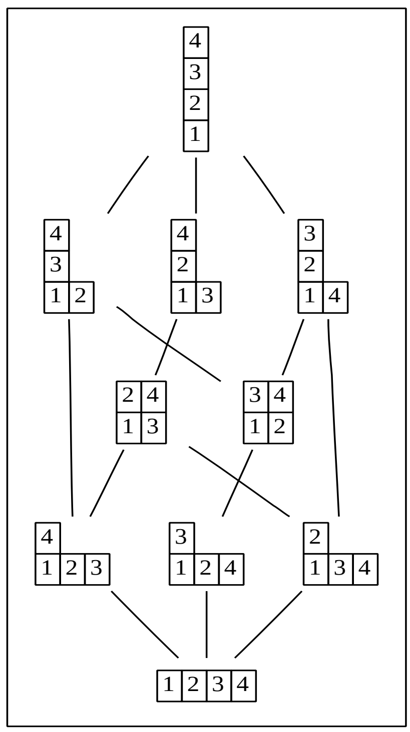
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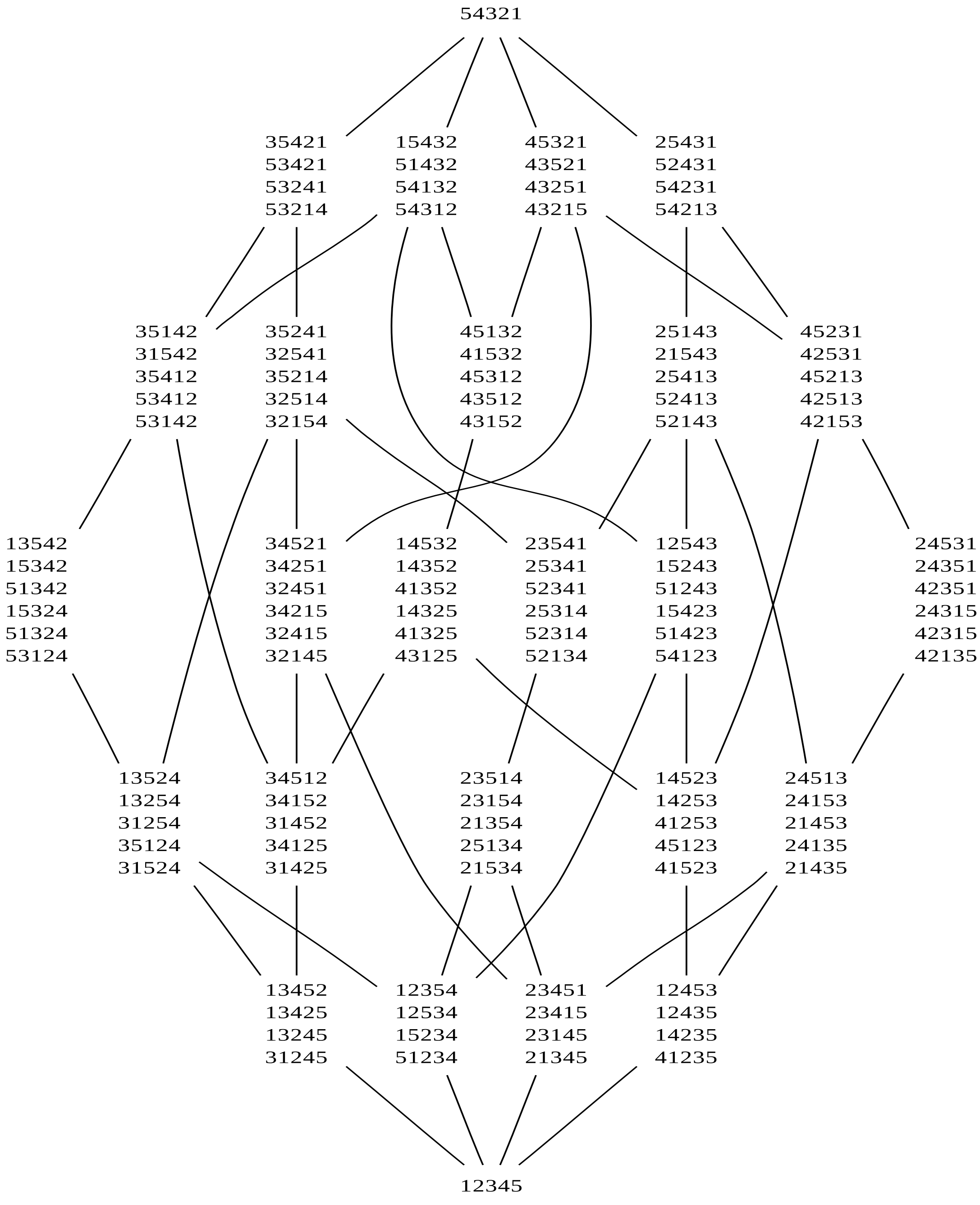
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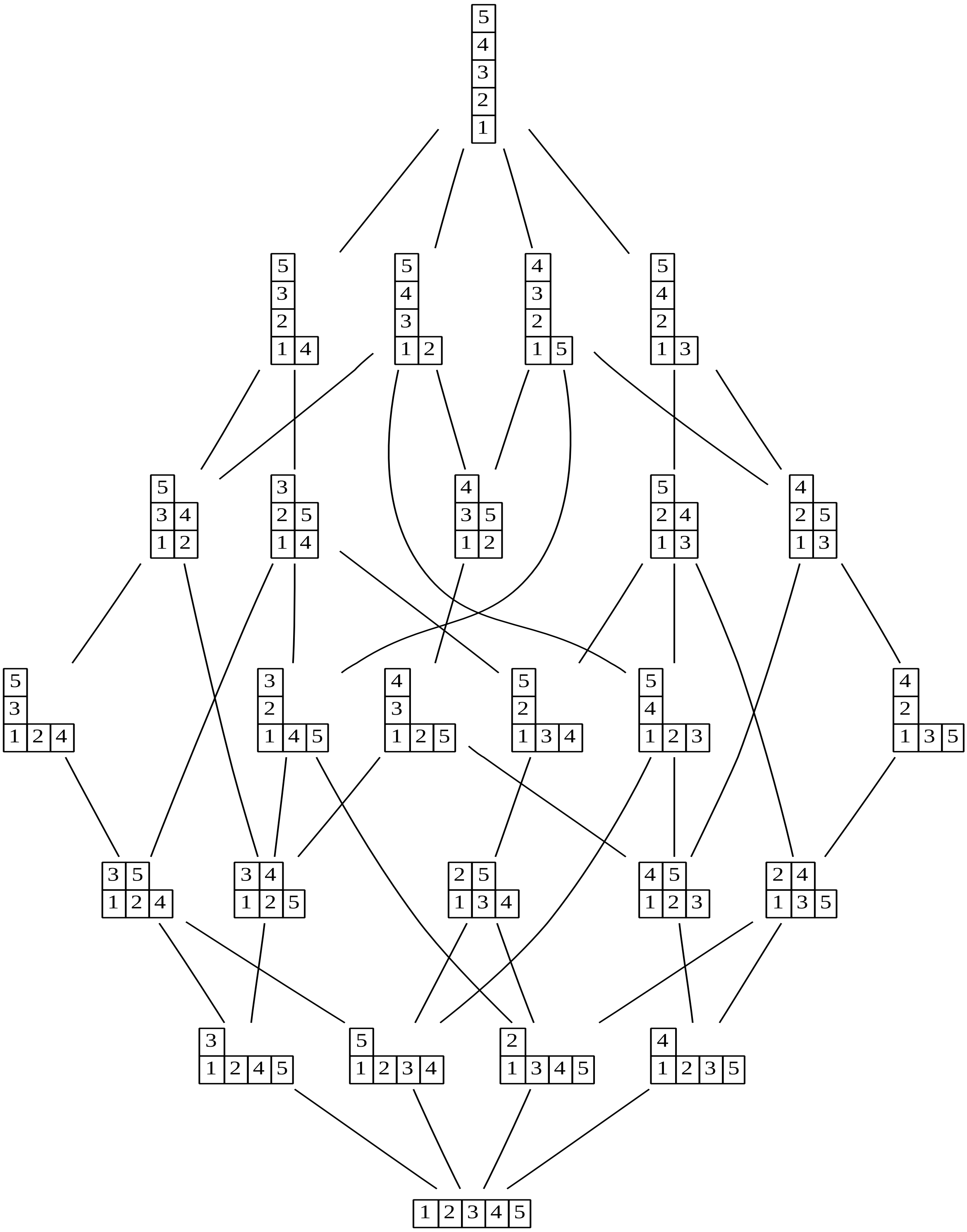
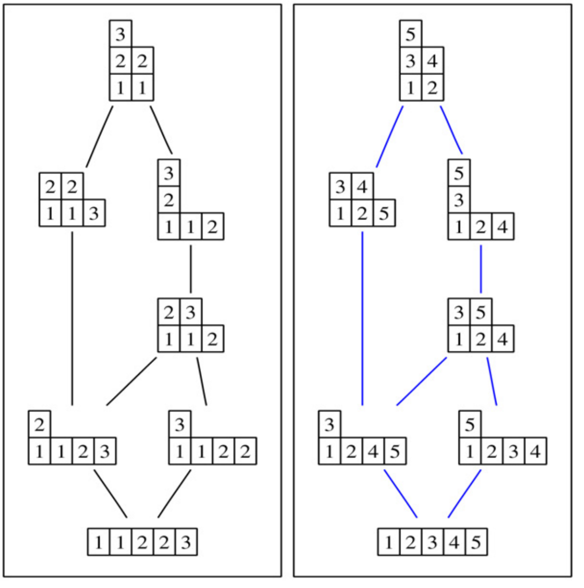




	5	41	32	311	221	2111	11111
5	1	1	1	1	1	1	1
41	.	1	1	2	2	3	4
32	.	.	1	1	2	3	5
311	.	.	.	1	1	3	6
221	.	.	.	.	1	2	5
2111	.	.	.	.	.	1	4
11111	.	.	.	.	.	.	1

The Kostka numbers $K_{\lambda,221}$ counts the number of standard Young tableaux of shape λ in the interval $[\hat{0}, \text{rowTab}(221)]$.

Fig. 9. Dominance order on semi-standard Young tableaux of content $\mu = 221$, order on standardized tableaux.

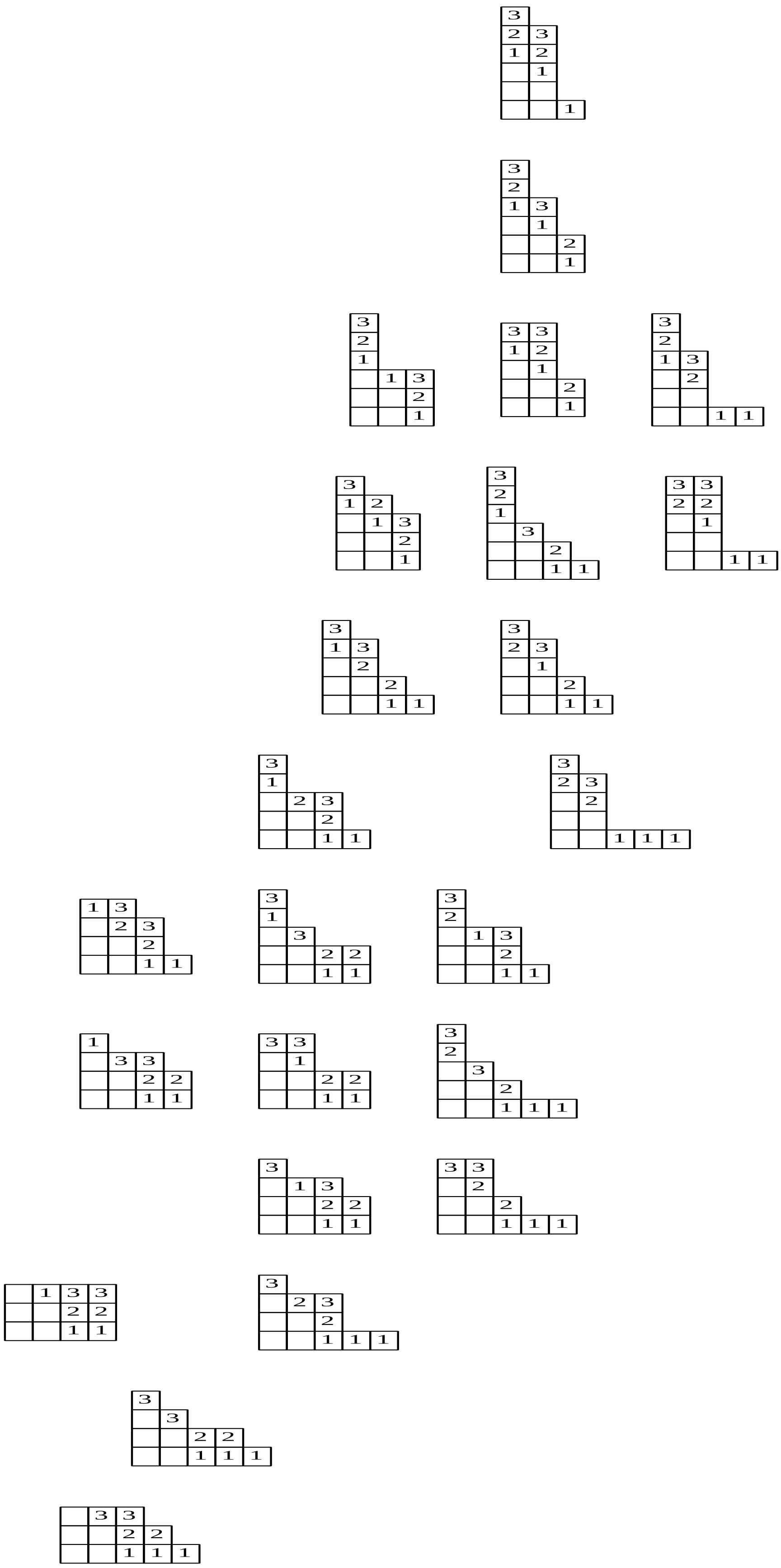
Theorem 7.4. *Let λ, μ be two partitions of the integer n, and let $\hat{0}$ be the standard Young tableau of shape (n). The Kostka number $K_{\lambda,\mu}$ is the number of standard Young tableaux of shape λ in the interval $[\hat{0}, \text{rowTab}(\mu)]$, for any one of the posets studied by Melnikov* [1] *and Taskin* [9].

Proof. Let $Tab(\mu)$ denotes the set of all semi-standard Young tableaux of content μ, that is each tableau in $Tab(\mu)$ has μ_i entries i for $i = 1, \ldots, \ell(\mu)$. The natural standardization process is a canonical bijection mapping $(Tab(\mu), \preccurlyeq_{dom})$ onto $([\hat{0}, \text{rowTab}(\mu)], \preccurlyeq_{dom})$ and this map is order preserving. So Theorem 7.4 holds for the dominance order on tableaux. From [9, Theorem 1.1] and the remark that $[\hat{0}, \text{rowTab}(\mu)] = \text{rowTab}(\mu_1) \times \text{rowTab}(\mu_2) \times \cdots \times \text{rowTab}(\mu_{\ell(\mu)})$, it follows that the set of tableaux in $[\hat{0}, \text{rowTab}(\mu)]$ does not depend on the choice of the partial order. □

Acknowledgments

The author is grateful to F. Hivert for his helpful comments and suggestions.
Experimentations and computations for this work were made with MuPAD-Combinat [7].
Figures were generated using MuPAD-Combinat [7], together with dot [5].

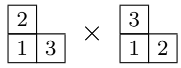
Please cite this article in press as: J. Nzeutchap, Young–Fibonacci insertion, tableauhedron and Kostka numbers, J. Combin. Theory Ser. A (2008), doi:10.1016/j.jcta.2008.05.010



3. The Malvenuto-Reutenauer Hopf algebra of tableaux

Denoted **FSym**, the Malvenuto-Reutenauer algebra [11] is a Hopf algebra structure on $\mathbb{Z}_T$, the $\mathbb{Z}$-module with basis the set of Young tableaux. It has also been studied by Lascoux et al. [2, 9]. There are several homomorphisms between **FSym** and the algebras of descents, of symmetric and quasi-symmetric functions.

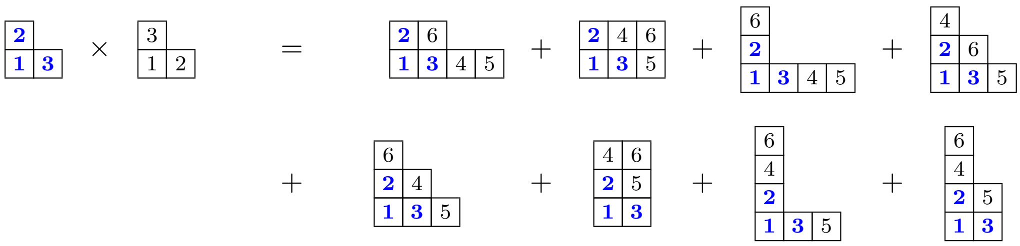
We assume the reader is familiar with the shuffle product on words. We do not give a formal definition of the product in **FSym**, but rather an example to illustrate the multiplication rule. We refer the reader to [2, 11] for a more formal description.

Example 3.1. *Suppose we want to perform the following product:* $t_1 \times t_2 =$ 

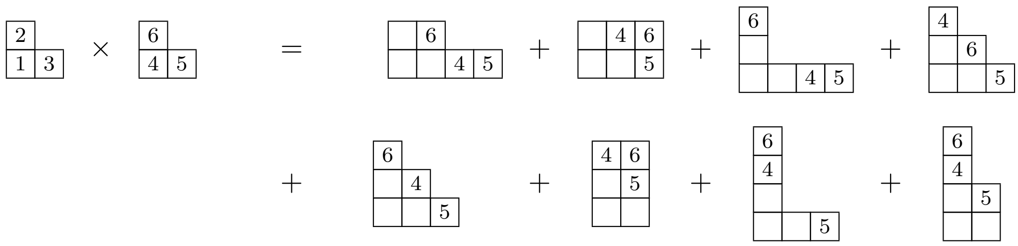
(1) *find a permutation* σ *in the plactic class of* t_1*: example* $\sigma = 213$*;*
(2) *list all the permutations in the plactic class of* t_2*:* $\mathrm{class}(t_2) = \{312, 132\}$*;*
(3) *for each* $\tau \in \mathrm{class}(t_2)$ *increase each digit of* τ *by the size of* t_1*:* $\mathrm{class}(\overrightarrow{t_2}) = \{645, 465\}$*;*
(4) *shuffle* σ *with this new set of permutations*

$$\sigma \sqcup\!\sqcup \ \mathrm{class}(\overrightarrow{t_2}) \ = \ \{213645, 216345, \ldots, 645213, 213465, 214365, \ldots, 465213\}$$

(5) $t_1 \times t_2$ *is the sum of tableaux obtained by applying the Schensted insertion to permutations appearing in the shuffle product above, writing each tableau only once.*



Note that each tableau appearing in the product $t_1 \times t_2$ has t_1 as inner tableau, and the standardized outer tableau rectifies to t_2 by jeu de taquin. We usually don't write the inner part of tableaux appearing in a product.



Let λ be the shape of t_1 and μ the shape of t_2, Lascoux and al. [2, Theorem 6.4.5] showed that the product of the corresponding Schur functions is deduced from the product of tableaux by summation of the shapes of the tableaux appearing in the product $t_1 \times t_2$. So for the example above, we recover the following product.

$$s_{21}s_{21} \ = \ s_{42} \ + \ s_{411} \ + \ s_{33} \ + \ 2\,s_{321} \ + \ s_{222} \ + \ s_{3111} \ + \ s_{2211}$$

So a good understanding of the product of tableaux may lead to an efficient algorithm to compute products of Schur functions. One of our motivations in studying the properties of the product of tableaux is a description of the product which will make no reference to Knuth classes, neither to shuffle product.

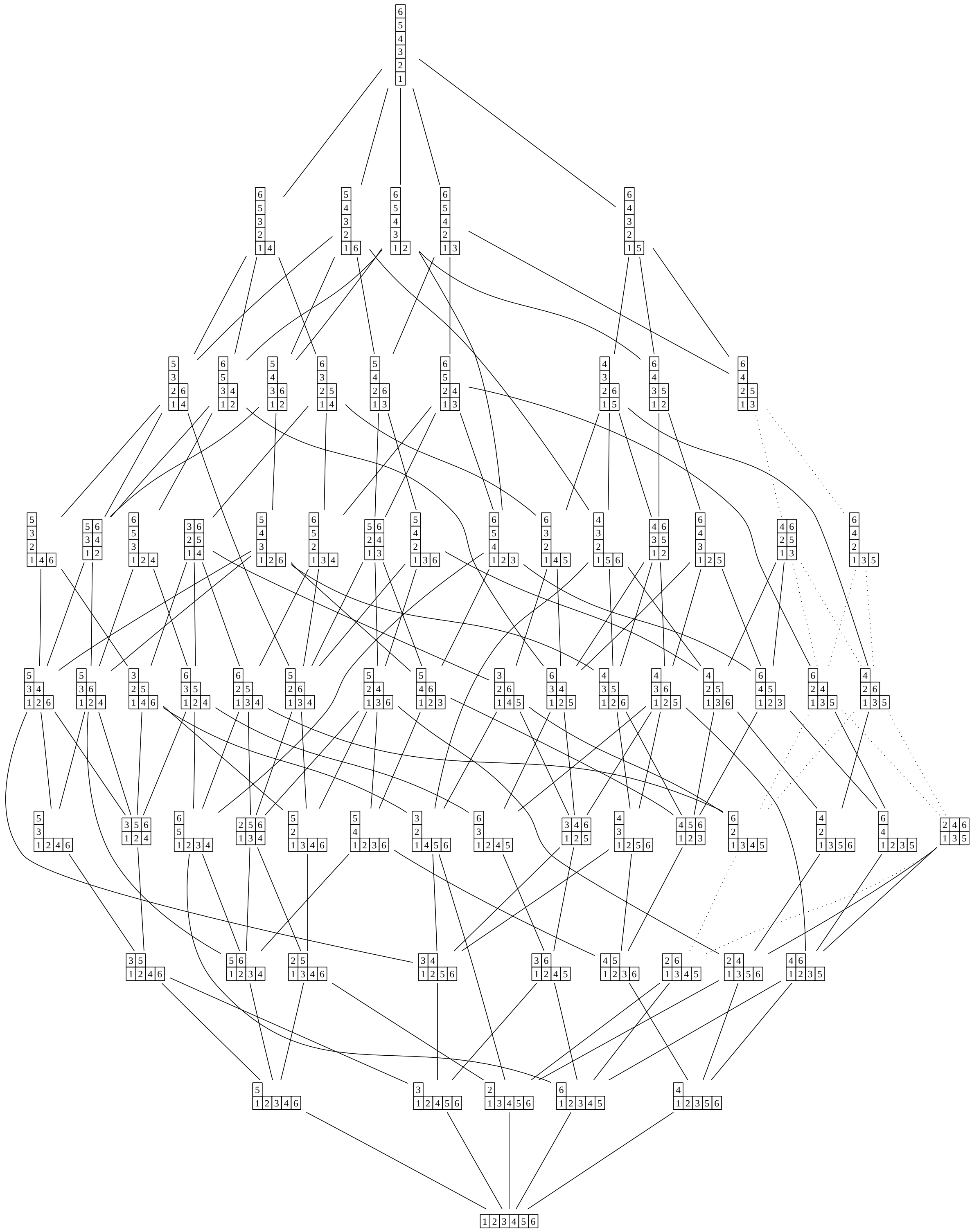


FIGURE 2. Hasse diagram of the Young tableauhedron of order $n = 6$.

Denoted $(\mathbb{YT}_n, \preceq_{weak})$, the Young tableauhedron of order n is the set $\mathbb{YT}_n$ of standard Young tableaux of size n, provided with the weak order on tableaux.

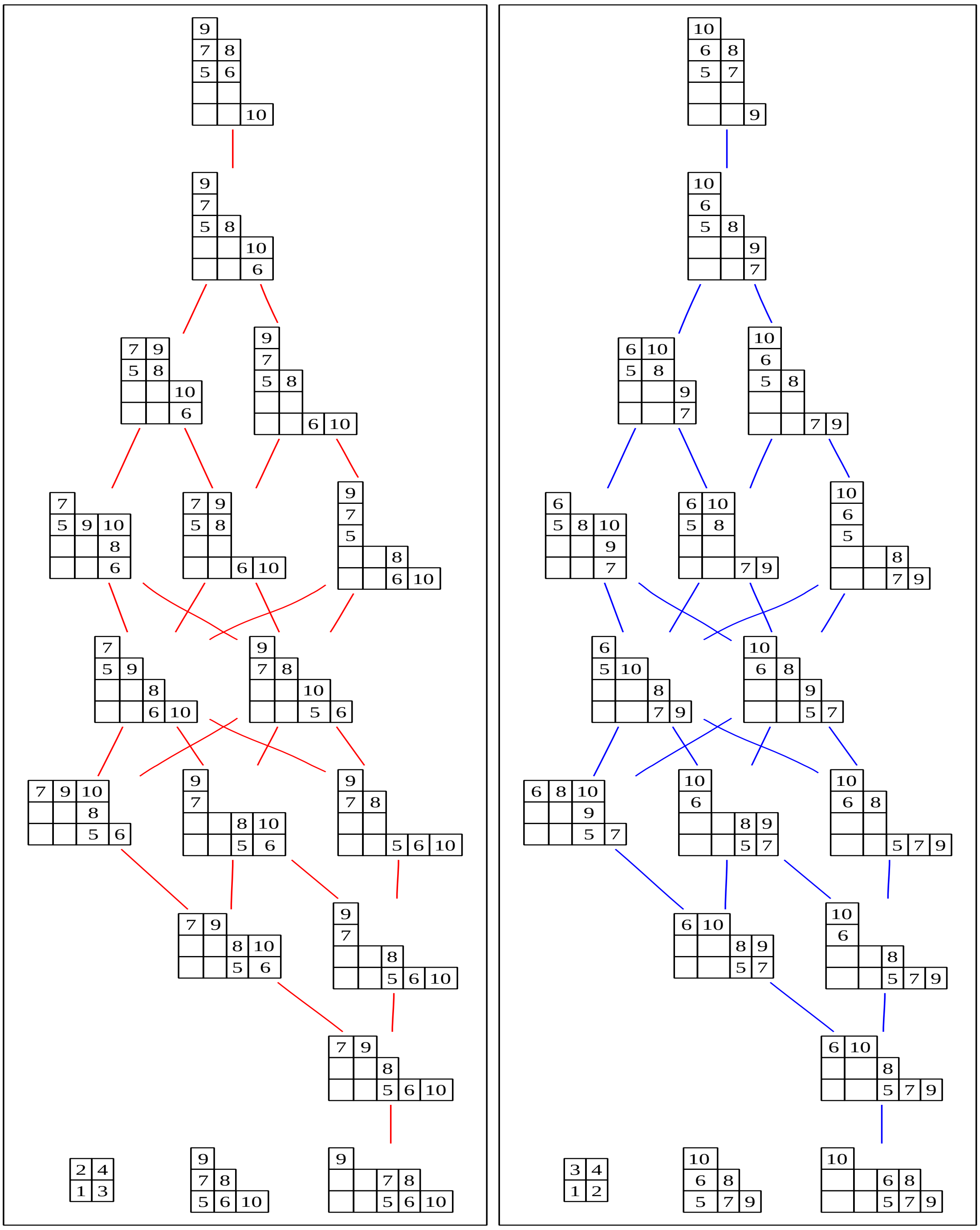


FIGURE 3. Products $t_1 \times t_2$ and $t_3 \times t_4$ where t_1 and t_3 (resp. t_2 and t_4) have the same shape.

We will now introduce a bijection φ which maps any tableau t of the blue poset to a tableau of the read poset. We prove that φ preserves the shapes of the tableaux, and also it is order preserving.

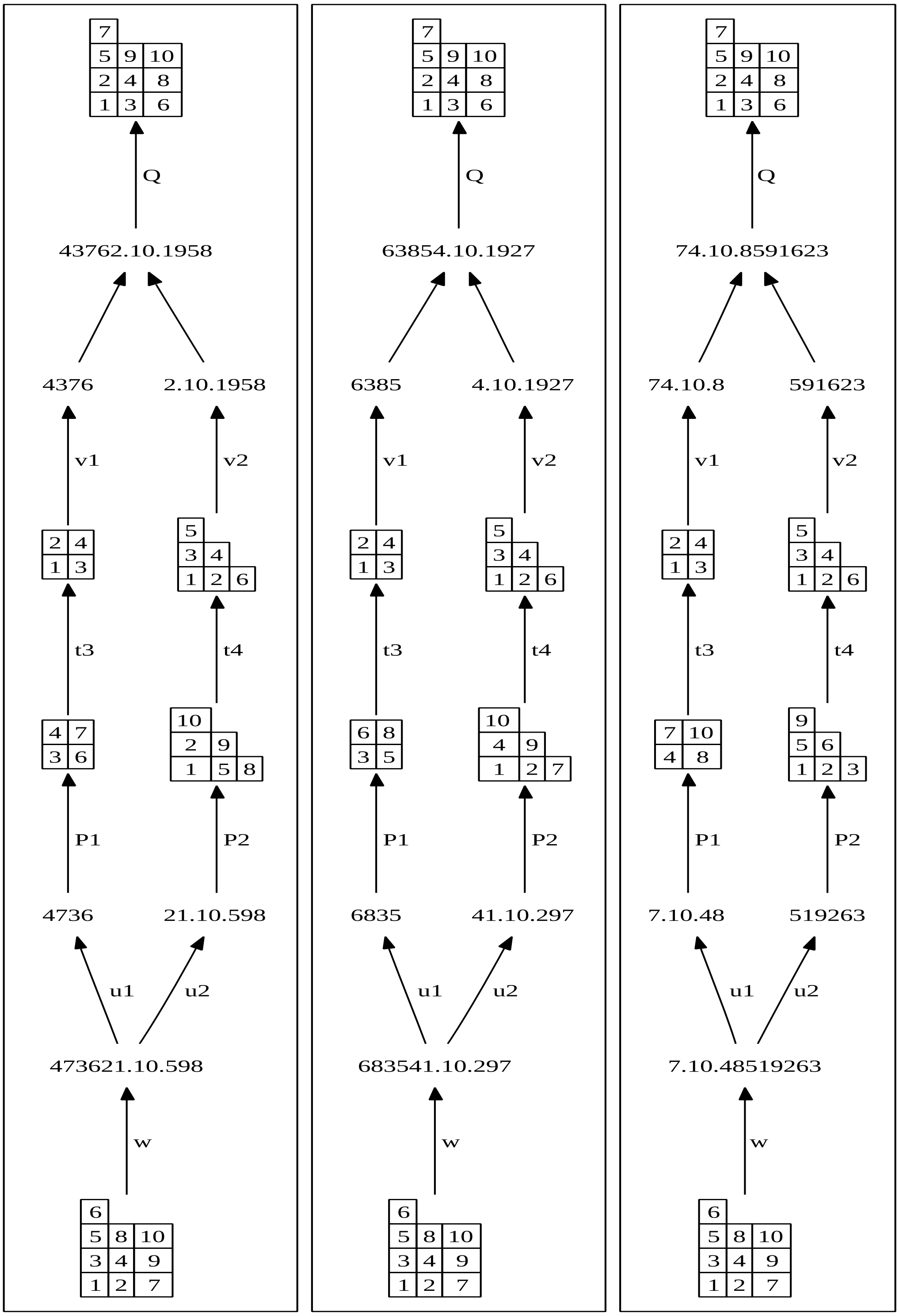


FIGURE 4. Image of a tableau through the bijection φ, using different choices of w.

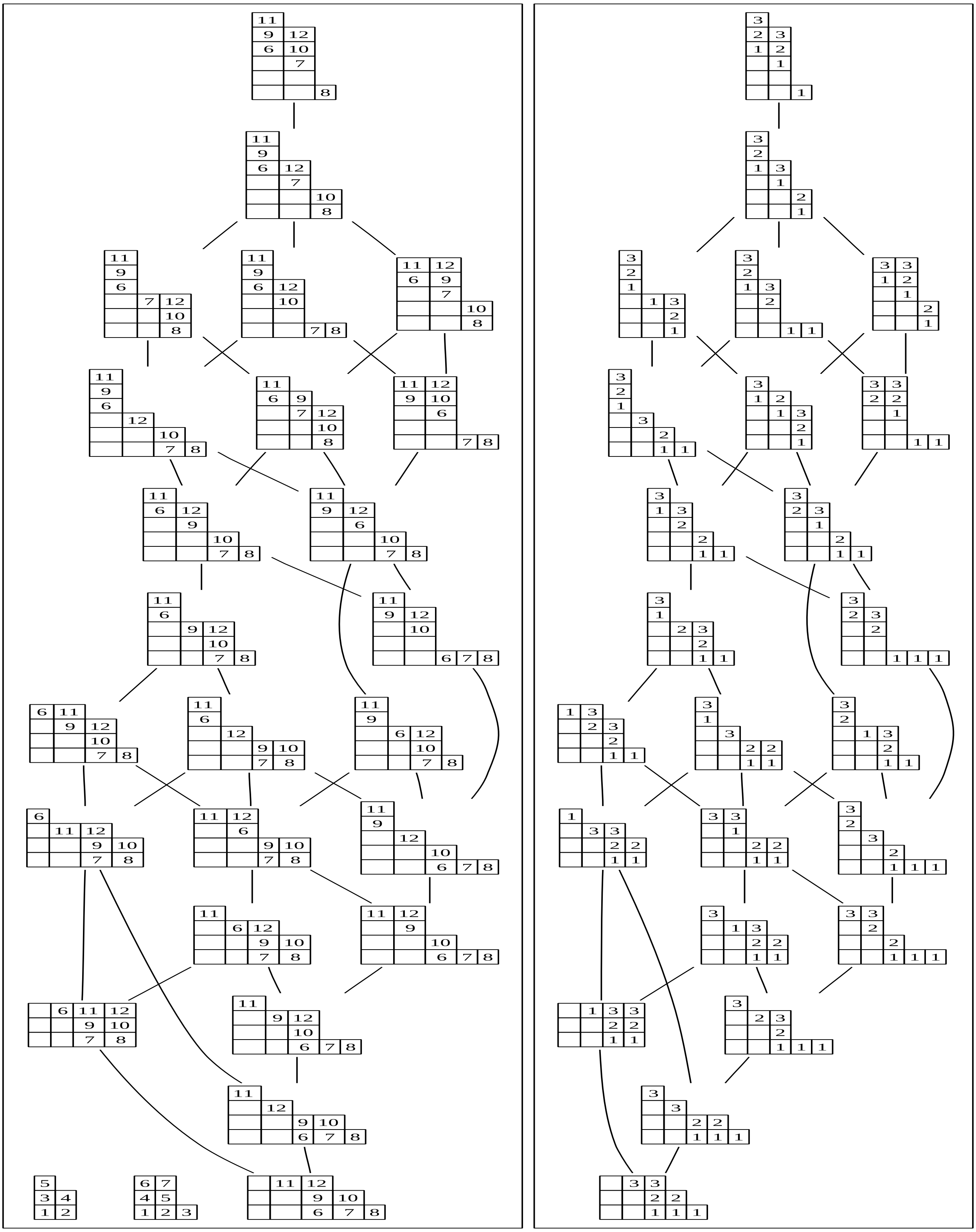


FIGURE 9. Product of canonical tableaux, and Yamanouchi poset for $\lambda = 221$, $\mu = 322$.