Sep. 21, 2007 Fields Institute Nantel Bergeron

Non commutative Symmetric Functions

SYM (symmetric functions)

S_n acts on $k[x_1, x_2, \cdots, x_n]$ (by permuting the variables) $k = \mathbb{Z}$ in mind

$$SYM_n = \{ P \in k[x_1, \cdots, x_n] \mid \sigma . P = P \quad \forall \sigma \in S_n \}$$

$$\sum_{i=1}^{n} e_i (x_1, \cdots, x_n) t^i = \prod_{i=1}^{n} (1 + x_i t)$$

THM $\boxed{SYM_n = k[e_1, \cdots, e_n]}$

Let $n \to \infty$ (in a control way)

$$SYM = k[e_1, e_2, \cdots]$$

where $E(t) = \sum_{i \geq 0} e_i t^i = \prod_{j \geq 1} (1 + x_j t)$

$$H(t) = \frac{1}{E(-t)} = \prod_{j \geq 1} \frac{1}{1 - x_j t} = \sum_{i \geq 0} h_i t^i$$

$$H(t) E(-t) = 1 \qquad (*) \boxed{\sum_{i=0}^{k} (-1)^{k-i} h_i e_{k-i} = 0} \quad k > 0$$

when $k=0$ it becomes $1 = 1$

$\Rightarrow SYM = k[h_1, h_2, \cdots]$ there is triangular relations between these two e_i, h_i

$$W : SYM \longrightarrow SYM \qquad e_i \longmapsto h_i$$

isomorphism of this algebra

so it's an involution

$$(*) \Rightarrow \quad h_i \longmapsto e_i$$

Other Basis

$[\, p_k = \sum_{i \geq 1} x_i^k$ power sum $\Rightarrow SYM = k[p_1, p_2, \cdots] \,]$

won't talk it in noncom. case

here you can not use $\mathbb{Z}$ since the relations with e_i or h_i involves $\mathbb{Q}$.

2.

Schur function S_λ :

$$\mathrm{SYM} \longleftrightarrow \bigoplus_{n \geq 0} \overset{\text{center}}{Z}(\mathbb{C} S_n)$$

class function when the type is λ it's 1 otherwise 0

$$p_\lambda \longleftrightarrow C_\lambda$$

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$$

$$S_\lambda \longleftrightarrow \chi^\lambda \quad \text{irred. rep}$$

$W(S_\lambda)$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$

$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$

$\lambda \vdash n$ (λ a partition of n)

flip along the diagonal $x=y$

$\lambda' = (\lambda_1', \lambda_2', \cdots)$

$\lambda = (7, 4, 3, 1)$

$\lambda' = (4, 3, 2, 2, 1, 1, 1)$

$$\boxed{W(S_\lambda) = S_{\lambda'}}$$

w function is $\otimes$ by sign rep in rep theory

SYM is a Hopf algebra (graded, connected)

- $\deg(h_i) = i$
- $\Delta(h_i) = \sum_{j=0}^{i} h_j \otimes h_{i-j}$
- antipode $S : \mathrm{SYM} \longrightarrow \mathrm{SYM}$, $h_i \longmapsto (-1)^i e_i$

$u : k \to H$
$\varepsilon : H \to k$

$\mu(Id \otimes S)\Delta = u\varepsilon \,(= \mu(S \otimes Id)\Delta)$

$(*) \Rightarrow S$ satisfies the equation for antipode

$$\bigoplus_{n\geq 0} Z(\mathbb{C}S_n)$$

mult. $\longleftrightarrow$ $\mathrm{Ind}_{\mathbb{C}S_n\otimes\mathbb{C}S_m}^{\mathbb{C}S_{n+m}} f\otimes g$

$\Delta \longleftrightarrow f\in Z(\mathbb{C}S_n)$, $\Delta(f) = \sum_{i=0}^{n} \mathrm{Res}^{\mathbb{C}S_n}_{\mathbb{C}S_i\otimes\mathbb{C}S_{n-i}} f$

$S \longmapsto (-1)^{n} f\otimes$ signrep

<u>NSYM</u>

$$NSYM = \underset{\uparrow}{k}\langle h_1, h_2, \dots\rangle$$ free alg. generated by h_i

noncommutative algebra

$$\Delta(h_i) = \sum_{j=0}^{i} h_i\otimes h_{i-j} \qquad \left[\begin{array}{l} S: NSYM \longrightarrow NSYM \\ \quad\;\; h_i \longmapsto (-1)^i e_i \end{array}\right]$$

A alphabet (countable $\overset{\text{totally}}{\text{ordered}}$ set) → want lex. order

$$\overrightarrow{\prod_{a\in A}} \underbrace{\frac{1}{1-ta}}_{1+ta+t^2a^2+\cdots}$$

noncommutative product

(a gives me which order in the product)

$$H(t) = \sum_{k\geq 0} h_k[A]\,t^k = \overrightarrow{\prod_{a\in A}} \frac{1}{1-ta}$$

$$E(t) = \overleftarrow{\prod_{a\in A}} (1+ta) = \sum_{k\geq 0} e_k[A]\,t^k$$

$$h_k[A] = \sum_{\substack{a_1\leq a_2\leq\cdots\leq a_k \\ a_i\in A}} a_1a_2\cdots a_k \qquad \boxed{a_1}\boxed{a_2}\boxed{\;}\boxed{\;}\boxed{\;}\boxed{\;}\boxed{\;}\boxed{\;}\boxed{a_k}$$

possible repeat

4.

$$e_k[A] = \sum_{\substack{a_1 > a_2 > \cdots > a_k \\ a_i \in A}} a_1 a_2 \cdots a_k \qquad \begin{array}{|c|} \hline a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_k \\ \hline \end{array}$$

no possible of repeat

$$H(t)E(-t) = 1 \quad ① \qquad \sum_{i=0}^{k} (-1)^{k-i} h_i e_{k-i} = 0 \quad k>0$$

$$E(t)H(-t) = 1 \quad ②$$

$$\omega : NSYM \longrightarrow NSYM$$

~~Def~~ : $h_i \longmapsto e_i$ from ①

Prop : $e_i \longmapsto h_i$ ②

~~then go back to~~ get the antipode

is there analogue for S_λ ? Yes. but first do someth dirty!

$$\begin{tikzcd} h_i \arrow[d, mapsto] & NSYM \arrow[d] & & NSYM^* = QSYM & \sum_{\alpha \sim \lambda} M_\alpha \\ h_i & SYM \arrow[rr, "\sim"] & & SYM^* \arrow[u, hook] & m_\lambda \arrow[u, mapsto] \end{tikzcd}$$

$$h_\lambda \longleftrightarrow m_\lambda$$

orbit of a monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ under S_∞

eg. $p_2 p_1 = (x_1^2 + x_2^2 + \cdots)(x_1 + x_2 \cdots)$

$m_{21} = \underbrace{x_1^2 x_2 + \cdots}_{M_{21}} + \underbrace{x_2^2 x_1 \cdots}_{M_{12}}$

$$M_\alpha = \sum_{i_1 < i_2 < \cdots i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k) \qquad \alpha_1 + \alpha_2 + \cdots + \alpha_k = n \qquad \alpha \models n$

$\alpha_i > 0, \ \alpha_i \in \mathbb{Z}$ strictly pos. int.

P-partition (Stanley)

$$F_\alpha = \sum_{\beta \leq \alpha} M_\beta \qquad \text{fundamental function}$$

$\beta \le \alpha$ if $\beta_1, \cdots, \beta_{s_1} \models \alpha_1$

$\beta_{s_1+1}, \cdots, \beta_{s_2} \models \alpha_2$

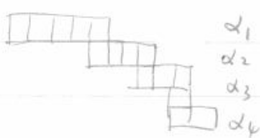
$\vdots$

$\alpha = (2, 3)$,

$\beta = (1, 1, 2, 1)$

$$\begin{array}{rcl} NSYM & \xleftrightarrow{\text{dual}} & QSYM \\ \sum_{\beta \le \alpha} S_\alpha = h_\beta & \longleftrightarrow & M_\beta \\ S_\alpha & \longleftrightarrow & F_\alpha = \sum_{\beta \le \alpha} M_\beta \end{array}$$

$\alpha \models n$



ribon No.

$S_{\text{ribbon}}[A] = \sum_{a_1 \le a_2 > a_3 \le \cdots} a_1 a_2 a_3 \cdots$

gives the order

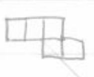
$S_\lambda \leftrightarrow$ filling of Young tab.
here $\leftrightarrow$ also a filling but of ribon

$$\{\alpha \models n\} \longleftrightarrow \{T \subseteq \{1, \cdots, n-1\}\}$$

$$(\alpha_1, \alpha_2, \cdots, \alpha_k) \xmapsto{D} \{\alpha_1, \alpha_1 + \alpha_2, \cdots, \alpha_1 + \cdots + \alpha_{k-1}\} = D(\alpha) \quad \text{descent set}$$

$$\le \longleftrightarrow \supseteq$$

$\omega(S_\alpha) = S_{\alpha^c}$ iso. $\omega(h_\alpha) \overset{\text{def}}{=} e_\alpha$



$\alpha^c \longleftrightarrow (D(\alpha))^c$

$D(\alpha) = \{3\}$ $\quad D(\alpha^c) = \{1, 2, 4\}$

$S_\alpha = \sum \{\} h_\beta$

$S_{\alpha^c} = \sum \{\} e_\beta$

6.

$$\begin{aligned} \tau: NSYM &\longrightarrow NSYM \\ h_\alpha &\longmapsto h_{\tilde{\alpha}} \end{aligned}$$

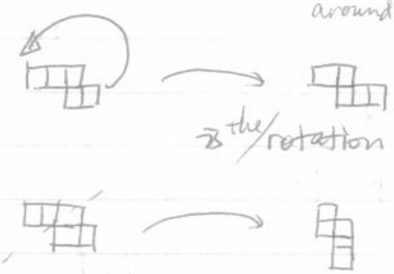
anti iso

$\tilde{\alpha}$ read it other way around

$w' := w \circ \tau = \tau \circ w$

$w'(S_\alpha) = S_{\alpha'}$

anti iso.



SYM

s_λ Representation theory of S_n

NSYM (QSYM)

Rep. th. of $H_n(0)$ $q_i = 0$ Not semisimple

$SYM \longleftrightarrow \bigoplus_{n \geq 0} Z(\mathbb{C}S_n)$

$s_\lambda \longleftrightarrow S^\lambda$

$G \longrightarrow K_0$(finitely generated modules)

first letter

$K \longrightarrow K_0$(finitely generated projective modules)

last letter

isomorphic classes of fin. gen. modules

$[M] \Big/ \langle [M] - [N] - [L] \rangle$ $\quad 0 \to L \to M \to N \to 0$

the quotient (relation)

impose this part to be zero

$L \hookrightarrow \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right] \twoheadrightarrow$ "M/L" N but $M \not\cong N \oplus L$

$G_0(\overset{\text{fin.gen.}}{H_n(0)}) \simeq \bigoplus_{\substack{\text{iso. classes of} \\ I \text{ simple} \\ \text{module index}}} \mathbb{Z}\, C_I$

$G(\oplus H_n(0)) \simeq \bigoplus_{I \models n} \mathbb{Z}\, C_I$

$[M] \in G(H_n(0))$

$[N] \in G(H_m(0))$

$\left[\mathrm{Ind}_{H_n \otimes H_m}^{H_{n+m}} [M] \otimes [N]\right] = \sum \{\ \}[\ ]$

$\downarrow$

QSYM

$G(\oplus H_n(0)) \xleftrightarrow{\text{Hopf morph.}} QSYM$

$[C_I] \longleftrightarrow F_I$

(Hivert & Thibon)

$K(\oplus H_n(0)) \xleftrightarrow{\text{Hopf morph.}} NSYM$

$[P_I] \longleftrightarrow \boxed{S_I}$ is our analogue of schur function

indecomposible ribbon

<u>Recall:</u> $NSYM = \mathbb{Z}\langle h_1, h_2, \cdots \rangle = \mathbb{Z}\langle e_1, e_2, \cdots \rangle$

$$\omega : h_i \longmapsto e_i$$

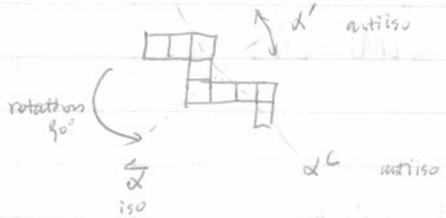
$$\omega(S_\alpha) = S_{\alpha^c}$$

$$\tau : NSYM \longrightarrow NSYM$$

$$h_\alpha \longmapsto h_{\overleftarrow{\alpha}}$$

$$\tau(S_\alpha) = S_{\overleftarrow{\alpha}}$$

$$\tau \circ \omega(S_\alpha) = S_{\alpha'}$$



$NSYM$ (Hopf algebra co-commutative)

$$S_\alpha S_\beta = S_{\alpha|\beta} + S_{\alpha\cdot\beta}$$

$$\alpha|\beta = (\alpha_1, \cdots, \alpha_{k-1}, \alpha_k + \beta_1, \beta_2, \cdots, \beta_\ell)$$

$$\alpha\cdot\beta = (\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_\ell)$$

$$\Delta(S_\alpha) = \sum_{T \subseteq \{1,\cdots,n\}} S_{\alpha(D(\sigma_T))} \otimes S_{\alpha(D(\sigma_{T^c}))}$$

$\alpha \models n$

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$

$\sigma_i > \sigma_{i+1} \iff i \in D(\alpha)$

$$\left(\Delta h_k = \sum_{i=0}^{k} h_i \otimes h_{k-i} \right)$$

antipode: $S(S_\alpha) = (-1)^{|\alpha|} S_{\alpha'}$

SYM $\qquad\qquad$ $NSYM$

SYM $\qquad\qquad\qquad\qquad$ NSYM

Hall-Littlewood sym. fun.

- $H_\lambda[X;t] = s_\lambda[X] + \sum_{\mu \rhd \lambda} K_{\mu,\lambda}(t)\, s_\mu[X]$

  $(\forall i \ \mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i)$

- $H_\lambda[X;1] = h_\lambda$

$B_m : \text{SYM} \longrightarrow \text{NSYM}$ natural operators

$s_\lambda \longmapsto S_{(m,\lambda)}$ $\quad$ $m \geq \lambda_1$; $m < \lambda_1$ there is twist to get ···

① $\quad \overset{\shortparallel}{\det(h_{\lambda_i+i-j})}$

$$B_{\lambda_1} \cdots B_{\lambda_{\ell-1}} \cdot B_{\lambda_\ell} \cdot 1 = s_\lambda$$

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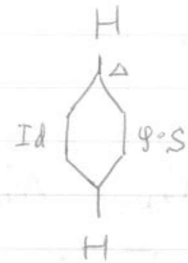
② $\varphi : H \longrightarrow H$ $\qquad$ H co-commutative then $\overline{\overline{\varphi}} = \varphi$

$\overline{\varphi} = Id * (\varphi \circ S)$



H graded by $\mathbb{N}$

$R^q : H \longrightarrow H$ $\qquad$ q is a parameter

$f \in H_n \longmapsto q^n f$

$$\left(\begin{array}{l} R^{q=0} = \underset{\text{counit}}{\varepsilon} \\ R^{q=1} = Id \end{array} \right)$$

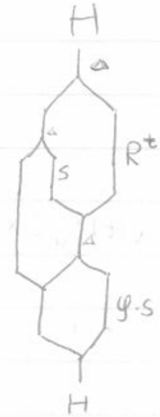
Define for $\varphi : H \longrightarrow H$

$$\widetilde{\varphi}^t = \overline{\overline{\varphi}\, R^t}$$

$\begin{cases} t=0 & \widetilde{\varphi}^0 = \varphi \\ t=1 & \widetilde{\varphi}^1 = \overline{\overline{\varphi} \cdot \underset{\text{mult}}{\mu}\varepsilon} \end{cases}$

THM [MIKE] $\boxed{\widetilde{B}^t_{\lambda_1} \cdots \widetilde{B}^t_{\lambda_{\ell-1}} \cdot \widetilde{B}^t_{\lambda_\ell} \cdot 1 = H_\lambda[X;t]}$

$\tilde{\varphi}^t$



<u>NSYM</u>

$$\boxed{\begin{aligned} A_m : NSYM &\longrightarrow NSYM \\ s_\alpha &\longmapsto s_{\alpha \cdot m} \end{aligned}}$$

$$\begin{aligned} B_m : NSYM &\longrightarrow NSYM \\ s_\alpha &\longmapsto s_{\alpha | m} \end{aligned}$$

$$\tilde{A}_m^t = A_m + B_m R^t$$

$$\tilde{A}_m^t(s_\alpha) = s_{\alpha \cdot m} + t^{|\alpha|} s_{\alpha | m}$$

Define

$$\begin{aligned} H_\alpha[X;t] &= \tilde{A}^t_{\alpha_k} \tilde{A}^t_{\alpha_{k-1}} \cdots \tilde{A}^t_{\alpha_1} \cdot 1 \\ &= s_\alpha + \sum_{\beta > \alpha} C^t_{\alpha\beta} s_\beta \end{aligned}$$

$t=0 \Rightarrow C^t_{\alpha\beta} = 0$

$t=1 \Rightarrow$ mult. of ..

$\updownarrow$

$t^{c(\alpha, \beta^c)}$

where $c(\alpha, \beta^c) = \sum_{i \in D(\alpha) \cap D(\beta^c)} i$, a kind of major index

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in SYM

$H_\lambda[X; q, t] = \cdots$

Mike will do it next time!

$\widetilde{H}_\mu[X;q,t]$ is a basis of the space of symmetric functions

- $\widetilde{H}_\mu[X(t-1);q,t] = \sum_{\lambda \leq \mu} r_{\lambda\mu}(q,t)\, S_\lambda[X]$
- $\widetilde{H}_\mu[X(1-q);q,t] = \sum_{\lambda \geq \mu} r'_{\lambda\mu}(q,t)\, S_\lambda[X]$
- $< \widetilde{H}_\mu[X;q,t], S_n[X] > = 1$

$\underset{\text{sym fun}}{f}[X(t-1)] = f[X]\Big|_{p_k \to (t^k-1)p_k}$ $\qquad f[X(1-q)] = f[X]\Big|_{p_k \to (1-q^k)p_k}$

eg. $\widetilde{H}_n[X;q,t]$

$\widetilde{H}_n[X(1-q);q,t] = \underset{\text{constant}}{C}\, S_n[X]$ $\qquad \widetilde{H}_\mu[X;q,t] = C\, S_n\left[\frac{X}{1-q}\right]$

$<C\, S_n[\frac{X}{1-q}], S_n[X]> = 1$

plug into computer to get the constant C

mike also can do this by hand

$$C < \sum P_\lambda[X] \Big/ \left(z_\lambda \prod_{i=1}^{\ell(\lambda)} (1-q^{\lambda_i})\right), \sum_\lambda P_\lambda[X] \big/ z_\lambda > = 1$$

$$= C \sum_\lambda \frac{1}{z_\lambda \prod_{i=1}^{\ell(\lambda)} (1-q^{\lambda_i})}$$

<u>n! Theorem</u> $\qquad \widetilde{H}_\mu[X;q,t] = \sum_\lambda \sum_{r,s} q^r t^s\, \mathrm{mult}_{\chi^\lambda}\left(M^\mu_{\substack{\deg x^r \\ \deg y^s}}\right) S_\lambda$

M^μ = linear span of derivitives of a determinant in the x & y variables

$\Delta_\mu = \det \left| x_i^{p_i}\, y_j^{q_j} \right|$ for (p_i, q_j) a coordinate of the partition μ.

$$\begin{array}{|c|c|c|} \hline 03 & & \\ \hline 02 & 12 & \\ \hline 01 & 11 & \\ \hline 00 & 10 & 20 \\ \hline \end{array}$$

$$\nabla\left(\widetilde{H}_\mu[X;q,t]\right) \overset{\text{def}}{=} q^{n(\mu')} t^{n(\mu)} \widetilde{H}_\mu[X;q,t]$$

$$n(\mu) = \sum_{i=1}^{\ell(\mu)} (i-1)\mu_i \qquad \begin{array}{|c|c|c|c|} \hline 3 \\ \cline{1-1} 2 \\ \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array} \quad \text{add all entries}$$

$\nabla(S_{1^n}[X])$ has been shown to be Schur positive and the graded Frob. characteristic of the "diagonal harmonics" $= \{ f(x_1, \cdots, x_n, y_1, \cdots, y_n)$ polynomial such that

$$\sum_{i=1}^{n} \partial_{x_i}^{k} \partial_{y_i}^{\ell} f(x_1, \cdots, x_n, y_1, \cdots, y_n) = 0 \quad \forall\ k+\ell \geq 1 \}$$

$\nabla(S_\lambda[X])$ is conjectured to be Schur positive or Schur negative.

$H_\mu[X;t]$ - Hall-Littlewood basis

$$H_{1^4}[X;t] = S_{1^4} + (t+t^2+t^3)S_{211} + (t^3+t^4)S_{22} + (t^3+t^4+t^5)S_{31} + t^6 S_4$$

$$H_{211}[X;t] = S_{211} + t S_{22} + (t+t^2) S_{31} + t^3 S_4$$

$$H_{22}[X;t] = S_{22} + t\, S_{31} + t^2 S_4$$

$$H_{31}[X;t] = S_{31} + t S_4$$

$$H_4[X;t] = S_4$$

$$H_\mu[X;q,t] \overset{\text{def}}{=} \widetilde{H}_\mu[X;q,1/t]\, t^{n(\mu)}$$

$$H_{1^4}[X;q,t] = s_{1^4} + (t+t^2+t^3)s_{211} + (t^2+t^4)s_{22} + (t^3+t^4+t^5)s_{31} + t^6 s_4$$

$$H_{211}[X;q,t] = q s_{1^4} + (1+qt+qt^2)s_{211} + (\cancel{t}+qt^2)s_{22} + (t+t^2+qt^3)s_{31} + t^3 s_4$$

$$H_{22}[X;q,t] = q^2 s_{1^4} + (q+q^2t+qt)s_{211} + (1+q^2t^2)s_{22} + (t+qt+qt^2)s_{31} + t^2 s_4$$

$$H_{31}[X;q,t] = q^3 s_{1^4} + (q+q^2+q^3t)s_{211} + (q+q^2t)s_{22} + (1+qt+q^3t)s_{31} + t s_4$$

$$H_4[X;q,t] = q^6 s_{1^4} + (q^3+q^4+q^5)s_{211} + (q^2+q^4)s_{22} + (q+q^2+q^3)s_{31} + s_4$$

set $q=0$

$t=\frac{1}{q}$, $q=\frac{1}{t}$ mult the highest power

$$H_\mu[X;0,t] = H_\mu[X;t] \quad \text{Hall-Littlewood}$$

$$H_\mu[X;q,0] = \omega H_{\mu'}[X;1/q]\, q^{n(\mu')}$$

Let $C_m(H_\mu[X;\frac{1}{q}]\,q^{n(\mu)}) \overset{def}{=} \begin{cases} H_{\mu+1^m}[X;q]\, q^{\overset{n(\mu+1^m)}{\overset{\parallel}{n(\mu)+\binom{m}{2}}}} & \text{if } m \geq \ell(\mu) \\ 0 & \text{otherwise} \end{cases}$

$$\tilde{H}_\mu[X;q,t] = \tilde{C}^t_{\mu_1}\tilde{C}^t_{\mu_2}\cdots\tilde{C}^t_{\mu_{\ell(\mu)}}\,1$$

check that RHS satisfied def of $\tilde{H}_\mu$

for an operator $V: \mathcal{H} \to \mathcal{H}$ ← Hopf alg.
$\tilde{V}^t$ was defined last time

$$NSym = \mathbb{Q}\langle h_1, h_2, \cdots\rangle$$

$$\tilde{\mathbf{HI}}^{qt}_\alpha = \sum_{\beta \models |\alpha|} t^{c(\alpha,\beta)} q^{c(\alpha',\overleftarrow{\beta})} \hat{S}_\beta$$

$$\mathbf{HI}^{qt}_\alpha = \sum_{\beta \models |\alpha|} t^{c(\alpha,\beta^c)} q^{c(\alpha',\overleftarrow{\beta})} \hat{S}_\beta$$

Open problem : find an "operator" definition of $\mathbf{HI}^{qt}_\alpha$

$$\tilde{\mathbf{HI}}^{q0}_\alpha = \tau\, \mathbf{HI}^{1/q}_{\alpha'}\, q^{n(\alpha')} = \sum_{\beta \geq \alpha} q^{c(\alpha',\overleftarrow{\beta})} \hat{S}_\beta$$

$$\tilde{H}_\alpha^{0t} = H_\alpha^{1/t}\, t^{n(\alpha)} = \sum_{\beta \geqslant \alpha} t^{c(\alpha,\beta)} S_\beta \quad \Bigg\} \text{ related to Hall-Littlewood}$$

where $H_\alpha^t = \sum_{\beta \geqslant \alpha} t^{c(\alpha,\beta^c)} S_\beta$

Luc & Jennifer's idea

Look at HL indexed by parts up to k

							2-Schur functions
H_{14}	1	$t+t^2+t^3$	t^2+t^4	$t^3+t^4+t^5$	t^6	$S^{(2)}_{14}$	
H_{211}		1	t	$t+t^2$	t^3	$S^{(2)}_{211}$	
H_{22}			1	t	t^2	$S^{(2)}_{22}$	

Find a basis such that $\{H_{14}, H_{211}, H_{22}\}$ expands positively and itself is Schur positive.

	1^4	211	22	31	4	conjugate
$S^{(2)}_{14}$	1	t	t^2			
$S^{(2)}_{211}$		1		t		
$S^{(2)}_{22}$			1	t	t^2	

$$H_{14} = S^{(2)}_{14} + (t^2+t^3) S^{(2)}_{211} + t^4 S^{(2)}_{22}$$

$$H_{211} = S^{(2)}_{211} + t\, S^{(2)}_{22}$$

$$H_{22} = S^{(2)}_{22}$$

$$H_{14}[X;q,t] = S^{(2)}_{14} + (t^2+t^3) S^{(2)}_{211} + t^4 S^{(2)}_{22}$$
$$H_{211}[X;q,t] = q S^{(2)}_{14} + (1+qt^2) S^{(2)}_{211} + t\, S^{(2)}_{22}$$
$$H_{22}[X;q,t] = q^2 S^{(2)}_{14} + (q+qt) S^{(2)}_{211} + S^{(2)}_{22}$$

$$H_{22}^{qt} = t^2 s_4 + q^3 t^2 s_{13} + s_{22} + q^3 s_{112} + q t^2 s_{31} + q^4 t^2 s_{121} + q s_{211} + q^4 s_{1111}$$

$$H_{211}^{qt} = t^5 s_4 + q^3 t^5 s_{13} + t^3 s_{22} + q^3 t^3 s_{112} + t^2 s_{31} + q^3 t^2 s_{121} + s_{211} + q^3 s_{1111}$$

$$H_{112}^{qt} = t^3 s_4 + t^2 s_{13} + t s_{22} + s_{112} + q t^3 s_{31} + q t^2 s_{121} + q t s_{211} + q s_{1111}$$

$$H_{1111}^{qt} = t^6 s_4 + t^5 s_{13} + t^4 s_{22} + t^3 s_{112} + t^3 s_{31} + t^2 s_{121} + t s_{211} + s_{1111}$$

$$H_\alpha^{qt} = \sum_{\beta \models |\alpha|} t^{c(\alpha, \beta^c)} q^{c(\alpha', \tilde{\beta})} S_\beta$$

① the k-Schur functions should be Schur positive & triangular (N.)

② the Hall-Littlewoods functions should be positive in k-Schur's & triangular (N.)

③ there is some symmetry with ω

ω(some k-Schur) = some other k-Schur | mult by t^*, $t \to 1/t$

Open problem : Define (in the commutative case) a set of conditions similar to ① ② & ③ which uniquely determine the k-Schur functions.

$$H_{111}^t = s_{111} + t s_{21} + t^2 s_{12} + t^3 s_3$$

$$\cancel{H_{21}^t = s_{21} + t^2 s_3}$$

$$H_{12}^t = s_{12} + t s_3$$

12 & 111

$$S_{111}^{(12)} = H_{111}^t - t^2 H_{12}^t = s_{111} + t s_{21}$$

$$S_{12}^{(12)} = H_{12}^t = s_{12} + t s_3$$

$$H_{22}^t = s_{22} + t^2 s_4$$

$$H_{112}^t = s_{112} + t s_{22} + t^2 s_{13} + t^3 s_4$$

$$\mathbb{H}^t_{211} = S_{211} + t^3 S_{22} + t^2 S_{31} + t^5 S_4$$
$$\mathbb{H}^t_{1111} = \mathbb{H}^{qt}_{1111}$$

$$S^{(22)}_{22} = S_{22} + t^2 S_4$$
$$S^{(22)}_{112} = \mathbb{H}^t_{112} - t S^{(22)}_{22} = S_{112} + t^2 S_{13}$$
sym. sym.
$$S^{(22)}_{211} = \mathbb{H}^t_{211} - t^3 S^{(22)}_{22} = S_{211} + t^2 S_{31}$$
$$S^{(22)}_{1111} = \mathbb{H}^t_{1111} - t^4 S^{(22)}_{22} - t S^{(22)}_{211} - t^3 S^{(22)}_{112}$$
$$= S_{1111} + t^2 S_{121}$$

Note with this example

$$\mathbb{H}^t_{22} = S^{(22)}_{22}$$
$$\mathbb{H}^t_{112} = S^{(22)}_{112} + t S^{(22)}_{22}$$
$$\mathbb{H}^t_{211} = S^{(22)}_{211} + t^3 S^{(22)}_{22}$$
$$\mathbb{H}^t_{1111} = S^{(22)}_{1111} + t S^{(22)}_{211} + t^3 S^{(22)}_{112} + t^4 S^{(22)}_{22}$$

<u>Define</u> $$S^{(\alpha)}_{\beta,t} = \sum_{\substack{\gamma \geq \beta \\ D(\beta)\setminus D(\gamma) \subseteq D(\alpha)}} t^{c(\beta,\gamma^c)} S_\gamma \quad \text{where } \beta \leq \alpha$$

① $S^{\alpha}_{\beta,t}$ expands positively in $S^{(\alpha')}_{\gamma,t}$ for α' finer than α

② if $\alpha \longrightarrow (1^n)$ then $S^{(1^n)}_{\beta,t} = S_\beta$

③ $S^{(\alpha)}_{\alpha} = \mathbb{H}^t_\alpha$

④ NC-Mac & NCHL expand positively in $S^{(\alpha)}_{\beta,t}$ indexed by $\gamma \leq \alpha$

⑤ $\omega\left(S^{(\alpha)}_{\beta,t}\right) = t^{n(\alpha)} S^{(\alpha)}_{\gamma,1/t}$ where $D(\gamma^c) = D(\beta)\setminus D(\alpha)$

⑥ $\tau\left(S^{(\alpha)}_{\beta,t=1}\right) = S^{(\overleftarrow{\alpha})}_{\overleftarrow{\beta},t=1}$

<u>Open problems</u>

Ⓐ coproduct should be positive in these elements

(B) product should be positive at $t=1$.

(C) should be a strange t product where at $t=1$ reduces to (B)

(D) does this represent the Schubert basis for some cohomology rings.

or

(D') What is the representation ring behind this?

About open problem from last week:

$$V^t_{1^r}\left(\mathbb{H}^t_\alpha\right) = \mathbb{H}^t_{(\alpha_1, \cdots, \alpha_k+1, \underbrace{1, 1, \cdots, 1}_{r-1})}$$

is $\widetilde{V}^{t\,q}_{1^{\alpha_1}} \cdots \widetilde{V}^{t\,q}_{1^{\alpha_k}}\, 1 \overset{?}{=} NCMac$